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SOME UNIQUENESS THEOREMS FOR $H^p(U^n)$ FUNCTIONS

Dedicated to Professor Gen-ichirô Sunouchi on his 60th birthday

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1. Let U^n denote the unit polydisc $\{(z_1, \cdots, z_n); |z_1| < 1, \cdots, |z_n| < 1\}$ in the n-dimensional complex vector space C^n, T^n the distinguished boundary of U^n and m_n the normalized Haar measure on T^n. A function $f(z)$, holomorphic in U^n, is said to be of class $H^p(U^n)$, $N(U^n)$ or $N_*(U^n)$, if

$$\|f\|_p = \sup_{0<r<1}\left(\int_{T^n}|f(rw)|^p dm_n(w)\right)^{1/p} < \infty ,$$

$\sup_{0<r<1}\int_{T^n}\log^+|f(rw)|dm_n(w) < \infty$ or $\{\log^+|f(rw)|, 0 < r < 1\}$ forms a uniformly integrable family on T^n respectively. It is known (see Rudin [6]) that if $f \in N(U^n)$, $\lim_{r\to 1}f(rw) = f^*(w)$ exists for almost all $w \in T^n$ and $\log|f^*| \in L^1(T^n)$ when $f \not\equiv 0$, and we have $N(U^n) \supset N_*(U^n) \supset H^p(U^n) \supset H^q(U^n)$ (if $0 < p < q \leqq \infty$). It is also known that an $f \in N_*(U^n)$ is of class $H^p(U^n)$ if and only if $f^* \in L^p(T^n)$. A function $f(z)$ is said to be outer if $f \in N_*(U^n)$ and $\log|f(0)| = \int_{T^n}\log|f^*(w)|dm_n(w)$. It can be shown easily that an f is outer if and only if f and $1/f \in N_*(U^n)$. This follows from the fact that an $f \in N(U^n)$ lies in $N_*(U^n)$ if and only if

$$\log|f(z)| \leqq \int P(z, w)\log|f^*(w)|dm_n(w) \qquad \text{in } U^n ,$$

where $P(z, w)$ is the Poisson kernel for U^n, i.e. $P(z, w) = \prod_{j=1}^n(1-r_j^2)/(1 - 2r_j\cos(\theta_j - \varphi_j) + r_j^2)$ if $z_j = r_je^{i\theta_j}$ and $w_j = e^{i\varphi_j}$, (Rudin [6], p. 47).

In this note we shall show three uniqueness theorems for $H^1(U^n)$ functions of which only the arguments on the boundary are given. Relating to these, we shall give geometric aspects of outer functions in section 2 and some uniqueness theorems for other classes of holomorphic functions in U^n in section 3. Some applications are given in section 5.

The main results are the followings.

THEOREM 1. [8]. *Let $f \in H^1(U^n)$, be outer and $1/f^* \in L^p(T^n)$ $(1/2 \leqq p \leqq 1)$.*

We use systematically the notations in Rudin [6].

Then if $g \in H^q(U^n)$ $(1/p + 1/q = 2)$, *and if* $g^*/f^* > 0$ a.e. *on* T^n, *it follows that* $f \in H^q(U^n)$ *and* $g = af$ *for some* $a > 0$, $(n \geqq 1)$.

THEOREM 2. [10]. *Let* $\{a_1, \cdots, a_k\}$ *be a finite set of distinct points on* T. *Let* $1 \leqq p \leqq \infty$ *and* $1/p + 1/q = 1$. *Let* $f(z) \in H^1(U)$, *be outer and assume futher* $(\prod_{j=1}^k (e^{i\theta} - a_j))/f(e^{i\theta}) \in L^p(T)$. *Then the conclusion of Theorem 1 for* $n = 1$ *is still valid.*

The proof of Theorem 2 is essentially contained in [10].

THEOREM 3. *Let* $f \in H^1(U^n)$, $\not\equiv 0$, *and* $\operatorname{Re} f^* \geqq 0$ a.e. *on* T^n. *Then if* $g \in H^1(U^n)$ *and* $g^*/f^* > 0$ a.e. *on* T^n, *it follows that* $g = af$ *for some* $a > 0$, $(n \geqq 1)$.

