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ON THE EXISTENCE OF SOLUTIONS OF MARTINGALE INTEGRAL EQUATIONS

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1. In the present paper we shall consider the following stochastic integral equation:

$$(1) \qquad X_t = x + \int_0^t f(X_{u-})dM_u + \int_0^t g(X_{u-})dA_u\,, \qquad X_0 = x \in R$$

where (M_t) is a local martingale and (A_t) is an increasing process. This is a continuation of [1] in which we assumed the square integrability of each M_t and the continuity of the process (A_t).

2. Let (Ω, F, P) be a complete probability space, given an increasing right continuous family (F_t) of sub σ-fields of F. We assume as usual that F_0 contains all the negligible sets. In addition, suppose the family (F_t) is quasi-left continuous; namely, for every stopping time T and every sequence (T_n) of stopping times such that $T_n \uparrow T$, the σ-field F_T is generated by the field $\bigcup_{n=1}^{\infty} F_{T_n}$. A notation such that "let $M = (M_t, F_t)$ be martingale" means that the martingale property is relative to the family (F_t). All martingales below are assumed to be right continuous.

By a normal change of time $C = (F_t, c_t)$ we mean a family of stopping times of the family (F_t), finite valued, such that for a・e ω the sample function $c.(\omega)$ is strictly increasing,

$$c_0(\omega) = 0,\ c_\infty(\omega) = \lim_{t\to\infty} c_t(\omega) = \infty$$

and continuous.

As usual, we do not distinguish two processes X and Y such that for a・e ω $X.(\omega) = Y.(\omega)$. This is important for the understanding of uniqueness statements below.

DEFINITION. A right continuous process $M = (M_t, F_t)$ is a local martingale if there exists a sequence of stopping times $T_n \uparrow \infty$ such that for every n the process $M_{t\wedge T_n}$ on the set $\{T_n > 0\}$ is a uniformly integrable martingale.

We assume in this paper that $M_0 = 0$.

3. We are now in a position to state our result.

THEOREM. *Let f and g be real valued bounded functions such that for all $x, y \in R$*

$$\text{(2)} \qquad \mathrm{Max}(|f(x)-f(y)|, |g(x)-g(y)|) \leqq \alpha|x-y|$$

where α is some constant. Then the equation (1) *has a unique solution.*

The key to the proof of this theorem is the following lemma which is closed to the Gundy decomposition of martingales. Since it is proved in [2], we omit the proof.

LEMMA. *Let M be a local martingale. Then there exist stopping times $R_n \uparrow \infty$ such that the process $M_{t\wedge R_n}$* can be written as

$$\text{(3)} \qquad M_{t\wedge R_n} = H_t + V_t,\ V_t = M_{R_n} I_{\{t \geqq R_n\}} + B_t^{(1)} - B_t^{(2)}$$

where (H_t) is an L^2-bounded martingale stopped at R_n and each $(B_t^{(i)})$, $i = 1, 2$, is a natural increasing process.

Of course, H and $B^{(i)}$ depend on R_n. Note that if the family (F_t) is quasi-left continuous, then any natural increasing process is continuous; so $B^{(i)}$ is continuous. This fact is important in the following.

PROOF OF THEOREM. Let us keep the notations used in the lemma. As is well known, there exists a unique continuous increasing process $\tilde{A}_t$ such that the process $A_t^* = A_t - \tilde{A}_t$ is a martingale. Then we can rewrite the equation (1) in the form

$$\text{(4)} \qquad X_t = x + \int_0^t f(X_{u-})dM_u + \int_0^t g(X_{u-})dA_u^* + \int_0^t g(X_{u-})d\tilde{A}_u\ .$$

Therefore, there is no loss of generality in assuming that the process A is continuous, as we now do.

First, we shall treat the equation (1) on the stochastic interval $[0, R[$, where R is one of the stopping times (R_n) in the lemma. On this interval we have

$$\text{(5)} \quad X_t = x + \int_0^t f(X_{u-})dH_u + \int_0^t f(X_{u-})dB_u^{(1)} - \int_0^t f(X_{u-})dB_u^{(2)} + \int_0^t g(X_{u-})dA_u\ .$$

As is well known, there exists a unique continuous increasing process $\langle H\rangle$ such that $H^2 - \langle H\rangle$ is a martingale.

