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POSITIVE JACOBI POLYNOMIAL SUMS

Dedicated to Professor Gen-ichirô Sunouchi on his 60th birthday

RICHARD ASKEY[1)]

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1. **Introduction.** Since Fejér [6] showed that the $(C, 1)$ summability of Fourier series follows from

$$\sum_{k=0}^{n} \sin\left(k+\frac{1}{2}\right)\theta \geqq 0\,, \qquad 0 \leqq \theta \leqq \pi\,, \tag{1.1}$$

and the $(C, 2)$ summability of Laplace series [7] depends on

$$\sum_{k=0}^{n} P_k(\cos\theta) \geqq 0\,, \qquad 0 \leqq \theta \leqq \pi\,, \tag{1.2}$$

($P_k(x)$ the Legendre polynomial of degree n), there has been interest in finding other nonnegative polynomials expansions which could be used to obtain new results in harmonic analysis. One of the early series was

$$\sum_{k=1}^{n} \frac{\sin k\theta}{k} > 0\,, \qquad 0 < \theta < \pi\,. \tag{1.3}$$

(1.3) was conjectured by Fejér in 1910 and proven by Jackson [15] and Gronwall [13]. This inequality was surprisingly hard to prove and the first simple proof was found by Landau [17]. The first proof which really explains why (1.3) holds seems to be due to Turán [25]. Actually this was not explained in [25] but in a later paper [26] where Turán proved

THEOREM A. *If*

$$\sum_{k=1}^{n} b_k \sin(2k-1)\theta \geqq 0\,, \qquad 0 < \theta < \pi \tag{1.4}$$

then

$$\sum_{k=1}^{n} \frac{b_k}{k} \sin k\varphi > 0\,, \qquad 0 < \varphi < \pi \tag{1.5}$$

unless all of the b_k are zero.

When $b_k = 1$, (1.4) is (1.1) and (1.5) is (1.3) and so (1.1) is a more

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basic inequality than (1.3). Other proofs of Theorem A were given by Hyltén-Cavallius [14] and Askey, Fitch, and Gasper [4].

In addition to proving (1.3) Turán [25] proved

$$\sum_{k=1}^{n} \frac{\sin k\theta}{k} < 2\sum_{k=1}^{\infty} \frac{\sin k\theta}{k} = \pi - \theta\,, \qquad 0 < \theta < \pi\,. \tag{1.6}$$

Hyltén-Cavallius [14] reproved (1.6) and stated both (1.3) and (1.6) in the following symmetrical form

$$\left| \sum_{k=n}^{\infty} \frac{\sin k\theta}{k} \right| < \sum_{k=1}^{\infty} \frac{\sin k\theta}{k}\,, \quad n = 2, 3, \cdots, 0 < \theta < \pi\,. \tag{1.7}$$

There have been extensions of (1.1), (1.2) and (1.3) to other orthogonal expansions. Let $P_n^{(\alpha,\beta)}(x)$ denote the Jacobi polynomial defined by

$$(1-x)^{\alpha}(1+x)^{\beta}P_n^{(\alpha,\beta)}(x) = \frac{(-1)^n}{2^n n\,!}\frac{d^n}{dx^n}[(1-x)^{n+\alpha}(1+x)^{n+\beta}]\,.$$

Feldheim [11] proved that

$$\sum_{k=0}^{n} \frac{P_k^{(\alpha,\alpha)}(x)}{P_k^{(\alpha,\alpha)}(1)} \geqq 0,\ \alpha \geqq 0,\ -1 \leqq x \leqq 1\,. \tag{1.8}$$

For $\alpha = 0$ (1.8) is (1.2) and for $\alpha = 1/2$ it is (1.3). Feldheim's proof was by a new fractional integral connecting $P_n^{(\alpha,\alpha)}(x)$ with $P_n^{(\beta,\beta)}(y)$ and so it really gives

THEOREM B. *If*

$$\sum_{k=0}^{n} a_k \frac{P_k^{(\alpha,\alpha)}(x)}{P_k^{(\alpha,\alpha)}(1)} \geqq 0,\ -1 \leqq x \leqq 1, \alpha > -1\,, \tag{1.9}$$

then

$$\sum_{k=0}^{n} a_k \frac{P_k^{(\beta,\beta)}(y)}{P_k^{(\beta,\beta)}(1)} \geqq 0,\ -1 \leqq y \leqq 1, \beta > \alpha\,. \tag{1.10}$$

