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ON THE MULTIPLIERS OF HANKEL TRANSFORM

Dedicated to Professor Gen-Ichirô Sunouchi on his 60th birthday

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The Jacobi polynomial of degree n, order (α, β), $\alpha, \beta > -1$, is defined by

$$(1-x)^{\alpha}(1+x)^{\beta}P_n^{(\alpha,\beta)}(x) = \frac{(-1)^n}{2^n n!}\frac{d^n}{dx^n}[(1-x)^{n+\alpha}(1+x)^{n+\beta}] .$$

$\{P_n^{(\alpha,\beta)}(\cos\theta)\}_{n=0}^{\infty}$ is an orthogonal system on $(0, \pi)$ with respect to the measure $(\sin\theta/2)^{2\alpha+1}(\cos\theta/2)^{2\beta+1}d\theta$.

For a function $f(\theta)$ integrable on $(0, \pi)$ with respect to such a measure define

$$\hat{f}(n) = \int_0^{\pi} f(\theta)P_n^{(\alpha,\beta)}(\cos\theta)\left(\sin\frac{\theta}{2}\right)^{2\alpha+1}\left(\cos\frac{\theta}{2}\right)^{2\beta+1}d\theta .$$

Put

$$\frac{1}{h_n^{(\alpha,\beta)}} = \int_0^{\pi}[P_n^{(\alpha,\beta)}(\cos\theta)]^2\left(\sin\frac{\theta}{2}\right)^{2\alpha+1}\left(\cos\frac{\theta}{2}\right)^{2\beta+1}d\theta .$$

Then we have formally

$$f(\theta) = \sum_{n=0}^{\infty}\hat{f}(n)h_n^{(\alpha,\beta)}P_n^{(\alpha,\beta)}(\cos\theta) .$$

For a sequence $\phi(n)$ on the non negative integers define a transformation T_ϕ by

$$T_\phi f(\theta) = \sum_{n=0}^{\infty}\phi(n)\hat{f}(n)h_n^{(\alpha,\beta)}P_n^{(\alpha,\beta)}(\cos\theta) .$$

For $p \geqq 1$ and the function f on $(0, \pi)$ we define a norm

$$\|f\|_p = \left(\int_0^{\pi}|f(\theta)|^p\left(\sin\frac{\theta}{2}\right)^{2\alpha+1}\left(\cos\frac{\theta}{2}\right)^{2\beta+1}d\theta\right)^{1/p}$$

and denote by $L^p_{(\alpha,\beta)}(0, \pi)$ the set of all measurable functions such that $\|f\|_p < \infty$. The operator norm of T_ϕ of $L^p_{(\alpha,\beta)}(0, \pi)$ to $L^p_{(\alpha,\beta)}(0, \pi)$ will be denoted by $\|T_\phi\|_p$ or $\|\phi(n)\|_p$.

Let $J_\alpha(x)$ be the Bessel function of the first kind. For a function $g(x)$ on $(0, \infty)$ the (modified) Hankel transform of order α is defined by

$$\hat{g}(y) = \int_0^\infty g(x) \frac{J_\alpha(xy)}{(xy)^\alpha} x^{2\alpha+1} dx$$

and the multiplier transformation associated with $\phi(y)$ is defined formally by

$$U_\phi g(x) = \int_0^\infty \phi(y)\hat{g}(y) \frac{J_\alpha(xy)}{(xy)^\alpha} y^{2\alpha+1} dy \, .$$

$L_\alpha^p(0, \infty)$ will denote the space of all measurable function g such that

$$|g|_p = \left(\int_0^\infty |g(x)|^p x^{2\alpha+1} dx\right)^{1/p} < \infty \, .$$

The operator norm of U_ϕ of $L_\alpha^p(0, \infty)$ to $L_\alpha^p(0, \infty)$ will be denoted by $|U_\phi|_p$ or $|\phi(y)|_p$.

The object of this paper is to study the relation of the multiplier transformations between Jacobi polynomial expansions and Hankel transformations.

THEOREM. *Let $1 \leqq p < \infty$ and $\alpha, \beta > -1$. Assume that ϕ is a function on $(0, \infty)$ continuous except on a null set and $\underline{\lim}_{\varepsilon \to +0} \|\phi(\varepsilon n)\|_p$ is finite, then $|\phi(x)|_p$ is finite and $|\phi(x)|_p \leqq \underline{\lim}_{\varepsilon \to +0} \|\phi(\varepsilon n)\|_p$.*