Define now:

$$\text{(6)} \qquad \lambda_t = t + \langle H\rangle_t + B_t^{(1)} + B_t^{(2)} + A_t\ , \qquad \theta_t = \inf\{u: \lambda_u > t\}\ .$$

Clearly (λ_t) is a continuous increasing process with $P(\lambda_0 = 0, \lambda_\infty = +\infty) = 1$.

Then an easy computation shows that $\Theta = (F_t, \theta_t)$ and $\Lambda = (F_{\theta_t}, \lambda_t)$ are normal change of time. It is obvious that λ_R is a stopping time with respect to the Family (F_{θ_t}) and the process $(t - \langle H \rangle_{\theta_t} - B^{(1)}_{\theta_t} - B^{(2)}_{\theta_t} - A_{\theta_t}, F_{\theta_t})$ is increasing. As $\theta_t < R$ on the set $\{t < \lambda_R\}$, we get from (5)

$$(7)\quad X_{\theta_t} = x + \int_0^t f(X_{\theta_{u-}})dM_{\theta_u} + \int_0^t g(X_{\theta_{u-}})dA_{\theta_u}$$

$$= x + \int_0^t f(X_{\theta_{u-}})dH_{\theta_u} + \int_0^t f(X_{\theta_{u-}})dB^{(1)}_{\theta_u} - \int_0^t f(X_{\theta_{u-}})dB^{(2)}_{\theta_u} + \int_0^t g(X_{\theta_{u-}})dA_{\theta_u}$$

on the stochastic interval $[0, \lambda_R[$ relative to the family (F_{θ_t}).

Therefore, in order to show the existence of a unique solution of the equation (1) on the interval $[0, R[$, it suffices to consider the equation (7) in stead of (1). Namely, there is no loss of generality in assuming that the process $(t - \langle H \rangle_t - B^{(1)}_t - B^{(2)}_t - A_t, F_t)$ is increasing, as we now do. For simplicity, the proof is spelled out for $0 \leqq t \leqq 1$ only.

Define in succession:

$$(8)\quad X^0_t = x$$

$$X^{n+1}_t = x + \int_0^t f(X^n_{u-})dM_u + \int_0^t g(X^n_{u-})dA_u\,, \qquad n = 1, 2, \cdots .$$

Clearly the processes $(f(X^n_t))$ and $(g(X^n_t))$ are right continuous.

Put now: $c^n_t = f(X^n_t) - f(X^{n-1}_t)$, $d^n_t = g(X^n_t) - g(X^{n-1}_t)$. For simplicity, we assume that $\alpha \leqq 1/4$. Then, by using the Schwarz inequality, we have

$$\begin{aligned} D_n(t) &\equiv E[(X^{n+1}_t - X^n_t)^2 I_{\{t<R\}}] \\ &= E\Big[\Big(\int_0^t c^n_{u-}dM_u + \int_0^t c^n_{u-}dB^{(1)}_u - \int_0^t c^n_{u-}dB^{(2)}_u + \int d^n_{u-}dA_u\Big)^2 I_{\{t<R\}}\Big] \\ &\leqq 4E\Big[\int_0^t (c^n_u)^2 I_{\{u<R\}} d\langle H \rangle_u + B^{(1)}_t \int_0^t (c^n_u)^2 I_{\{u<R\}} dB^{(1)}_u \\ &\quad + B^{(2)}_t \int_0^t (c^n_u)^2 I_{\{u<R\}} dB^{(2)}_u + A_t \int_0^t (d^n_u)^2 I_{\{u<R\}} dA_u\Big] \\ &\leqq (4\alpha)^2 \int_0^t E[(X^n_u - X^{n-1}_u)^2 I_{\{u<R\}}]du \\ &\leqq \int_0^t D_{n-1}(u)du;\ D_0(t) \leqq (4K)^2 t, \text{ where } K = \mathrm{Max}\,(\|f\|_\infty, \|g\|_\infty)\,. \end{aligned}$$

