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GENERATORS OF W^*-ALGEBRAS II

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In [1] it was shown that generators of properly infinite W^*-algebras can be chosen from rather restricted classes of operators. Here we shall show that generators of properly infinite W^*-algebras abound in another sense. Let $\mathfrak{A}$ be a W^*-algebra on the separable Hilbert space H with no summand of type II_1, then the set of generators of $\mathfrak{A}$ is a norm dense set in $\mathfrak{A}$. Moreover any operator $T \in \mathfrak{A}$ can be written as the sum of two generators of $\mathfrak{A}$. These results have been obtained previously for $B(H)$, the algebra of all bounded linear operators on H [5, 7, 8, 9]. These results are also valid for certain W^*-algebras of type II_1, for example the hyperfinite factor of type II_1. Throughout all Hilbert spaces will be separable and all W^*-algebras are assumed to act on separable Hilbert spaces. For a W^*-algebra $\mathfrak{A}$ we denote the algebra of all k by k matrices with entries from $\mathfrak{A}$ by $M_k(\mathfrak{A}) = \mathfrak{A} \otimes M_k$. In this notation M_∞ stands for the algebra of all bounded linear operators on a separable infinite dimensional Hilbert space. For $A_1, A_2, \ldots \in B(H)$, the W^*-algebra generated by $A_1, A_2, \ldots$ will be denoted by $\mathfrak{R}(A_1, A_2, \ldots)$. For $T \in B(H)$, $T = A + iB$ will always stand for the decomposition of T into its real and imaginary part. The spectrum of an operator T will be denoted by $\operatorname{Sp} T$. For a W^*-algebra $\mathfrak{A}$ let $\mathfrak{A}_h$ ($\mathfrak{A}_+$) be the set of hermitean (positive) elements of $\mathfrak{A}$.

LEMMA 1. a) *Let* $A_1, A_2 \in B(H)$ *with* $Sp\, A_1 \cap Sp\, A_2 = \emptyset$ *then* $CA_1 = A_2C$ *implies* $C = 0$.

b) *Let* $A_1, A_2 \in B(H)_h$ *and* $C \in B(H)$ *positive and invertible, then* $CA_1 = A_2C$ *implies* $A_1 = A_2$.

PROOF. a) This is an easy consequence of a result of Rosenblum [10].

b) $CA_1 = A_2C$ implies $A_1C = CA_2$ and $C^2A_1 = CA_2C = A_1C^2$. Since C is positive A_1 commutes also with C. Thus $A_1C = CA_1 = A_2C$ and $A_1 = A_2$, because C is invertible.

LEMMA 2. *Let* $\mathfrak{A}$ *be an abelian* W^**-algebra, then the set of hermitean*

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generators of $\mathfrak{A}$ *is dense in* $\mathfrak{A}_h$.

PROOF. Let $A \in \mathfrak{A}_h$ and $\varepsilon > 0$, then there exist an invertible operator $A' \in \mathfrak{A}_h$ with a finite spectrum and $|A - A'| < \varepsilon$. Let $A' = \sum_{i=1}^n \lambda_i e_i$ be the spectral decomposition of A' and let η be the smallest distance between the points in the set $\{0, \lambda_1, \cdots, \lambda_n\}$. Each abelian W^*-algebra $\mathfrak{A}e_i$, $1 \leqq i \leqq n$, has a hermitean generator a_i with $|a_i| < \min(\varepsilon, \eta/2)$. Then $A'' = \sum_{i=1}^n e_i(\lambda_i + a_i)$ is a hermitean generator of $\mathfrak{A}$ and $|A - A''| < 2\varepsilon$.

LEMMA 3. *Let the* W^**-algebra* $\mathfrak{A}$ *be the* (*countable*) *direct sum of* W^**-algebras* $\mathfrak{A}_i$, $\mathfrak{A} = \sum \oplus \mathfrak{A}_i$, *and let* $T = \sum \oplus T_i \in \mathfrak{A}$. *Assume for each* i *and for some* $\varepsilon > 0$ *there exists a generator* T_i' *of* $\mathfrak{A}_i$, *with* $|T_i - T_i'| < \varepsilon$; *then there exists a generator* T'' *of* $\mathfrak{A}$ *with* $|T - T''| < 2\varepsilon$.

PROOF. Let $T' = \sum \oplus T_i'$ and let $T' = A' + iB'$ be the decomposition of T' into its real and imaginary part. B' lies in some maximal abelian subalgebra $\mathfrak{B}$ of $\mathfrak{A}$, and by Lemma 2 there exists a generator $B'' \in \mathfrak{B}_h$ of $\mathfrak{B}$ with $|B' - B''| < \varepsilon$. Then $T'' = A' + iB''$ is the desired operator. $|T - T''| < 2\varepsilon$ is obvious. Let be the central projection onto $\mathfrak{A}_i$, $\mathfrak{A}_i = \mathfrak{A}z_i$. Then $z_i \in \mathfrak{B}$ and any $D = D^* \in \mathfrak{R}(T'')'$ commutes with all z_i. Hence $D = \sum \oplus D_i$ and $D_i \in \mathfrak{R}(T''z_i)' = \mathfrak{R}(T_i')' = \mathfrak{A}_i'$. This shows $D \in \mathfrak{A}'$ or $\mathfrak{R}(T'') = \mathfrak{A}$.