Using a more general fractional integral connecting two different Jacobi polynomials Askey and Fitch have shown

THEOREM C. *If*

$$\sum_{k=0}^{n} a_k \frac{P_k^{(\alpha,\beta)}(x)}{P_k^{(\alpha,\beta)}(1)} \geqq 0,\ -1 \leqq x \leqq 1, \alpha, \beta > -1\,, \tag{1.11}$$

then

$$\sum_{k=0}^{n} a_k \frac{P_k^{(\alpha+\mu,\beta-\nu)}(y)}{P_k^{(\alpha+\mu,\beta-\nu)}(1)} \geqq 0,\ -1 \leqq y \leqq 1, 0 \leqq \nu \leqq \mu, \beta - \nu > -1\,. \tag{1.12}$$

See [1, (4. 11)].

From Theorem C and other results Askey and Fitch proved

$$\sum_{k=0}^{n} \frac{P_k^{(\alpha,\beta)}(x)}{P_k^{(\alpha,\beta)}(1)} \geqq 0 \,, \quad -1 \leqq x \leqq 1 \,, \tag{1.13}$$

for

$$\alpha \geqq \beta \geqq 0; \alpha \geqq 0, -\frac{1}{2} \leqq \beta < 0; \alpha \geqq (-\beta - 1 + \sqrt{(\beta^2 - 2\beta - 1)})/2 \,,$$
$$-1 < \beta < -\frac{1}{2} \,.$$

This last condition is given incorrectly in [3].

Unfortunately no applications were given for (1.13) in [3], so we were unaware that it was not a particularly useful generalization of Feldheim's result (1.8). A more useful inequality would be

$$\sum_{k=0}^{n} \frac{P_k^{(\alpha,\beta)}(x)}{P_k^{(\beta,\alpha)}(1)} \geqq 0 \,. \tag{1.14}$$

2. **Jacobi sums.** Before proving (1.14) for some (α, β) let us consider an application. It is well known that the Poisson kernel for Fourier series is positive, i.e.

$$\frac{1}{2} + \sum_{n=1}^{\infty} r^n \cos n(\theta - \varphi) > 0 \,, \qquad -1 < r < 1 \,. \tag{2.1}$$

The even part of this series is also positive, so

$$\frac{1}{2} + \sum_{n=1}^{\infty} r^n \cos n\,\theta \cos n\,\varphi > 0 \,, \qquad 0 \leqq r < 1 \,, \qquad 0 \leqq \theta,\ \varphi \leqq \pi \,. \tag{2.2}$$

(2.2) was generalized to Jacobi polynomials by Bailey [5], who proved

$$\sum_{n=0}^{\infty} r^n \frac{(2n + \alpha + \beta + 1)\Gamma(n + \alpha + \beta + 1)\Gamma(n + 1)}{\Gamma(n + \alpha + 1)\Gamma(n + \beta + 1)} P_n^{(\alpha,\beta)}(x) P_n^{(\alpha,\beta)}(y) > 0 \,, \tag{2.3}$$
$$0 \leqq r < 1 \,, -1 \leqq x, y \leqq 1, \alpha, \beta > -1 \,.$$

Bailey actually summed the series (2.3) and the positivity was obvious from his formula. The case $\alpha = \beta = -1/2$ is (2.2). The positivity of (2.3) gives the following strong maximum principle (as is well known).

THEOREM 1. *Let $f(x)$ be measurable, $f(x) \geqq 0$ a.e. and*

$$\int_{-1}^{1} |f(x)| (1 - x)^{\alpha}(1 + x)^{\beta} dx < \infty \,.$$

If

$$a_n = \frac{\int_{-1}^{1} f(x)P_n^{(\alpha,\beta)}(x)(1-x)^\alpha(1+x)^\beta dx}{\int_{-1}^{1} [P_n^{(\alpha,\beta)}(x)]^2(1-x)^\alpha(1+x)^\beta dx},$$

and

$$f_r(x) = \sum_{n=0}^{\infty} a_n r^n P_n^{(\alpha,\beta)}(x), \qquad 0 \leqq r < 1, \tag{2.4}$$

then $f_r(x) > 0$ *unless* $f(x) = 0$ a.e.