We shall prove Theorem 3 in section 4. We should remark that the above theorems are mutually independent. In fact, to $(1 - z)(i - z)$ only Theorem 2 is applicable. To $1 - B(z)$ only Theorem 3 is applicable when $B(z) \in H^\infty(U)$, $|B^*| = 1$ a.e. on T and T is the natural boundary of $B(z)$. To $(1 - B(z))^{1/2}(2 + z)^4$ only Theorem 1 is applicable.

2. Geometric aspects of $H^p(U^n)$ functions and outer functions.

We notice that Littlewood's subordination principle is still applicable in U^n. (Consider n-subharmonic function in place of subharmonic function). Hence we have two lemmas which are given by Cargo for the case $n = 1$.

LEMMA 1. *A holomorphic function in* U^n *whose range is contained in a (closed or open) wedge of angular measure* $\alpha\pi$ $(0 < \alpha < 2)$ *is in* $H^p(U^n)$ *if* $0 < p < 1/\alpha$.

LEMMA 2. *A holomorphic function in* U^n *whose range is contained in an open and simply connected set with at least two boundary points is in* $H^p(U^n)$ *if* $0 < p < 1/2$.

Using these lemmas we have some sufficient conditions for a holomorphic function to be outer.

PROPOSITION 1. *Let* I *be a closed simple arc on the complex plane* C, *such that one of the end points is on the origin. Then if an* $f \in N_*(U^n)$ *omits every point of* I, f *is outer.*

PROOF. By assumption $1/f$ is holomorphic in U^n and the range of $1/f$ does not intersect I' which is the image of I by the mapping $1/z$. Since $C - I'$ is open, simply connected and has at least two boundary points, $1/f$ lies in $H^p(U^n)$ $(0 < p < 1/2)$ by virtue of Lemma 2. Hence using a fact in section 1 we have that f is outer. We should remark that the hypothesis $f\in N_*(U^n)$ is not superfluous. In fact,

$$f(z)=\exp\{(1+z)/(1-z)\}$$

omits every point of the closure of U but does not belong to $N_*(U)$. Proposition 1 is clearly false when I is replaced by the origin. Such an example is given by $\exp\{-(1+z)/(1-z)\}$.

As an immediate consequence we have

COROLLARY 1. *Let I be a closed simple arc joining the origin and the point at infinity. Then a holomorphic function in U^n which omits every point of I is outer.*

3. Uniqueness theorems for other classes. Note first that if the range of an f, holomorphic in U^n, is contained in an open wedge of angular measure $\alpha\pi$ $(0<\alpha<2)$, then its radial limit f^* exists a.e. on T^n by means of Lemma 1. We can state then the following theorem.

THEOREM 4. *If the range of f and g, holomorphic in U^n, are contained in some open wedge of angular measure $\alpha\pi$ $(0<\alpha<2)$ with vertex at the origin, then the propositon $f^*/g^*>0$ a.e. on T^n implies that $f=ag$ for some $a>0$.*

This is a special case of the following theorem.

THEOREM 5. *Under the same assumptions as in Theorem 4, the proposition f^*/g^* is real a.e. on T^n and $m_n\{w\in T^n; u<f^*/g^*(w)<v\}=0$ for some $u<v\leqq 0$ implies that $f=ag$ for some non-zero real a.*

PROOF. We may assume that the above wedge is of the form $\{z\in C; |\arg z|<\beta\pi\}$ $(0<\beta<1)$ without loss of generality. Then it follows that $\log f/g$ is holomorphic in U^n and $\arg f/g$ is n-harmonic[1] and $|\arg f/g|<2\beta\pi$ in U^n. Hence we have $\arg f^*/g^*=\pm\pi$ or 0 a.e. on T^n. We may assume therefore that $|\arg f/g|<\pi$ in U^n. In fact, if $\arg f/g(z)\geqq\pi$ for some $z\in U^n$, it follows from the maximum principle for bounded n-harmonic functions that $\arg f/g=\pi$ in U^n and so $f/g=a$ for some $a<0$, which will show the theorem. The same situation takes place in the case $\arg f/g(z)\leqq-\pi$ for some $z\in U^n$. Now let $M=-(u+v)/2>0$. Then, since $f^*/g^*\leqq u$ or $\geqq v$, we have $|f^*/g^*+M|\geqq(v-u)/2>0$. Let us consider the function $f/g+M=(f+Mg)/g$. Since $|\arg f/g|<\pi$ in U^n, we have also $|\arg(f/g+M)|<\pi$. Applying Corollary 1 we see that $(f+Mg)/g$ is outer and so $g/(f+Mg)$ is in $N_*(U^n)$. Since $|g^*/(f^*+Mg^*)|\leqq 2/(v-u)<\infty$, we have $g/(f+Mg)\in H^\infty(U^n)$ by using a fact in section 1.