As $\sup_{0\leqq t\leqq 1} D_0(t) \leqq (4K)^2$, we derive the estimate

$$(9)\qquad D_n(t) \leqq (4K)^2 \frac{t^{n+1}}{(n+1)!}\,.$$

Since the process $\left(\int_0^t c^n_{u-}dH_u, F_t\right)$ is an L^2-bounded martingale, the extension of Kolmogorov's inequality to martingales shows that for any $\varepsilon > 0$

$$
\begin{aligned}
(10)\qquad P\Big(\sup_{0\leqq t\leqq 1}\Big|\int_0^t c_{u-}^n dH_u\Big| \geqq \varepsilon\Big) &\leqq \varepsilon^{-2}E\Big[\int_0^t (c_u^n)^2 d\langle H\rangle_u\Big] \\
&\leqq \varepsilon^{-2}E\Big[\int_0^t (c_u^n)^2 I_{\{u<R\}} d\langle H\rangle_u\Big] \qquad (\because H_t = H_{t\wedge R}) \\
&\leqq \alpha^2\varepsilon^{-2}\int_0^1 D_{n-1}(u)du\ .
\end{aligned}
$$

Next, we get by using the Schwarz inequality

$$
\begin{aligned}
(11)\qquad P\Big(\sup_{0\leqq t\leqq 1}\Big|\int_0^t c_{u-}^n dB_u^{(i)}\Big| \geqq \varepsilon\Big) &= P\Big(\sup_{0\leqq t\leqq 1}\Big(\int_0^t c_{u-}^n dB_u^{(i)}\Big)^2 \geqq \varepsilon^2\Big) \\
&\leqq P\Big(\sup_{0\leqq t\leqq 1} B_t^{(i)}\int_0^t (c_{u-}^n)^2 dB_u^{(i)} \geqq \varepsilon^2\Big) \\
&\leqq P\Big(\int_0^1 (c_{u-}^n)^2 I_{\{u<R\}} du \geqq \varepsilon^2\Big) \qquad (\because B_t^{(i)} = B_{t\wedge R}^{(i)}) \\
&\leqq \alpha^2\varepsilon^{-2}\int_0^1 D_{n-1}(u)du\ .
\end{aligned}
$$

Similarly we obtain

$$
(12)\qquad P\Big(\sup_{0\leqq t\leqq 1}\Big|\int_0^t d_{u-}^n dA_{u\wedge R}\Big| \geqq \varepsilon\Big) \leqq \alpha^2\varepsilon^{-2}\int_0^1 D_{n-1}(u)du\ .
$$

Thus $P(\sup_{0\leqq t\leqq 1}|X_t^{n+1} - X_t^n| I_{\{t<R\}} \geqq 4\varepsilon) \leqq \text{Const.} \times \varepsilon^{-2}/(n+1)!$. Pick $\varepsilon^{-2} = (n-1)!$. Then $\varepsilon^{-2}/(n+1)!$ is the general term of a convergent sum, and so the Borel-Cantelli lemma shows that the processes $(X_t^n I_{\{t<R\}})$ converge uniformly almost surely for $0 \leqq t \leqq 1$ to some right continuous process $X^R = (X_t^R, F_t)$. Furthermore by using the extension of Kolmogorov's inequality to martingales we have

$$
\begin{aligned}
&P\Big(\sup_{0\leqq t\leqq 1}\Big|\int_0^t f(X_{u-}^n)dH_u - \int_0^t f(X_{u-}^R)dH_u\Big| \geqq \varepsilon\Big) \\
&\qquad \leqq \varepsilon^{-2}E\Big[\int_0^1 \{f(X_u^n) - f(X_u^R)\}^2 I_{\{u<R\}} d\langle H\rangle_u\Big]\ .
\end{aligned}
$$