When $\alpha = \beta = -1/2$ Fejér [8] showed that the partial sums of (2.3) are always nonnegative for $0 \leqq r \leqq 1/2$, but not for $r > 1/2$. Then he observed that the partial sums of (2.4) are also nonnegative for $0 \leqq r \leqq 1/2$ when $f(x) \geqq 0$, $-1 \leqq x \leqq 1$. For $\alpha = \beta = 0$ Szegö [22] showed that the partial sums of (2.3) are nonegative for $0 \leqq r \leqq 1/3$, but not for larger r. We will prove the following conditional theorem.

THEOREM 2. *If* $\alpha \geqq \beta \geqq -1/2$ *and*

$$\sum_{k=0}^{n} \frac{P_k^{(\alpha,\beta)}(x)}{P_k^{(\beta,\alpha)}(1)} \geqq 0 \tag{2.5}$$

then

$$\sum_{k=0}^{n} r^k \frac{(2k+\alpha+\beta+1)\Gamma(k+\alpha+\beta+1)\Gamma(k+1)}{\Gamma(k+\alpha+1)\Gamma(k+\beta+1)} P_k^{(\alpha,\beta)}(x)P_k^{(\alpha,\beta)}(y) \geqq 0, \tag{2.6}$$

$$0 \leqq r \leqq 1/(\alpha+\beta+3), \ -1 \leqq x, y \leqq 1.$$

(2.6) *fails for* $n = 1$, $x = -1$, $y = 1$ *if* $r > 1/(\alpha+\beta+3)$.

The last remark is easy, since (2.6) becomes

$$\frac{(\alpha+\beta+1)\Gamma(\alpha+\beta+1)}{\Gamma(\alpha+1)\Gamma(\beta+1)} + \frac{(\alpha+\beta+3)\Gamma(\alpha+\beta+2)\Gamma(2)}{\Gamma(\alpha+2)\Gamma(\beta+2)} r(1+\alpha)(-1)(1+\beta)$$
$$= \frac{\Gamma(\alpha+\beta+2)}{\Gamma(\alpha+1)\Gamma(\beta+1)}[1 - r(\alpha+\beta+3)].$$

We have used

$$P_n^{(\alpha,\beta)}(1) = \binom{n+\alpha}{n} = \frac{(\alpha+1)_n}{n!}$$
$$P_n^{(\alpha,\beta)}(-x) = (-1)^n P_n^{(\beta,\alpha)}(x).$$

In general, series with two variables are too complicated to handle and so it is surprising that (2.6) can be proven. There is a recent result of Gasper [12] which allows us to suppress one of the variables and then recover it. It clearly is necessary to have (2.6) for $y = 1$ if we wish to have it for $-1 \leqq y \leqq 1$. Gasper's result says that this is also sufficient. A special case is

THEOREM D. *Let $f(x)$ be a continuous function, $\alpha \geqq \beta \geqq -1/2$,*

$$f(x) \sim \sum_{n=0}^{\infty} a_n P_n^{(\alpha,\beta)}(x) ,$$

i.e.

$$a_n = \int_{-1}^{1} f(x) P_n^{(\alpha,\beta)}(x)(1-x)^{\alpha}(1+x)^{\beta}dx \Big/ \int_{-1}^{1} [P_n^{(\alpha,\beta)}(x)]^2 (1-x)^{\alpha}(1+x)^{\beta}dx ,$$

and assume

$$\sum_{n=0}^{\infty} | a_n | (n+1)^{\alpha} < \infty .$$

If $f(x) \geqq 0$, $-1 \leqq x \leqq 1$ then $f(x, y) \geqq 0$, $-1 \leqq x, y \leqq 1$, where

$$f(x, y) = \sum_{n=0}^{\infty} a_n P_n^{(\alpha,\beta)}(x) P_n^{(\alpha,\beta)}(y) / P_n^{(\alpha,\beta)}(1) .$$