COROLLARY. *Let* $\mathfrak{A} = \sum \oplus \mathfrak{A}_i$ *be such that each* $\mathfrak{A}_i$ *has a dense set of generators. Then* $\mathfrak{A}$ *has a dense set of generators.*

Thus the W^*-algebra $\mathfrak{A}$ has a dense set of generators, if its parts of type I_n, type II and type III each have this property. Apart from this rather obvious application the lemma will also be used in the following way. Let $\mathfrak{A}$ be a W^*-algebra, $T \in \mathfrak{A}$ and $\varepsilon > 0$ arbitrary. Then determine a countable central decomposition of $\mathfrak{A}$, $\mathfrak{A} = \sum \oplus \mathfrak{A}_i$ and $T = \sum \oplus T_i$, which may depend on T, such that for each i the operator T_i can be approximated within ε by a generator T_i' of $\mathfrak{A}_i$. Then the lemma shows the existence of a generator T'' of $\mathfrak{A}$ with $|T - T''| < 2\varepsilon$.

PROPOSITION 1. *A finite* W^**-algebra* $\mathfrak{A}$ *of type I on a separable Hilbert space has a norm dense set of generators.*

PROOF. By Lemma 3 and the remarks following it we may assume $\mathfrak{A}$ to be homogeneous of type I_n, with $n < \infty$. Then we can write $\mathfrak{A} = \mathfrak{Z} \otimes M_n$, where $\mathfrak{Z}$ is the center of $\mathfrak{A}$. Let $\varepsilon > 0$ and let $T = A + iB \in \mathfrak{A}$. By [3, Cor. 3.3] we may assume A to be diagonal, $A = \operatorname{diag}(a_1, \ldots, a_n)$ with $a_1, \ldots, a_n \in \mathfrak{Z}$. Let $\mathfrak{B}$ be the algebra of all diagonal operators $C = \operatorname{diag}(c_1, \ldots, c_n)$, with $c_1, \ldots, c_n \in \mathfrak{Z}$. $\mathfrak{B}$ is a maximal abelian subalgebra of $\mathfrak{A}$, and by Lemma 2 there exists a hermitean generator A' of $\mathfrak{B}$ with

$|A - A'| < \varepsilon/2$. Let $B = (b_{i,j})_{i,j=1}^n$, then we can find invertible operators $b'_{i,j} = b'^*_{j,i} \in \mathfrak{Z}$ with $|b_{i,j} - b'_{i,j}| < 1/2 \cdot \varepsilon \cdot n^{-2}$, with $1 \leqq i, j \leqq n$. Set now $B' = (b'_{i,j})_{i,j=1}^n$ and $T' = A' + iB'$. Then $|T - T'| < \varepsilon$. To show $\mathfrak{R}(T') = \mathfrak{A}$ let $D = D^* \in \mathfrak{R}(T')'$. Since in particular $D \in \mathfrak{B}'$, D is diagonal, $D = \operatorname{diag}(d_1, \ldots, d_n)$ with $d_i \in \mathfrak{Z}$. Then $DB' = B'D$ gives $d_1 b'_{1,i} = b'_{1,i} d_i = d_i b'_{1,i}$ or $d_1 = d_i$ for $1 \leqq i \leqq n$, because the $b'_{1,i}$ are invertible. $D \in \mathfrak{A}'$ is now obvious. Hence $\mathfrak{R}(T')' = \mathfrak{A}'$ or $\mathfrak{R}(T') = \mathfrak{A}$.

We state now a number of results on operators in W^*-algebras, which will be needed later. Most of these results are based on the polar decomposition of operators [2].

Let $\mathfrak{A}$ be a W^*-algebra and $T \in \mathfrak{A}$. We say T has finite rank if there exists a finite projection $P \in \mathfrak{A}$ with $TP = T$. In a purely infinite W^*-algebra only the 0 operator is of finite rank. Clearly $T \in \mathfrak{A}$ has finite rank if and only if T^* has finite rank. Let $\{Q_i\}_{i=1}^n$ with $n = 1, 2, \ldots, \infty$, be a family of equivalent orthogonal projections in $\mathfrak{A}$ with $\sum Q_i = 1$. The $\{Q_i\}_{i=1}^n$ induce a tensor decomposition of $\mathfrak{A}$ [2, ch. I § 2], $\mathfrak{A} = \mathfrak{B} \otimes M_n$. In this notation $T \in \mathfrak{A}$ has the matrix form $T = (t_{i,j})_{i,j=1}^n$ with $t_{i,j} = V_i^* T V_j$ where $V_i V_i^* = Q_i$ and $V_i^* V_i = Q_1$. Let P be a finite projection in $\mathfrak{A}$, then PV_j has finite rank. Thus if $T \in \mathfrak{A}$ has finite rank the $t_{i,j}$ have finite rank too. The converse holds if n is finite.