According to the bounded convergence theorem, the right hand side of this inequality converges to 0 as $n \to \infty$. Thus the processes $\Big(\int_0^t f(X_{u-}^{n_k})dH_u\Big)$ converge uniformly almost surely to the process $\Big(\int_0^t f(X_{u-}^R)dH_u\Big)$ for some subsequence (n_k). It is not difficult to see that $\Big(\int_0^t f(X_{u-}^n)dB_u^{(i)}\Big)$ and $\Big(\int_0^t g(X_{u-}^n)dA_{u\wedge R}\Big)$ converge uniformly almost surely to $\Big(\int_0^t f(X_{u-}^R)dB_u^{(i)}\Big)$ and $\Big(\int_0^t g(X_{u-}^R)dA_{u\wedge R}\Big)$ respectively. Consequently the process X^R satisfies the following equality:

$$(13)\quad X_t^R = x + \int_0^t f(X_{u-}^R)dH_u + \int_0^t f(X_{u-}^R)dB_u^{(1)} - \int_0^t f(X_{u-}^R)dB_u^{(i)} + \int_0^t g(X_{u-}^R)dA_u \quad \text{on } [0, R[\ .$$

That is, X^R satisfies the equation (1) on the interval $[0, R[$.

Now let X and Y be two solutions of the equation (1) on $[0, R[$. Then we can obtain as in the proof of (9)

$$(14)\qquad D(t) \equiv E[(X_t - Y_t)I_{\{t<R\}}] \leqq \int_0^t D(u)du\ , \qquad \sup_{0\leqq t\leqq 1} D(t) \leqq 32K^2\ ,$$

from which $D(t) = 0$. This implies that $X = Y$ on $[0, R[$.

Next, for each n, let $X^{R_n} = (X_t^{R_n}, F_t)$ be a solution of the equation (1) on the stochastic interval $[0, R_n[$. $X^{R_{n+1}}$ being also a solution of (1) on $[0, R_n[$, we get $X^{R_n} = X^{R_{n+1}}$ on $[0, R_n[$. This relation therefore defines a right continuous process X such that

$$(15)\qquad X = X^{R_n} \text{ on } [0, R_n[,\ n = 1, 2, , \cdots$$

Furthermore, for each n,

$$E\left[\left\{\int_0^t (fX_{u-}^{R_n}) - f(X_{u-}))dH_u\right\}^2\right] = E\left[\int_0^t \{f(X_{u-}^{R_n}) - f(X_{u-})\}^2 I_{\{u<R_n\}} d\langle H\rangle_u\right] = 0\ ,$$

from which $\int_0^t f(X_{u-})dH_u = \int_0^t f(X_{u-}^{R_n})dH_u$ on $[0, R_n[$.

Obviously we have on the interval $[0, R_n[$

$$\int_0^t f(X_{u-})dB_u^{(i)} = \int_0^t f(X_{u-}^{R_n})dB_u^{(i)} \text{ and } \int_0^t g(X_{u-})dA_u = \int_0^t g(X_{u-}^{R_n})dA_u\ .$$

Thus, the process X satisfies the equation (1) on each $[0, R_n[$. As $R_n \uparrow \infty$, X is a solution of (1).

Finally, we are going to show its uniqueness. If X and Y are two solutions of (1), then these processes satisfy the equation (1) on each interval $[0, R_n[$. Therefore $X = Y$ on $[0, R_n[$ for each n. Hence $X = Y$. This completes the proof.

4. In the following, instead of the quasi-left continuity of the family (F_t), we assume that the local martingale M and the increasing process A are continuous.

PROPOSITION. *Let ρ and κ be positive increasing function defined on $(0, \infty)$. Suppose that*

$$(16)\qquad \int_{0+} \rho^{-2}(u)du = +\infty\ , \quad \int_{0+} \kappa^{-1}(u)du = +\infty$$

$$|f(x) - f(y)| \leqq \rho(|x-y|)\ , \quad |g(x) - g(y)| \leqq \kappa(|x-y|)\ , \quad \forall x, y \in R$$

and κ is concave.

Then the uniqueness holds for the equation (1).

By using a normal change of time, this proposition can be proved in the same way as Theorem 1 of [3].

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