When $y = 1$, it is sufficient to show

$$\sum_{k=0}^{n} c_k \frac{P_k^{(\alpha,\beta)}(x)}{P_k^{(\beta,\alpha)}(1)} = \sum_{k=0}^{n} r^k \frac{(2k+\alpha+\beta+1)\Gamma(k+\alpha+\beta+1)}{\Gamma(k+1)} \frac{P_k^{(\alpha,\beta)}(x)}{P_k^{(\beta,\alpha)}(1)} \geqq 0 .$$

A summation by parts gives the nonnegativity of (2.6) if (2.5) holds and

$$c_{k+1} \leqq c_k = r^k (2k+\alpha+\beta+1)\Gamma(k+\alpha+\beta+1)/\Gamma(k+1) .$$

This is equivalent to

$$r \leqq \frac{(k+1)(2k+\alpha+\beta+1)}{(k+\alpha+\beta+1)(2k+\alpha+\beta+3)} , \qquad k = 0, 1, \cdots, n-1; n = 1, 2, \cdots .$$

A routine calculation shows that the right hand side is an increasing function of k and so (2.6) holds for $0 \leqq r \leqq 1/(\alpha+\beta+3)$.

When $\alpha = \beta = 0$ this argument was used by Szegö [22]. In the usual fashion Theorem 2 can be applied to a function $f(x) \geqq 0$, as in Theorem 1. For $\alpha = \beta =$ "half an integer" it can be applied to spherical harmonic expansions. Szegö gives the details for the case $\alpha = \beta = 0$ and Laplace series on the sphere in 3-space. As it stands Theorem 2 is not very satisfactory, since the condition (2.5) must be checked. Some values of (α, β) for which it holds are given in

THEOREM 3.

$$\sum_{k=0}^{n} \frac{P_k^{(\alpha,\beta)}(x)}{P_k^{(\beta,\alpha)}(1)} \geqq 0, \ -1 \leqq x \leqq 1 , \qquad -1 < \alpha \leqq \beta + 1, \alpha + \beta \geqq 0 .$$

First consider $\alpha = \beta + 1$.

$$\sum_{k=0}^{n} \frac{P_k^{(\beta+1,\beta)}(x)}{P_k^{(\beta,\beta+1)}(1)} = \sum_{k=0}^{n} \frac{(k+\beta+1)}{(\beta+1)} \frac{P_k^{(\beta+1,\beta)}(x)}{P_k^{(\beta+1,\beta)}(1)} . \tag{2.7}$$

Recall that

$$\frac{P_k^{(\alpha,\beta)}(x)}{P_k^{(\alpha,\beta)}(1)} - \frac{P_{k+1}^{(\alpha,\beta)}(x)}{P_{k+1}^{(\alpha,\beta)}(1)} = \frac{(2k+\alpha+\beta+2)}{2(\alpha+1)} \frac{P_k^{(\alpha+1,\beta)}(x)}{P_k^{(\alpha+1,\beta)}(1)} (1-x) , \tag{2.8}$$

[24, (4.5.4)] .

Use of (2.8) in (2.7) gives

$$(1-x) \sum_{k=0}^{n} \frac{P_k^{(\beta+1,\beta)}(x)}{P_k^{(\beta,\beta+1)}(1)} = \sum_{k=0}^{n} \left[\frac{P_k^{(\beta,\beta)}(x)}{P_k^{(\beta,\beta)}(1)} - \frac{P_{k+1}^{(\beta,\beta)}(x)}{P_{k+1}^{(\beta,\beta)}(1)} \right] = 1 - \frac{P_{n+1}^{(\beta,\beta)}(x)}{P_{n+1}^{(\beta,\beta)}(1)} . \tag{2.9}$$

But

$$\left| \frac{P_k^{(\alpha,\beta)}(x)}{P_k^{(\alpha,\beta)}(1)} \right| \leqq 1,\ -1 \leqq x \leqq 1,\ \alpha \geqq -\frac{1}{2},\ \beta > -1 , \quad [24, (7.32.2)] , \tag{2.10}$$

so the right hand side of (2.9) is nonnegative, which proves (2.5) for

$$\alpha = \beta + 1,\ \beta \geqq -\frac{1}{2} .$$

Then Bateman's integral [3, (3.4)]

$$(1+x)^{\beta+\mu} \frac{P_n^{(\alpha-\mu,\beta+\mu)}(x)}{P_n^{(\beta+\mu,\alpha-\mu)}(1)} = \frac{\Gamma(\beta+\mu+1)}{\Gamma(\beta+1)\Gamma(\mu)} \int_{-1}^{x} (1+y)^{\beta} \frac{P_n^{(\alpha,\beta)}(y)}{P_n^{(\beta,\alpha)}(1)} (x-y)^{\mu-1} dy , \tag{2.11}$$

$\mu > 0,\ \alpha - \mu > -1$ gives (2.6) for $-1 < \alpha \leqq \beta + 1,\ \alpha + \beta \geqq 0$.