In any W^*-algebra $\mathfrak{A}$ a partial isometry is a restriction of an isometry or a coisometry. Using this and the polar decomposition of operators in $\mathfrak{A}$, it is easy to see that any $T \in \mathfrak{A}$ can be approximated in the norm by an operator S such that SS^* or S^*S are invertible. This result is optimal as the example of a nonunitary isometry shows. However since partial isometries of finite rank are restrictions of unitary operators, operators of finite rank can be approximated by invertible operators. Let $\mathfrak{A}$ be a W^*-algebra and P a projection in $\mathfrak{A}$. Then there exists a unique central projection Z in $\mathfrak{A}$ such that ZP is finite and such that $(1 - Z)P$ is properly infinite.

Let $A \in \mathfrak{A}$ be a selfadjoint invertible operator with a finite spectrum and let $A = \sum_{i=1}^n \lambda_i P_i$ be the spectral resolution of A. Let Z_i with $1 \leqq i \leqq n$, be the unique central projection such that $(1 - Z_i)P_i$ is properly infinite. The projections Z_i generate a finite dimensional algebra $\mathfrak{Z}_0$ of central operators. Let Z be a minimal nonzero projection in $\mathfrak{Z}_0$. Then the spectral projections of the operator $AZ = \sum_{i=1}^n \lambda_i P_i Z \in \mathfrak{A}Z$ are either finite or properly infinite.

LEMMA 4. *Let $\mathfrak{A}$ be a properly infinite W^*-algebra on a separable*

Hilbert space and let $A \in \mathfrak{A}_h$. Then for any $\varepsilon > 0$ there exists a generator T of $\mathfrak{A}$ with $|T - A| < \varepsilon$.

PROOF. a) Since the invertible hermitean operators with finite spectrum are dense in $\mathfrak{A}_h$, we may assume without loss of generality that A is invertible and has a finite spectrum. Then by our remarks above $\mathfrak{A}$ has a finite decomposition by central projections Z_i of $\mathfrak{A}$ such that the spectral projections of AZ_i in $\mathfrak{A}Z_i$ are either finite or properly infinite. Because of our remarks following Lemma 3 we may thus assume that the spectral projections of A are either finite or infinite. Let $A = \sum_{i=1}^{n} \lambda_i P_i$ be the spectral resolution of A such that the projections $P_1, \ldots, P_k$ are properly infinite and such that $P_{k+1}, \ldots, P_n$ are finite. Since $\mathfrak{A}$ is properly infinite and $1 = \sum_{i=1}^{n} P_i$ we get $k \geqq 1$. If $k = 1$ we decompose P_1 into two equivalent orthogonal projections P_1' and P_1'' and replace A by the operator $A' = P_1'\lambda_1 + P_1''\lambda_1' + \sum_{i=2}^{n} \lambda_i P_i$ with $\lambda_1' \neq \lambda_i$, $\lambda_1' \in \boldsymbol{R}$ and $|\lambda_1' - \lambda_1| < \varepsilon/2$. For the operator A' we have $k \geqq 2$. Thus we may assume without loss of generality $k \geqq 2$.

b) Now let $Q_1 = P_1, \ldots, Q_{k-1} = P_{k-1}$ and $Q_k = P_k + \cdots + P_n$. Then the projections $Q_1, \ldots, Q_k$ are properly infinite, orthogonal, equivalent and satisfy $\sum_{i=1}^{k} Q_i = 1$. Thus they induce a tensor decomposition of $\mathfrak{A}$, $\mathfrak{A} \cong \mathfrak{B} \otimes M_k$. In this decomposition A is diagonal, $A = \operatorname{diag}(\lambda_1, \ldots, \lambda_{k-1}, a) = \operatorname{diag}(a_1, \ldots, a_k)$. By construction the operator $a - \lambda_k$ has finite rank in $\mathfrak{B}$. This will be of importance later. Furthermore we should point out that $\operatorname{Sp} a_i \cap \operatorname{Sp} a_j = \varnothing$ for $i \neq j$.

c) The algebra $\mathfrak{B}$ is generated by two positive invertible operators b and c, which we may choose such that $|b|, |c| \leqq \varepsilon/2$. Then set

$$B = \begin{bmatrix} b & d & \cdots & d \\ d & c & & \\ \vdots & & 0 & \\ \vdots & & & \ddots \\ d & & & & 0 \end{bmatrix}$$

with $d = \varepsilon/4k \cdot 1$ and let $T = A + iB$.

d) Clearly $|A - T| < \varepsilon$. To show $\mathfrak{R}(T) = \mathfrak{A}$ let $C = C^* \in \mathfrak{R}(T)'$. C can also be considered as a matrix, $C = (c_{i,j})_{i,j=1}^{k}$. Since $\operatorname{Sp} a_i \cap \operatorname{Sp} a_j = \varnothing$ for $i \neq j$ $CA = AC$ implies by Lemma 1 (a) that C is diagonal, $C = \operatorname{diag}(c_1, \ldots, c_k)$. Then $CB = BC$ gives $c_1 d = c_i d$ for $1 \leqq i \leqq k$, or $c_1 = \cdots = c_k$. $CB = BC$ gives further $c_1 b = bc_1$ and $c_1 c = cc_1$ or $c_1 \in \mathfrak{B}'$. Thus $\mathfrak{R}(T)' = \mathfrak{B}' \otimes 1$ or $\mathfrak{R}(T) = \mathfrak{A}$.