PROOF. Let g be an infinitely differentiable function with compact support in a finite interval $(0, M)$ and put $g_\lambda(\theta) = g(\lambda\theta)$ where $\lambda > 0$ is so large that the support of $g_\lambda(\theta)$ is contained in $(0, \pi)$. Then we have by the assumption

$$(1) \qquad \left\|\sum_{n=0}^\infty \phi\left(\frac{n}{\lambda}\right)\hat{g}(n) h_n^{(\alpha,\beta)} P_n^{(\alpha,\beta)}(\cos\theta)\right\|_p \leqq \left\|\phi\left(\frac{n}{\lambda}\right)\right\|_p \|g\|_p \, .$$

Changing variable we get

$$\lambda^{(2\alpha+2)/p} \|g_\lambda\|_p = \left(\int_0^M |g(\tau)|^p \left(\lambda \sin\frac{\tau}{2\lambda}\right)^{2\alpha+1} \left(\cos\frac{\tau}{2\lambda}\right)^{2\beta+1} d\tau\right)^{1/p} ,$$

which tends to

$$\left(\frac{1}{2^{2\alpha+1}} \int_0^\infty |g(\tau)|^p \tau^{2\alpha+1} d\tau\right)^{1/p}$$

as $\lambda \to \infty$. Apply the similar argument to the left hand side of (1). Then we get by Fatou's lemma

$$(2) \qquad \left(\frac{1}{2^{2\alpha+1}} \int_0^\infty \underline{\lim}_{\lambda\to\infty} \left|\sum_{n=0}^\infty \phi\left(\frac{n}{\lambda}\right)\hat{g}_\lambda(n) h_n^{(\alpha,\beta)} P_n^{(\alpha,\beta)}\left(\cos\frac{\tau}{\lambda}\right)\right|^p \tau^{2\alpha+1} d\tau\right)^{1/p}$$
$$\leqq \underline{\lim}_{\lambda\to\infty} \left\|\phi\left(\frac{n}{\lambda}\right)\right\|_p \left(\frac{1}{2^{2\alpha+1}} \int_0^\infty |g(\tau)|^p \tau^{2\alpha+1} d\tau\right)^{1/p} .$$

Now we proceed to the computation of the left hand side of (2). First we remark that (2) holds for $p = 2$ and

$$\left\|\phi\left(\frac{n}{\lambda}\right)\right\|_2 \leqq \left\|\phi\left(\frac{n}{\lambda}\right)\right\|_p .$$

Thus for a sequence $\lambda_1 < \lambda_2 < \cdots < \lambda_j \to \infty$

$$G(\tau, \lambda) = \sum_{n=0}^{\infty} \phi\left(\frac{n}{\lambda}\right)\hat{g}_\lambda(n) h_n^{(\alpha,\beta)} P_n^{(\alpha,\beta)}\left(\cos\frac{\tau}{\lambda}\right)$$

converges weakly to a function $G(\tau)$ in $L_\alpha^2(0, K)$ for every $K > 0$ and $G(\tau)$ satisfies the inequality

$$(3) \qquad \left(\int_0^\infty |G(\tau)|^p \tau^{2\alpha+1} d\tau\right)^{1/p} \leqq \varliminf_{\lambda\to\infty} \left\|\phi\left(\frac{n}{\lambda}\right)\right\|_p \left(\int_0^\infty |g(\tau)|^p \tau^{2\alpha+1} d\tau\right)^{1/p} .$$

To show that $G(\tau)$ is the Hankel transform of $\phi\hat{g}$ put

$$\begin{aligned} G(\tau, \lambda) &= \left(\sum_{n=0}^{N[\lambda]} + \sum_{n=N[\lambda]+1}^{\infty}\right)\phi\left(\frac{n}{\lambda}\right)\hat{g}_\lambda(n) h_n^{(\alpha,\beta)} P_n^{(\alpha,\beta)}\left(\cos\frac{\tau}{\lambda}\right) \\ &= G^N(\tau, \lambda) + H^N(\tau, \lambda), \text{ say,} \end{aligned}$$

for $N = 1, 2, \cdots$.

Since

$$\begin{aligned} &\frac{d}{d\theta}\left[\left(\sin\frac{\theta}{2}\right)^{2\alpha+2}\left(\cos\frac{\theta}{2}\right)^{2\beta+2} P_{n-1}^{(\alpha+1,\beta+1)}(\cos\theta)\right] \\ &\qquad = n\left(\sin\frac{\theta}{2}\right)^{2\alpha+1}\left(\cos\frac{\theta}{2}\right)^{2\beta+1} P_n^{(\alpha,\beta)}(\cos\theta) \end{aligned}$$

(cf. [5, p. 97]), integrating by parts we get

$$\hat{g}_\lambda(n) = -\frac{\lambda}{n}\int_0^\pi \frac{g'(\lambda\theta)}{\sin\theta/2\cos\theta/2} P_{n-1}^{(\alpha+1,\beta+1)}(\cos\theta)\left(\sin\frac{\theta}{2}\right)^{2\alpha+3}\left(\cos\frac{\theta}{2}\right)^{2\beta+3} d\theta .$$