Unfortunately (2.6) does not hold for all $(\alpha, \beta), \alpha \geqq \beta \geqq -1/2$. As Szegö [11] pointed out it fails for $n = 2$ when $\alpha = \beta = -1/2$. However (2.6) holds for $\alpha = \beta = -1/2$, so it probably holds for all $\alpha, \beta \geqq -1/2$, even when (2.5) fails.

As a generalization of (1.7) we prove

THEOREM 4.

$$\left| \sum_{k=n}^{\infty} r^k \frac{P_k^{(\alpha,\alpha)}(x)}{P_k^{(\alpha,\alpha)}(1)} \right| \leqq \sum_{k=0}^{\infty} r^k \frac{P_k^{(\alpha,\alpha)}(x)}{P_k^{(\alpha,\alpha)}(1)} , \quad \alpha \geqq \frac{1}{2},\ 0 \leqq r < 1,\ -1 \leqq x \leqq 1 .$$

When $-1 < x < 1$ *this result also holds for* $r = 1$.

From [3, (3.8)]

$$\frac{(1+x)^{\beta+\mu}}{(1-x)^{n+\beta+1}} \frac{P_n^{(\alpha,\beta+\mu)}(x)}{P_n^{(\beta+\mu,\alpha)}(1)} = \frac{2^{\mu}\Gamma(\beta+\mu+1)}{\Gamma(\beta+1)\Gamma(\mu)} \int_{-1}^{x} \frac{(1+y)^{\beta}(x-y)^{\mu-1}}{(1-y)^{n+\beta+\mu+1}} \frac{P_n^{(\alpha,\beta)}(y)}{P_n^{(\beta,\alpha)}(1)} dy , \quad \mu > 0 \tag{2.12}$$

and the positivity of the Poisson kernel (2.3) one can show that

$$\frac{P_n^{(\alpha,\beta+\mu)}(x)}{P_n^{(\beta+\mu,\alpha)}(1)} = \int_{-1}^{1} \frac{P_n^{(\alpha,\beta)}(y)}{P_n^{(\beta,\alpha)}(1)}\, d\mu_x(y) , \qquad d\mu_x(y) \geqq 0 . \tag{2.13}$$

See a similar argument in [3, (4.18)].

When $\mu = 1$, (2.13) gives

$$D(x, y) = \sum_{k=0}^{\infty} r^k \frac{P_k^{(\alpha+1,\alpha+1)}(x)}{P_k^{(\alpha+1,\alpha+1)}(1)} \frac{\Gamma(k+2\alpha+2)}{\Gamma(k+\alpha+1)} P_k^{(\alpha+1,\alpha)}(y) \geqq 0 , \quad 0 \leqq r < 1 . \tag{2.14}$$

The Christoffel-Darboux formula [24, (4.5.3)] gives

$$D(x, y) = A \sum_{k=0}^{\infty} r^k \frac{P_k^{(\alpha+1,\alpha+1)}(x)}{P_k^{(\alpha+1,\alpha+1)}(1)} \sum_{j=0}^{k} \frac{(2j+2\alpha+1)\Gamma(j+2\alpha+1)}{\Gamma(j+\alpha+1)} P_j^{(\alpha,\alpha)}(y) ,$$

where $A = A(\alpha) > 0$. Integration gives

$$\left| \int_{-1}^{1} D(x, y) \frac{P_n^{(\alpha,\alpha)}(y)}{P_n^{(\alpha,\alpha)}(1)} (1-y^2)^\alpha dy \right| = \left| \sum_{k=n}^{\infty} r^k \frac{P_k^{(\alpha+1,\alpha+1)}(x)}{P_k^{(\alpha+1,\alpha+1)}(1)} \right| .$$