LEMMA 5. *Let $\mathfrak{A}$ be a properly infinite W^*-algebra on a separable*

Hilbert space and let $T = A + iB \in \mathfrak{A}$, with B of finite rank. Then for any $\varepsilon > 0$ there exists a generator T' of $\mathfrak{A}$ with $| T - T' | < 2\varepsilon$.

PROOF. a) Using the same arguments as in the proof of Lemma 4 (a) and (b), we may assume that $\mathfrak{A}$ has the form $\mathfrak{A} = \mathfrak{B} \otimes M_k$ and that A is diagonal, $A = \operatorname{diag}(\lambda_1, \ldots, \lambda_{k-1}, a) = \operatorname{diag}(a_1, \ldots, a_k)$ with $\operatorname{Sp} a_i \cap \operatorname{Sp} a_j = \varnothing$ for $i \neq j$. B has the form $B = (b_{i,j})_{i,j=1}^k$ and each $b_{i,j}$ for $1 \leqq i, j \leqq k$, has finite rank in $\mathfrak{B}$. Let η be the smallest distance between the points in the set $\{0\} \cup \operatorname{Sp} A$. By Lemma 4 there exists a positive operator $b \in \mathfrak{B}$ and a selfadjoint operator $b'_{1,1} \in \mathfrak{B}$ with $|b'_{1,1} - b_{1,1}| < \varepsilon/2$, $\mathfrak{R}(b'_{1,1}, b) = \mathfrak{B}$ and $| b | < \min(\eta/2, \varepsilon)$. Similarly there exist invertible operators $b'_{1,i} = b'^*_{i,1} \in \mathfrak{B}$ with $2 \leqq i \leqq k$ such that $| b_{1,i} - b'_{1,i} | < \varepsilon/4k$. Now set $b'_{i,j} = b_{i,j}$ for the remaining indices and define $B' = (b'_{i,j})_{i,j=1}^k$. The operator A' is defined by $A' = \operatorname{diag}(\lambda_1 + b, \lambda_2, \ldots, \lambda_{k-1}, a)$. Then the operator $T' = A' + iB'$ satisfies $| T - T' | < 2\varepsilon$.

b) Let $C = C^* \in \mathfrak{R}(T')'$. $CA' = A'C$ shows as before with the aid of Lemma 1 that C is diagonal, $C = \operatorname{diag}(c_1, c_2, \ldots, c_k)$. $CA' = A'C$ and $CB' = B'C$ give further $c_1 b = b c_1$ and $c_1 b'_{1,1} = b'_{1,1} c_1$. Thus $c_1 \in \mathfrak{B}'$. Then $CB' = B'C$ shows $b'_{1,i} c_1 = c_1 b'_{1,i} = b'_{1,i} c_i$ or $c_1 = c_i$ for $1 \leqq i \leqq k$, because the $b'_{1,i}$ are invertible. Hence $\mathfrak{R}(T')' = \mathfrak{B}' \otimes 1$ or $\mathfrak{R}(T) = \mathfrak{A}$.

With this lemma we can now prove the general result.

PROPOSITION 2. *The set of generators in a properly infinite W^*-algebra $\mathfrak{A}$ on a separable Hilbert space is norm dense.*

PROOF. a) Let $\varepsilon > 0$ and let $T = A + iB \in \mathfrak{A}$. We shall find a generator T' of $\mathfrak{A}$ with $| T - T' | < 2\varepsilon$. Arguing as in the Lemmas 4 and 5 there is no loss of generality if we assume $\mathfrak{A} = \mathfrak{B} \otimes M_k$, with $k \geqq 2$, and $A = \operatorname{diag}(\lambda_1, \ldots, \lambda_{k-1}, a)$. We may further assume $\lambda_i \neq \lambda_j$ for $i \neq j$ and $\lambda_i \notin \operatorname{Sp} a$ for $1 \leqq i, j \leqq k - 1$. By construction we know further that there exists a constant λ_k such that $\lambda_k - a$ has finite rank. B has the form $B = (b_{i,j})_{i,j=1}^k$. Let again η be the smallest distance between the points in $\{0\} \cup \operatorname{Sp} A$. By Lemma 4 we can find positive operators b_i for $1 \leqq i \leqq k - 1$ and selfadjoint operators $b'_{i,i}$ such that $| b_{i,i} - b'_{i,i} | < \varepsilon/2$, $| b_i | \leqq \min(\varepsilon, \eta/2)$ and $\mathfrak{R}(b'_{i,i}, b_i) = \mathfrak{B}$. Since a has essentially finite rank we can find by Lemma 5 selfadjoint operators $a', b'_{k,k} \in \mathfrak{B}$ with $| a - a' | < \min(\varepsilon, \eta/2)$, $| b'_{k,k} - b_{k,k} | < \varepsilon/2$ and $\mathfrak{R}(a', b'_{k,k}) = \mathfrak{B}$. In addition we choose operators $b'^*_{i,1} = b'_{1,i}$ such that $| b_{i,1} - b'_{i,1} | < \varepsilon/4k$ and such that $b'_{1,i} b'^*_{1,i}$ or $b'^*_{1,i} b_{1,i}$ is invertible. Set now $b'_{i,j} = b_{i,j}$ for the remaining indices and define $B' = (b'_{i,j})$. With $A' = \operatorname{diag}(\lambda_1 + b_1, \cdots, \lambda_{k-1} + b_{k-1}, a')$ and $T' = A' + iB'$ we have obviously $| T - T' | < 2\varepsilon$.

