Tôhoku Math. Journ.
24 (1972), 207-221.

ON POWER SERIES WITH NEGATIVE ZEROS*

Dedicated to Professor Gen-ichirô Sunouchi on his 60th birthday

W. B. JURKAT (Syracuse) and A. PEYERIMHOFF (Ulm)

(Received Oct. 1, 1971)

It is well known [3] that the functions $f_\kappa(z) = \sum_{n=0}^{\infty} n^\kappa z^n\,(\kappa > 0)$ and $g_\kappa(z) = \sum_{n=0}^{\infty}(1-c^n)^\kappa z^n(\kappa > 0,\ 0 < c < 1)$ admit (unique) analytic extensions onto $C^* = \{z = x+iy \mid y \neq 0 \text{ if } x \geqq 1\}$. Both functions have a finite number of zeros only. Moreover, all zeros are $\leqq 0$ and simple, and their number is k, where $k-1 < \kappa \leqq k$, $k = 1, 2, \cdots$. In this paper we will give some general theorems on the zeros of power series, and these results contain the information on f_κ and g_κ as special cases. Further examples are mentioned in Section 4.

We remark that our functions need not be meromorphic (like f_κ) or may have infinitely many zeros and poles on $(1, \infty)$ (like g_κ) so that known results on zeros of analytic functions like those in [1] cannot be applied. Theorem 1 gives an upper estimate for the number of zeros of certain power series $\sum a_n z^n$, and Theorem 2 gives a lower estimate. In Theorem 1 we require that certain differences of the coefficients a_n form a completely monotone sequence (we use the definitions given in [6]). In discussing special cases it will be more convenient to require that $a_n = a(n)$, where $a(x)$ satisfies a linear differential equation with completely monotone right hand side (Theorems 3 and 4).

0. In what follows we will denote by $[x_1, \cdots, x_n]_{f(v)}$ the divided differences of f (see [2]). If C is a simple closed curve containing $x_1, \cdots, x_n$ in its interior, and if f is holomorphic inside and on C, then

$$[x_1, \cdots, x_n]_{f(v)} = \frac{1}{2\pi i}\int_c \frac{f(z)}{p(z)}\,dz\ , \qquad p(z) = \prod_1^n (z-x_i)\ .$$

The differential equation

$$(1) \qquad \left\{\prod_1^k (D-x_i)\right\} y(x) = \varphi(x)\ , \qquad x_i \text{ constant},\ D = \frac{d}{dx},\ k = 0, 1, \cdots$$

where

* The research of both authors was supported in part by the National Science Foundation.

$$(2)\qquad \varphi(x) = \int_{+0}^{1} w^x\, dg(w)\ ,\quad \begin{cases} g \in V[\varepsilon, 1] \text{ for every } \varepsilon > 0\ , \\ \int_{+0}^{1} w^x\, |dg(w)| < \infty \text{ for every } x > 0 \end{cases}$$

has the particular solution

$$(3)\qquad y(x) = \int_{+0}^{1} w dg(w)\, [x_1, \cdots, x_k, \log w]_{e^{(x-1)v}}\, ,\ x > 0\ .$$

In order to prove this we first show that (3) exists. Let C be a simple closed curve containing $x_1, \cdots, x_k$, and let $\log w$ be outside of C for $0 < w \leqq w_0 < 1$. Writing $p(z) = \prod_1^k (z - x_i)$ we have for $0 < w \leqq w_0$

$$(4)\qquad \begin{cases} [x_1, \cdots, x_k, \log w]_{e^{(x-1)v}} = \dfrac{w^{x-1}}{p(\log w)} + \dfrac{1}{2\pi i}\displaystyle\int_c \frac{e^{(x-1)z}}{p(z)} \frac{dz}{z - \log w} \\ \qquad = \dfrac{w^{x-1}}{p(\log w)} + O\left(\dfrac{1}{\log w}\right)\ (w \longrightarrow 0) \end{cases}$$

where the O-term is uniform in x when x is restricted to a compact interval. This shows that (3) exists. It follows from

$$(5)\qquad (D - \alpha)\, [x_1, \cdots, x_n, \alpha]_{e^{xv} f(v)} = [x_1, \cdots, x_n]_{e^{xv} f(v)}$$

that

$$\left\{\prod_1^k (D - x_i)\right\} y(x) = \int_{+0}^{1} w dg(w) \left\{\prod_1^k (D - x_i)\right\} [x_1, \cdots, x_k, \log w]_{e^{xv} e^{-v}}$$

$$= \int_{+0}^{1} w [\log w]_{e^{(x-1)v}}\, dg(w) = \varphi(x)\ ,$$

and this shows that (3) is a solution of (1).

1. Given a sequence $\{t_n\}_0^\infty$, let

$$\Delta(c) t_n = \begin{cases} t_n - c t_{n-1} & \text{for } n \geqq 1\ , \\ t_0 & \text{for } n = 0\ . \end{cases}$$

THEOREM 1. *Let $\{a_n\}_0^\infty$ be a real sequence and such that for certain integers $0 \leqq k \leqq p$ and constants $c_i \in (0, 1]$*

$$\left\{\prod_{i=1}^k \Delta(c_i)\right\} a_{n+p} = b_n\ (n = 0, 1, \cdots)$$

defines a completely monotone sequence $\{b_n\}_0^\infty$. Then $f(z) = \sum_0^\infty a_n z^n$ defines on C^*(uniquely) a holomorphic function which has at most p zeros unless $f \equiv 0$.