This, if $K > 0$ is any fixed number and $\pi\lambda > K$, then

$$\begin{aligned} &\int_0^K |H^N(\tau, \lambda)|^2 \left(\lambda \sin\frac{\tau}{2\lambda}\right)^{2\alpha+1}\left(\cos\frac{\tau}{2\lambda}\right)^{2\beta+1} d\tau \\ &\qquad \leqq \int_0^\pi |H^N(\tau, \lambda)|^2 \left(\lambda \sin\frac{\tau}{2\lambda}\right)^{2\alpha+1}\left(\cos\frac{\tau}{2\lambda}\right)^{2\beta+1} d\tau \\ &\qquad = \lambda^{2\alpha+2}\int_0^\pi |H^N(\lambda\tau, \lambda)|^2 \left(\sin\frac{\tau}{2}\right)^{2\alpha+1}\left(\cos\frac{\tau}{2}\right)^{2\beta+1} d\tau . \end{aligned}$$

By Parseval's relation the last term equals

$$\lambda^{2\alpha+2}\sum_{n=N[\lambda]+1}^{\infty}\left|\phi\left(\frac{n}{\lambda}\right)\right|^2|\hat{g}_\lambda(n)|^2 h_n^{(\alpha,\beta)} .$$

Since $h_n^{(\alpha,\beta)} = 2n + O(1)$ as $n \to \infty$ and ϕ is uniformly bounded, the above is dominated by

$$A\lambda^{2\alpha+2}\left(\frac{\lambda}{N\lambda}\right)^2\sum_{n=N[\lambda]+1}^{\infty}\left|\frac{n}{\lambda}\hat{g}_\lambda(n)\right|^2 h_{n-1}^{(\alpha+1,\beta+1)} ,$$

where A is a constant independent on λ and N. By Bessel's inequality this is bounded by

$$\begin{aligned} &A\frac{\lambda^{2\alpha+2}}{N^2}\int_0^\pi\left|\frac{g'(\lambda\theta)}{\sin\theta/2\cos\theta/2}\right|^2\left(\sin\frac{\theta}{2}\right)^{2\alpha+3}\left(\cos\frac{\theta}{2}\right)^{2\beta+3}d\theta \\ &= \frac{A}{N^2}\int_0^M|g'(\theta)|^2\left(\lambda\sin\frac{\theta}{2\lambda}\right)^{2\alpha+1}\left(\cos\frac{\theta}{2\lambda}\right)^{2\beta+1}d\theta \\ &= O\left(\frac{1}{N^2}\right) \end{aligned}$$

uniformly in λ.

Thus we get

$$\int_0^\pi|H^N(\tau,\lambda)|^2\tau^{2\alpha+1}d\tau = O\left(\frac{1}{N^2}\right)$$

uniformly in λ.

Thus by the diagonal argument there exists a subsequence $\{\lambda_{k_j}\}$ of $\{\lambda_j\}$ such that $H^N(\tau,\lambda_{k_j})$ converges weakly to a function $H^N(\tau)$ in $L^2_\alpha(0, K)$ for every $N = 1, 2, \cdots$ and

$$\int_0^K|H^N(\tau)|^2\tau^{2\alpha+1}d\tau = O\left(\frac{1}{N^2}\right) .$$

For a subsequence $\{N_j\}$, $H^{N_j}(\tau)$ converges to zero almost everywhere.

Since

$$G^N(\tau,\lambda) = G(\tau,\lambda) - H^N(\tau,\lambda) ,$$

$G^N(\tau,\lambda_{k_j})$ converges weakly in $L^2_\alpha(0, K)$ to a limit $G^N(\tau)$ as $j \to \infty$ and $G(\tau) = G^N(\tau) + H^N(\tau)$ for $N = 1, 2, \cdots$. Thus $G^{N_j}(\tau)$ converges to $G(\tau)$ almost everywhere.

We prove that $G^N(\tau,\lambda)$ converges pointwise to a function as $\lambda \to \infty$. Then the limit function coincides with $G^N(\tau)$.

First we note that

$$\left(\sin\frac{\theta}{2}\right)^\alpha\left(\cos\frac{\theta}{2}\right)^\beta P_n^{(\alpha,\beta)}(\cos\theta)$$

$$= \tilde{n}^{-\alpha}\frac{\Gamma(n+\alpha+1)}{n^\alpha}\left(\frac{\theta}{\sin\theta}\right)^{1/2}J_\alpha(\tilde{n}\theta) + \begin{cases} O(\theta^{1/2}n^{-3/2}) & \text{for } Cn^{-1} \leqq \theta \leqq \pi - \varepsilon \\ O(\theta^{\alpha+2}n^\alpha) & \text{for } 0 < \theta \leqq Cn^{-1}, \end{cases}$$

where $\tilde{n} = n + (\alpha+\beta+1)/2$, and ε and C are fixed positive numbers ([5, p. 197]).