b) Let $C = C^* \in \mathfrak{R}(T')'$. C has the matrix form $C = (c_{i,j})_{i,j=1}^k$. As before $CA = AC$ shows that C is diagonal, $C = \operatorname{diag}(c_1, \ldots, c_k)$. Then $c_i \in \mathfrak{B}'$, for $1 \leqq i \leqq k$, follows as above from $CA = AC$ and $CB = BC$. Using the off-diagonal terms of $CB = BC$ we get

$$b'_{1,i}c_1 = c_1 b'_{1,i} = b'_{1,i}c_i = c_i b'_{1,i} ,$$

for $1 \leqq i \leqq k$. If $b'^*_{1,i}b'_{1,i}$ is invertible multiply this equation from the left by $b'^*_{1,i}$. One gets $b'^*_{1,i}b'_{1,i}c_1 = b'^*_{1,i}b'_{1,i}c_i$ or $c_1 = c_i$. Otherwise multiply by $b'^*_{1,i}$ from the right. Again $c_1 b'_{1,i}b'^*_{1,i} = c_i b'_{1,i}b'^*_{1,i}$ shows $c_1 = c_i$. Thus we have $c_1 = c_i$ for $1 \leqq i \leqq k$, because $b^*_{1,i}b_{1,i}$ or $b_{1,i}b^*_{1,i}$ are invertible. The remainder is now obvious.

Summing up we have:

THEOREM 1. *Any W^*-algebra $\mathfrak{A}$ on a separable Hilbert space with no direct summand of type II_1 has a norm dense set of generators.*

PROOF. Apply Lemma 3 to Propositions 1 and 2.

Next we want to extend Theorem 1 to W^*-algebras of type II_1. Since it is not yet known whether factors of type II_1 are singly generated, we introduce a class $\mathscr{A}$ of W^*-algebras of type II_1 on a separable Hilbert space with:

i) $\mathfrak{A} \in \mathscr{A}$ then $\mathfrak{A}$ is singly generated

ii) $\mathfrak{A} \in \mathscr{A}$ then there exists a $\mathfrak{B} \in \mathscr{A}$ with $\mathfrak{A} = \mathfrak{B} \otimes M_2$.

If every factor of type II_1 on a separable Hilbert space is singly generated, also every W^*-algebra of type II_1 on a separable Hilbert space is singly generated, because the direct integral of singly generated W^*-algebras is again singly generated (P. Willing, private communication). In that case $\mathscr{A}$ may be chosen to be the class of all W^*-algebras of type II_1 on a separable Hilbert space. In any case we may always assume that $\mathscr{A}$ contains the hyperfinite factor.

LEMMA 6. *Let $\mathfrak{A} \in \mathscr{A}$ and let $A \in \mathfrak{A}_h$. Then for any $\varepsilon > 0$ there exists a generator T of $\mathfrak{A}$ with $| T - A | < \varepsilon$.*

PROOF. a) Without loss of generality we may assume that A is invertible and has a finite spectrum. Let $A = \sum_{i=1}^{n} \lambda_i P_i$ be the spectral decomposition of A. Let $\natural$ be the natural center valued map [2]. Since $1 = \sum_{i=1}^{n} P_i = \sum_{i=1}^{n} P_i^\natural$ there exist finitely many orthogonal central projections $Z_1, \cdots, Z_m$ with $\sum Z_i = 1$ such that for each Z_j there exists a P_i with $P_i^\natural Z_j \geqq 2^{-n} Z_j$. Using again the remarks following Lemma 3, we may assume without loss of generality $P_1^\natural \geqq 2^{-n}$. Then by [4, Theorem 1] we can find a family $\{Q_i\}_{i=1}^{2^n}$ of orthogonal equivalent projections with $\sum_{i=1}^{2^n} Q_i = 1$, $Q_i A = A Q_i$ and $Q_1 \leqq P_1$. The $\{Q_i\}_{i=1}^{2^n}$ induce a tensor decomposition of $\mathfrak{A}$, $\mathfrak{A} = \mathfrak{B} \otimes M_{2^n}$. Then A has the form