But $D(x, y) \geqq 0$ and $|P_n^{(\alpha,\alpha)}(y)| \leqq |P_n^{(\alpha,\alpha)}(1)|$, this was (2.10), so

$$\left| \sum_{k=n}^{\infty} r^k \frac{P_k^{(\alpha+1,\alpha+1)}(x)}{P_k^{(\alpha+1,\alpha+1)}(1)} \right| \leqq \sum_{k=0}^{\infty} r^k \frac{P_k^{(\alpha+1,\alpha+1)}(x)}{P_k^{(\alpha+1,\alpha+1)}(1)}, \ \alpha \geqq -\frac{1}{2} , \quad 0 \leqq r < 1, \ -1 \leqq x \leqq 1 . \tag{2.15}$$

When $-1 < x < 1$ the function $\sum_{k=0}^{\infty} (P_k^{(\alpha+1,\alpha+1)}(x))/(P_k^{(\alpha+1,\alpha+1)}(1))$ is continuous and so we may let $r \to 1$ in (2.15) to complete the proof of Theorem 4.

A more complicated argument can be used to show that equality only holds in (2.15) for $n = 0$. This follows from a more detailed study of the measure $d\mu_x(y)$ in (2.13). It is not only nonnegative but strictly positive. Since I know of no applications of this refinement it will not be given here.

3. Comments and other problems. The main problem in this area is to show the positivity of the $(C, \alpha+\beta+2)$ means of (2.3), $\alpha, \beta \geqq -1/2$. For $\alpha = \beta$ this was done by Kogbetliantz [16]. Earlier Fejér had proven this for $\alpha = \beta = -1/2$ and $\alpha = \beta = 0$, and somewhat later he proved this for $\alpha = 1/2$, $\beta = -1/2$ [10]. This last result suggested the $(C, \alpha+\beta+2)$ conjecture, and for $\alpha = -\beta$ the conjecture follows from the case $\alpha = 1/2 = -\beta$ when the Bateman integral (2.11) is used. This is a general phenomenon, i.e. the result for $(\alpha+\beta+1/2, -1/2)$ could be used to prove this conjecture for (α, β).

This conjecture is the only missing fact in the proof of L^p convergence for Lagrange interpolation polynomials at zeros of $P_n^{(\alpha,\beta)}(x)$.

One other result which would follow from the $(C, \alpha+\beta+2)$ conjecture is a generalization of (2.6) to the (C, γ) means, and not just the $(C, 0)$ means of (2.3). Fejér [9] gives the details for the $(C, 0)$ and $(C, 1)$ means of a series whose $(C, 2)$ means are positive, and Schur and Szegö [20] treat the (C, γ) means of (2.1). Actually this type of result is not really that interesting and so the details will not be given here. A more important problem is to find the largest $r = r(n)$ for which (2.6) holds for a given n. The behavior of $r(n)$ as $n \to \infty$ has been found for $\alpha = \beta = -1/2$, [20], $\alpha = \beta = 0$, [22] and, $\alpha = \beta = 1/2$, [21]. This last result has been generalized to (C, γ) means for $\gamma = 1, 2, 3$ by Robertson [18].

(2.5) can be applied to prove the positivity of some Cotes numbers. This will be given in a paper dealing with quadrature problems.

In trying to prove (2.5) for other values of (α, β) it is worthwhile observing that positivity of (2.5) for $(\alpha+\beta+1/2, -1/2)$ implies the positivity for (α, β). This follows from the Bateman integral (2.11). The next easiest case to consider should be $(3/2, -1/2)$. In this case (2.5) reduces to showing

$$\cos\frac{\theta}{2}\sum_{k=1}^{n}\frac{\sin k\theta}{k}+\sin\frac{\theta}{2}-\sin\left(n+\frac{1}{2}\right)\theta \geqq 0,\ 0 \leqq \theta \leqq \pi . \tag{3.1}$$

Since ultraspherical polynomials are usually easier to deal with than Jacobi polynomials it should be pointed out that (2.5) for $\beta = -1/2$ is equivalent to

$$\sum_{k=0}^{n}\frac{P_k^{(\alpha,-1/2)}(1)}{P_k^{(-1/2,\alpha)}(1)}\frac{P_{2k}^{(\alpha,\alpha)}(x)}{P_{2k}^{(\alpha,\alpha)}(1)} \geqq 0,\ -1 \leqq x \leqq 1 . \tag{3.2}$$

(3.2) fails for $\alpha < 1/2$ when $n = 2$. It probably holds for $\alpha \geqq 1/2$.