Let K be a fixed number and $0 < \tau \leqq K$. For n, $0 \leqq n \leqq N[\lambda]$, we have

$$\begin{aligned}
&\frac{h_n^{(\alpha,\beta)}}{\lambda^\alpha}P_n^{(\alpha,\beta)}\left(\cos\frac{\tau}{\lambda}\right) \\
&= h_n^{(\alpha,\beta)}\tilde{n}^{-\alpha}\frac{\Gamma(n+\alpha+1)}{n^\alpha}\left(\frac{\tau/\lambda}{\sin\tau/\lambda}\right)^{1/2}J_\alpha\left(\frac{\tilde{n}}{\lambda}\tau\right)\frac{1}{(\lambda\sin\tau/2\lambda)^\alpha(\cos\tau/2\lambda)^\beta} + O\left(\frac{n^{\alpha+1}}{\lambda^{\alpha+2}}\right) \\
&= h_n^{(\alpha,\beta)}J_\alpha\left(\frac{\tilde{n}}{\lambda}\tau\right)\left(\frac{2}{\tau}\right)^\alpha + o(n) \\
&= 2nJ_\alpha\left(\frac{n}{\lambda}\tau\right)\left(\frac{2}{\tau}\right)^\alpha + o(n)\,.
\end{aligned}$$

On the other hand

$$\begin{aligned}
\lambda^\alpha\hat{g}_\lambda(n) &= \frac{1}{\lambda^{\alpha+2}}\int_0^M g(\theta)P_n^{(\alpha,\beta)}\left(\cos\frac{\theta}{\lambda}\right)\left(\sin\frac{\theta}{2\lambda}\right)^{2\alpha+1}\left(\cos\frac{\theta}{2\lambda}\right)^{2\beta+1}d\theta \\
&= \frac{1}{\lambda^2}\int_0^M g(\theta)\tilde{n}\frac{\Gamma(n+\alpha+1)}{\alpha}\left(\frac{\theta/\lambda}{\sin\theta/\lambda}\right)^{1/2}J_\alpha\left(\frac{\tilde{n}}{\lambda}\theta\right)\left(\lambda\sin\frac{\theta}{2\lambda}\right)^{\alpha+1}\left(\cos\frac{\theta}{2\lambda}\right)^{\beta+1}d\theta \\
&\quad + o\left(\frac{1}{\lambda^2}\right) \\
&= \frac{1}{\lambda^2}\frac{1}{2^{\alpha+1}}\int_0^\infty g(\theta)J_\alpha\left(\frac{n}{\lambda}\theta\right)\theta^{\alpha+1}d\theta + o\left(\frac{1}{\lambda^2}\right).
\end{aligned}$$

Thus

$$\begin{aligned}
&\lim_{\lambda\to\infty}\sum_{n=0}^{N[\lambda]}\phi\left(\frac{n}{\lambda}\right)\hat{g}_\lambda(n)h_n^{(\alpha,\beta)}P_n^{(\alpha,\beta)}\left(\cos\frac{\tau}{\lambda}\right) \\
&= \lim_{\lambda\to\infty}\left\{\sum_{n=0}^{N[\lambda]}\phi\left(\frac{n}{\lambda}\right)\int_0^\infty g(\theta)J_\alpha\left(\frac{n}{\lambda}\theta\right)\theta^{\alpha+1}d\theta J_\alpha\left(\frac{n}{\lambda}\tau\right)\frac{1}{\tau^\alpha}\frac{n}{\lambda}\frac{1}{\lambda} + o(1)\frac{n}{\lambda^2}\right\} \\
&= \int_0^N \phi(v)\hat{g}(v)\frac{J_\alpha(v\tau)}{(v\tau)^\alpha}v^{2\alpha+1}dv\,.
\end{aligned}$$

Thus we get

$$G(\tau) = \int_0^\infty \phi(v)\hat{g}(v)\frac{J_\alpha(v\tau)}{(v\tau)^\alpha}v^{2\alpha+1}dv \quad \text{a.e.} \tag{4}$$

From (3) it follows that

$$|\phi(x)|_p \leqq \varliminf_{\lambda\to\infty} \left\|\phi\left(\frac{n}{\lambda}\right)\right\|_p ,$$

which proves the theorem.

Our theorem proves the mean convergence, mean Cesàro summability, the multiplier theorems of Marcinkiewicz' type and decomposition theorem for Hankel transform by the theorems in [4], [1] and [2].

We remark that our theorem is reduced to a theorem in [3] when $\alpha = \beta = -1/2$.

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