$$A = \operatorname{diag}(\lambda_1, a_2, \ldots, a_{2^n}) .$$

Now we replace each a_i with $2 \leqq i \leqq 2^n$ by a selfadjoint operator a'_i with the same spectral projections as a_i, such that $|a_i - a'_i| < \varepsilon/2$ and such that $\operatorname{Sp} a'_i \cap \operatorname{Sp} a'_j = \emptyset$ for $i \neq j$, $1 \leqq i, j \leqq 2^n$, $\lambda_1 = a_1$. Then the operator $A' = \operatorname{diag}(\lambda_1, a'_2, \ldots, a'_{2^n})$ satisfies $|A - A'| < \varepsilon/2$.

b) $\mathfrak{B}$ is generated by the positive invertible operators b, c with $|b|, |c| \leqq \varepsilon/4$, because $\mathfrak{B} \in \mathscr{A}$. Then let B be given by the matrix

$$B = \begin{bmatrix} b & d & & & 0 \\ d & c & d & & \\ & d & 0 & d & \\ 0 & & \ddots & \ddots & d \\ & & & d & 0 \end{bmatrix}$$

with $d = \varepsilon/8 \cdot 1$.

c) The operator $T = A' + iB$ satisfies clearly $|A - T| < \varepsilon$. Let $C = C^* \in \mathfrak{R}(T)'$. Then $CA' = A'C$ shows as above by Lemma 1 that C is diagonal, $C = \operatorname{diag}(c_1, \ldots, c_{2^n})$. From this and $CB = BC$ one obtains by the same methods as before $c_1 = c_2 = \ldots = c_{2^n} \in \mathfrak{B}'$. Thus $\mathfrak{R}(T) = \mathfrak{A}$.

THEOREM 2. *Let $\mathfrak{A} \in \mathscr{A}$, then the set of generators of $\mathfrak{A}$ is dense in $\mathfrak{A}$.*

PROOF. a) Let $\varepsilon > 0$ and $T = A + iB \in \mathfrak{A}$. Arguing as in the proof of Lemma 6 we may assume $\mathfrak{A} = \mathfrak{B} \otimes M_{2^n}$ and

$$A = A^* = \operatorname{diag}(\lambda_1, a_2, \cdots, a_{2^n}) .$$

We may further assume that spectrum of A is finite and that

$$\operatorname{Sp} a_i \cap \operatorname{Sp} a_j = \emptyset$$

for $i \neq j$, $1 \leqq i, j \leqq 2^n$ and $\lambda_1 = a_1$. Let η be the smallest distance between the points in $\operatorname{Sp} A$ and let $B = (b_{i,j})_{i,j=1}^{2n}$. By Lemma 6 there exists a positive operator $b \in \mathfrak{B}$ and a selfadjoint operator $b'_{1,1}$ with $|b| \leqq \min(\eta/2, \varepsilon)$, $|b_{1,1} - b'_{1,1}| < \varepsilon$ and $\mathfrak{R}(b, b'_{1,1}) = \mathfrak{B}$. We can further find invertible operators $b'_{1,i} = b'^*_{i,1}$ for $1 < i \leqq 2^n$ with $|b_{1,i} - b'_{1,i}| < \varepsilon \cdot 2^{-n}$. For the remaining indices set $b_{i,j} = b'_{i,j}$. Then let $B' = (b'_{i,j})_{i,j=1}^{2n}$ and $A' = \operatorname{diag}(\lambda_1 + b, a_2, \ldots, a_{2^n})$.

b) Clearly $T' = A' + iB'$ satisfies $|T - T'| < 4\varepsilon$. Let $C = C^* \in \mathfrak{R}(T')'$, then $CA' = A'C$ implies again by Lemma 1 $C = \operatorname{diag}(c_1, \ldots, c_{2^n})$. $T'C = CT'$ gives further $c_1 b = b c_1$ and $c_1 b'_{1,1} = b'_{1,1} c_1$ or $c_1 \in \mathfrak{B}'$. Then $b'_{1,i} c_i =$

$c_1 b'_{1,i} = b'_{1,i} c_1$ for $1 \leqq i \leqq 2^n$, or $c_1 = c_i$, because $b'_{1,i}$ is invertible. Again $C = \operatorname{diag}(c_1, \ldots, c_1)$ shows $\Re(T') = \mathfrak{A}$.

We want to show now that in most W^*-algebras on a separable Hilbert space each operator can be written as the sum of two generators. This is known for $B(H)$ [5, 8]. We begin with a general lemma, which in some sense is an analogue of Lemma 3.