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[1] A continuous real-(or complex-) valued function in an open set in C^n is said to be n-harmonic if it is harmonic in each complex variable z_j separately.

Since every $H^1(U^n)$ function can be represented by the Poisson integral of its boundary function and since only real constants are real-valued holomorphic functions, we can assert that $g/(f + Mg)$ is a constant and consequently that f/g is a non-zero real constant in virtue of $f/g \neq 0$. This completes the proof.

REMARK. The above theorem is false when $\alpha = 2$. Indeed, $\{(1 - z)/(1 + z)\}^2$ and $\{(1 + z)/(1 - z)\}^2$ omit every point of negative real axis in U, while their boundary values are negative except at $z = \pm 1$.

4. **Proof of Theorem 3.** Now we can prove Theorem 3 easily. Since $f \in H^1(U^n)$ and $\operatorname{Re} f^* \geqq 0$, $\operatorname{Re} f(z) \geqq 0$ in U^n. We may assume further that $\operatorname{Re} f(z) > 0$ in U^n, since otherwise by the maximum principle for n-harmonic functions $\operatorname{Re} f(z) = 0$ in U^n and so $f(z) = b$ for some pure imaginary b, and thus it can be reduced to the above case. The situation is analogous for g, since $g^*/f^* > 0$ a.e. on T^n and $g \in H^1(U^n)$. Now apply Theorem 4.

REMARK (a). That the hypothesis $g \in H^1(U^n)$ is not superfluous is shown by the example; $f(z) = 1 - z$ and $g(z) = -(1 + z)^2/(1 - z)$. In fact, $f^*/g^* > 0$ on T except at $z = \pm 1$, while $g \in H^p(U)$ only for $0 < p < 1$.

REMARK (b). Theorem 3 is false when a closed half-plane is replaced by a wedge of angular measure $\alpha\pi$ $(1 < \alpha < 2)$ with vertex at the origin. Indeed, let $f(z) = (1 - z)^\alpha$ and $g(z) = -(1 + z)^2/(1 - z)^{2-\alpha}$. Then $f^*/g^* > 0$ on T except at $z = \pm 1$, while $|\arg f^*| < \alpha\pi/2$ and $g \in H^1(U)$ because of $1 < 1/(2 - \alpha)$.

5. **Applications.** An $f \in H^\infty(U^n)$ is said to be inner if $|f^*| = 1$ a.e. on T^n. As an application of Theorem 3 we have

PROPOSITION 2. *Let f be inner and not identically -1. Then if $g \in H^1(U^n)$ and $g^*/(1 + f^*) > 0$ a.e. on T^n, it follows that $g = a(1 + f)$ for some $a > 0$.*

We remark that $1/(1 + f) \in H^p(U^n)$ $(0 < p < 1)$, but not in $H^1(U^n)$ if f is not constant, and we can not apply Theorem 1.

As an application of Theorem 5 we have

PROPOSITION 3. *If $\operatorname{Re} f_j > 0$ in U^n $(j = 1, 2, 3)$ and $f_1^* f_2^* f_3^*$ is real a.e. on T^n and $m_n\{u < f_1^* f_2^* f_3^* < v\} = 0$ for some $u < v \leqq 0$, then we have that $f_1 f_2 f_3$ is constant.*

This proposition is false for four functions, which is shown by $f_j = (1 + z)/(1 - z)$, $(j = 1, 2, 3, 4)$. Similar formulations can be done for other wedges, but the above is very simple and seems to us interesting.

6. Localizations of the preceding results and generalizations to more general domains will be given elsewhere. We remark only that Theorem 5 is closely related to a special case of R. Nevanlinna and W. Seidel's continuation theorem for inner functions in U.

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