While $U_n(r, \theta) = 1/2 + \sum_{k=1}^{n} r^k \cos k\theta \geqq 0$ only holds for $0 \leqq r \leqq 1/2$ it holds for $0 \leqq r \leqq 1$ in an average sense. For

$$\int_0^1 U_n(r, \theta)dr = \frac{1}{2}+\frac{\cos\theta}{2}+\frac{\cos 2\theta}{3}+\cdots+\frac{\cos n\,\theta}{n+1}$$

and Rogosinski and Szegö [19] have shown the positivity of this series.

A different generalization of (1.2) was given by Szegö. (1.2) could have been given as

$$\sum_{k=0}^{n} P_k(\cos\theta) > 0 , \qquad 0 < \theta < \pi .$$

The positivity of the Poisson kernel gives

$$\sum_{k=0}^{n} r^k P_k(\cos\theta) > 0 ,\quad 0 < \theta < \pi ,\quad -1 \leqq r \leqq 1 . \tag{3.3}$$

Szegö [23] proved that $\sum_{k=1}^{n} z^k P_k(\cos\theta) \neq 0$, $|z| \leqq 1$, $0 < \theta < \pi$, where z is now complex. Unfortunately his proof uses two integral representations of a very special nature and it does not generalize to any other (α, β). It is quite likely that

$$\sum_{k=0}^{n} z^k \frac{P_k^{(\alpha,\alpha)}(\cos\theta)}{P_k^{(\alpha,\alpha)}(1)} \neq 0\,, \quad |z| \leqq 1\,, \quad 0 < \theta < \pi\,, \quad \alpha > 0\,. \tag{3.4}$$

It is true when $\alpha \to \infty$ since

$$\lim_{\alpha\to\infty} \frac{P_n^{(\alpha,\alpha)}(x)}{P_n^{(\alpha,\alpha)}(1)} = x^n\,.$$

These problems and results can be dualized to integrals of Jacobi polynomials. There are also related problems for integrals of Bessel functions and integrals of Laguerre polynomials. The integral for Bessel functions which corresponds to (2.5) is

$$\int_0^y x^{-\beta} J_\alpha(x)dx\,, \quad y > 0\,. \tag{3.5}$$

The condition $-\beta + \alpha > -1$ is now necessary to insure convergence at zero. The integrals in (3.5) have arisen in many different contexts and we hope to be able to completely solve the positivity problem in the near future. It is likely that

$$\int_0^y x^{-\beta} J_\alpha(x)dx > 0$$

when $\beta > \beta(\alpha)$, where $\beta(\alpha) = -1/2$, $\alpha \geqq 1/2$ (this is known) and $\beta(\alpha)$ for $-1 < \alpha < 1/2$ is the root of the transcendental equation

$$\int_0 j_{\alpha,2} x^{-\beta(\alpha)} J_\alpha(x)dx = 0\,,$$

with $j_{\alpha,2}$ the second positive zero of $J_\alpha(x)$.

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Theorem 3 for $\alpha=\beta+1$, $\beta=(k-1)/2$, $k=0, 1, \cdots$, was proven by M. H. Taibleson, Mean and pointwise convergence of logarithmic means of Laplace series, Scripta Mathematica 28 (1965), 197-203. The proof is the same, but the notation is different, since he only discusses ultraspherical polynomials.

The application of Theorem 3 to prove the positivity of some Cotes numbers is given in R. Askey, Positivity of the Cotes numbers for some Jacobi abscissas, Numersche Mathematik, 19 (1972), 46-48.

The positivity of the (C, $\alpha+\beta+2$) means for Jacobi series has been proven for $\beta-1\leqq\alpha\leqq\beta+1$, $\alpha+\beta\geqq 0$; and $\beta-2\leqq\alpha\leqq\beta+2$, $\alpha+\beta\geqq 3$. This will appear in the near future.

UNIVERSITY OF WISCONSIN, MADISON
MADISON, WISCONSIN, U.S.A.