LEMMA 7. *Let the W^*-algebra $\mathfrak{A}$ be a finite direct sum of W^*-algebras $\mathfrak{A}_i$, $\mathfrak{A} = \sum_{i=1}^n \oplus \mathfrak{A}_i$. Assume in each $\mathfrak{A}_i$ every operator is the sum of two generators of $\mathfrak{A}_i$, then every $T \in \mathfrak{A}$ can likewise be written as the sum of two generators of $\mathfrak{A}$.*

PROOF. Let $T = \sum \oplus T_i \in \mathfrak{A}$ and let $T_i = U_i + V_i$, where U_i and V_i are generators of $\mathfrak{A}_i$. We write

$$T = \sum \oplus (U_i + K_i) + \sum + (V_i - K_i) = U + V ,$$

where the K_i are scalars. Since n is finite we can choose the K_i such that $\operatorname{Sp}(U_i + K_i) \cap \operatorname{Sp}(U_j + K_j) = \emptyset$ and $\operatorname{Sp}(V_i - K_i) \cap \operatorname{Sp}(V_j - K_j) = \emptyset$ for $i \neq j$. Let $C = C^* \in \Re(U)'$, then we can write $C = (c_{i,j})_{i,j=1}^n$, and we obtain $c_{i,j}(U_j + K_j) = (U_i + K_i)c_{i,j}$. For $i \neq j$ Lemma 1 (a) shows $c_{i,j} = 0$. Hence $C = \sum \oplus c_i$ and $c_i(U_i + K_i) = (U_i + K_i)c_i$ or $c_i \in \Re(U_i)' = \mathfrak{A}'_i$. Thus $\Re(U)' = \sum \oplus \mathfrak{A}'_i$ and $\Re(U) = \mathfrak{A}$. Similarly one shows $\Re(V) = \mathfrak{A}$.

PROPOSITION 3. *Let $\mathfrak{A}$ be a properly infinite W^*-algebra on a separable Hilbert space. Then any element in $\mathfrak{A}$ can be written as the sum of two generators of $\mathfrak{A}$.*

PROOF. a) Let $T = A + iB \in \mathfrak{A}$. Then there exist four equivalent orthogonal projections $F_i \in \mathfrak{A}$, with $1 \leqq i \leqq 4$, such that $F_i A = A F_i$ and $\sum F_i = 1$ [4, th. 3]. The $\{F_i\}_{i=1}^4$ induce a tensor decomposition of $\mathfrak{A}$, $\mathfrak{A} = \mathfrak{B} \otimes M_4$ and $A = \operatorname{diag}(a_1, a_2, a_3, a_4)$. Let $K = 3\,|\,T\,|$ and let $A' = \operatorname{diag}(0, K, 2K, 3K)$. Since $\mathfrak{A}$ is properly infinite, it is generated by the positive invertible operators c and d with $c, d \geqq K \cdot 1$. B is represented by the matrix $B = (b_{i,j})_{i,j=1}^4$, then let B' be given by the matrix

$$B' = \begin{bmatrix} c - b_{1,1} & d - b_{1,2} & 0 & 0 \\ d - b_{1,2}^* & c & d & 0 \\ 0 & d & 0 & K \\ 0 & 0 & K & 0 \end{bmatrix} .$$

Now write $T = T_1 + T_2 = [(A + A') + i(B + B')] - [A' + iB']$.

b) Let $C = C^* \in \Re(T_1)'$, then $C(A + A') = (A + A')C$ and Lemma 1 (a) imply that C is diagonal, $C = \operatorname{diag}(c_1, c_2, c_3, c_4)$. Then

$$C(B' + B) = (B' + B)C$$

gives $c_1 c = c c_1$ and $c_1 d = d c_2$. By Lemma 1 (b) this shows $c_1 = c_2 \in \mathfrak{B}'$, because $\mathfrak{R}(c, d) = \mathfrak{B}$. $CT_1 = T_1 C$ gives further $(d + b_{2,3})c_3 = c_2(d + b_{2,3}) = (d + b_{2,3})c_2$ or $c_2 = c_3$, because $d + b_{2,3}$ is invertible. The relation

$$(K + b_{3,4})c_4 = c_3(K + b_{3,4}) = (K + b_{3,4})c_3$$

finally gives $c_3 = c_4$ by the same argument. Thus $C = c_1 \otimes 1$ with $c_1 \in \mathfrak{B}'$ or $\mathfrak{R}(T_1) = \mathfrak{A}$.

c) $\mathfrak{R}(T_2) = \mathfrak{A}$ is shown as in (b).

To show this result also for finite W^*-algebras of type I, we need some preparations.

LEMMA 8. *Let $\mathfrak{A}$ be an abelian W^*-algebra then any $T \in \mathfrak{A}$ is the sum of two generators of $\mathfrak{A}$.*

PROOF. Let $T = A + iB$ and let C be a selfadjoint generator of $\mathfrak{A}$. Then $T = [C + i(B - C)] + [(A - C) + iC]$ is the desired decomposition.

Now let $\mathfrak{A}$ be a W^*-algebra of type I_n, then $\mathfrak{A}$ can be represented as $\mathfrak{A} = \mathfrak{Z} \otimes M_n$, where $\mathfrak{Z}$ is the center of $\mathfrak{A}$. Let $T \in \mathfrak{A}$, then we may assume that T has upper triangular form [3], $T = (t_{i,j})_{i,j=1}^n$ and $t_{i,j} = 0$ for $j < i$. The diagonal part of such a T we denote by diag T, diag $T = \text{diag}(t_{1,1}, \ldots, t_{n,n})$. Let $\mathfrak{B}$ be the maximal abelian subalgebra of $\mathfrak{A}$, which consists of all diagonal operators. Of course $\mathfrak{B}$ depends on the given matrix representation of $\mathfrak{A}$. In the next lemma we exhibit a large class of generators of $\mathfrak{A}$.

LEMMA 9. *Let $\mathfrak{A}, \mathfrak{B}$ and T as above, then $\mathfrak{R}(T) = \mathfrak{A}$ if*

i) $\mathfrak{R}(\text{diag } T) = \mathfrak{B}$

ii) $t_{i,i+1}$ *is invertible for all* $1 \leqq i \leqq n - 1$.

PROOF. We may represent $\mathfrak{A}$ on a suitable Hilbert space H such that $\mathfrak{B}$ is a maximal abelian subalgebra of $B(H)$. Let $C = C^* \in \mathfrak{R}(T)'$ and $C = (c_{i,j})_{i,j=1}^n$. Computing the $(n, 1)$ matrix element of $CT = TC$ one finds $c_{n,1}t_{1,1} = t_{n,n}c_{n,1}$. Let C' be the operator, which one obtains from C by setting all matrix elements except the one in the $(n, 1)$ position equal to zero. Then C' diag $T =$ diag TC' and by the Fuglede theorem [6] $C' \in \mathfrak{R}(\text{diag } T)' = \mathfrak{B}' = \mathfrak{B}$. Hence $C' = c_{n,1} = c_{1,n} = 0$. With the same method applied to $(CT)_{n,i} = (TC)_{n,i}$ one shows by induction $c_{n,i} = c_{i,n} = 0$ for all $i < n$ and $c_{n,n} \in \mathfrak{Z}$. Further induction finally yields $C = c \otimes 1 \in \mathfrak{A}'$. Thus $\mathfrak{R}(T)' = \mathfrak{A}'$ or $\mathfrak{R}(T) = \mathfrak{A}$.

Lemma 9 can be extended to finite W^*-algebras of type I. To do this let $\mathfrak{A} = \sum \oplus \mathfrak{A}_n$, with $\mathfrak{A}_n = \mathfrak{Z}_n \otimes M_n$, be a finite W^*-algebra of type

I. Let $T = \sum \oplus T_n \in \mathfrak{A}$. Then we may assume that each $T_n = (t_{i,j}^{(n)})$ is upper triangular [3]. With respect to this representation let again $\mathfrak{B}_n$ denote the diagonal part of $\mathfrak{A}_n$ and let $\mathfrak{B} = \sum \oplus \mathfrak{B}_n$.

LEMMA 10. *Let $\mathfrak{A}, \mathfrak{B}$ and T as above, then $\mathfrak{R}(T) = \mathfrak{A}$ if*
i) $\mathfrak{R}(\sum \oplus diag\ T_n) = \mathfrak{B}$ *and*
ii) $t_{i,i+1}^{(n)}$ *is invertible for all* $1 \leqq i < n-1$ *and for all* n.

PROOF. Again we may find a representation of $\mathfrak{A}$ on a suitable Hilbert space H such that $\mathfrak{B}$ is a maximal abelian subalgebra of $B(H)$. Let $C = C^* = (C_{l,k}) \in \mathfrak{R}(T)'$. Then $T_k C_{k,l} = C_{k,l} T_l$ and because of Lemma 9 it suffices to show that this implies $C_{k,l} = 0$ for $k \neq l$. To do this write T_k, T_l and $C_{k,l}$ as matrices and use the same methods as in Lemma 9.

THEOREM 3. *Let $\mathfrak{A}$ be a W^*-algebra on a separable Hilbert space with no summand of type II_1 then any $T \in \mathfrak{A}$ can be written as the sum of two generators of $\mathfrak{A}$.*

PROOF. a) Let $\mathfrak{A} = \sum \oplus \mathfrak{A}_n$ be a finite W^*-algebra of type I and let $T = \sum \oplus T_n \in \mathfrak{A}$. Write each T_n in upper triangular from [3] and determine the corresponding $\mathfrak{B}_n$ and $\mathfrak{B}$. Now it is easy to see that T can be written as the sum of two operators in $\mathfrak{A}$, each of which satisfies the conditions (i) and (ii) of Lemma 10.

b) Now the theorem follows from Lemma 7, Proposition 3 and (a).

Theorem 3 holds also for certain W^*-algebras of type II_1.

THEOREM 4. *Let $\mathfrak{A}$ be a W^*-algebra of type II_1 and let $\mathfrak{A} \in \mathscr{A}$, then any $T \in \mathfrak{A}$ can be written as the sum of two generators of $\mathfrak{A}$.*

PROOF. The proof of Proposition 3 will also show this result.

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