PROOF. This follows from the identity

$$\left(\prod_1^k (1 - c_i z)\right) \sum_0^\infty a_n z^n = \sum_1^\infty z^n \left(\prod_1^k \Delta(c_i)\right) a_n = \sum_0^{p-1} z^n \left(\prod_1^k \Delta(c_i)\right) a_n + z^p \sum_0^\infty b_n z^n$$

by Theorem 1 of [3].

Our next theorem gives a lower estimate for the number of zeros of functions of the type

$$F_m(z) = F_m(z;\tau) = \begin{cases} \sum_{n=0}^{\infty} (n+\tau)^m c(n+\tau) z^n , & \tau \in (0,1) \\ \sum_{n=1}^{\infty} n^m c(n) z^n , & \tau = 0 , \end{cases}$$

$(m=0, 1, \cdots)$, where the function $c(x)$, $x>0$, satisfies a differential equation

$$\left\{\prod_{i=1}^{r} (D-\xi_i)\right\} c(x) = \int_{+0}^{1} w^x dh(w),\ \xi_i \text{ constant},\ h \in V[\varepsilon, 1]$$

for every $\varepsilon > 0$, $r = 0, 1, \cdots$.

THEOREM 2. *Let* $\xi_i \leqq 0$, *and assume that*

$$x \int_{+0}^{1/x} w \frac{|\,dh(w)\,|}{(\log 1/w)^r} = o(1) \quad (x \to \infty) . \tag{6}$$

Then F_m *is (uniquely) defined on* C^*. *Let* F_0 *have (at least) the following zeros*:

$$z_\nu < z_{\nu-1} < \cdots < z_1 < 0 < z_1' < z_2' < \cdots < z_\mu' < 1 \ (\nu, \mu=0, 1, \cdots) .$$

Then $F_m(m=1, 2, \cdots)$ *has (at least) zeros of the following kind*

$$\zeta_{m+\nu} < \cdots < \zeta_1 \leqq 0 < \zeta_1' < \cdots < \zeta_\mu' < 1 ,$$

and $\zeta_1 < 0$ *if* $\tau \in (0, 1)$.

PROOF. We mention first two consequences of (6):

$$x^\tau \int_{+0}^{1/x} w^\tau \frac{|\,dh(w)\,|}{(\log 1/w)^r} = o\,(1) \quad (x \longrightarrow \infty, \tau \in (0, 1)) , \tag{7}$$

$$x^{\tau-1} \int_{1/x}^{1} w^{\tau-1} \frac{|\,dh(w)\,|}{(1+\log 1/w)^r} = o(1) \quad (x \longrightarrow \infty,\ \tau \in [0, 1)) . \tag{8}$$

Writing $dh^*(w)$ for $(|\,dh(w)\,|)/((1+\log 1/w)^r)$ the relation (7) follows from (6) and

$$x^\tau \int_{+0}^{1/x} w^\tau dh^*(w) = x^\tau \int_{+0}^{1/x} w^{\tau-1} d \int_{+0}^{w} t dh^*(t)$$

$$= x \int_{+0}^{1/x} t\, dh^*(t) + (1-\tau) x^\tau \int_{+0}^{1/x} w^{\tau-1} \Big(\frac{1}{w} \int_{+0}^{w} t\, dh^*(t)\Big)\, dw ,$$

and the relation (8) follows from (6) and

$$x^{\tau-1}\int_{1/x}^{1} w^{\tau-1}dh^*(w) = x^{\tau-1}\int_{1/x}^{1} w^{\tau-2}\, d\int_{+0}^{w} tdh^*(t)$$
$$= x^{\tau-1}\int_{+0}^{1} tdh^*(t) - x\int_{+0}^{1/x} tdh^*(t)$$
$$+ (2-\tau)x^{\tau-1}\int_{1/x}^{1} w^{\tau-2}\Big(\frac{1}{w}\int_{+0}^{w} tdh^*(t)\Big)\, dw\ .$$

We note that (7) implies

$$\int_{+0}^{1} w^{x}\,|\, dh(w)\,| < \infty \quad \text{for every } x > 0\,[1]\,. \tag{9}$$

Next we wish to show that F_m exists on C^* and that

$$\begin{cases} F_m(-x;\,0) \longrightarrow 0 \quad \text{for} \quad x \longrightarrow \infty\ , \quad m = 1, 2, \cdots\ , \\ x^{\tau}F_m(-x;\,\tau) \longrightarrow 0 \quad \text{for} \quad x \longrightarrow \infty\ , \quad \tau \in (0, 1)\ , \quad m = 0, 1, \cdots\ . \end{cases} \tag{10}$$

In order to prove this we note first that (3) and (9) imply

$$\begin{cases} c(x) = c_0(x) + \int_{+0}^{1} wdh(w)[\xi_1, \cdots, \xi_r, \log w]_{e^{(x-1)v}},\ x > 0\ , \\ \Big\{\prod_{1}^{r} (D-\xi_i)\Big\}c_0(x) = 0\ . \end{cases} \tag{11}$$

Denoting by P_j polynomials of degree $\leqq j$, we have for $\rho = 0, 1, \cdots$ and $\alpha \in (-\infty, \infty)$ a representation

$$\sum_{n=0}^{\infty} (n+\alpha)^{\rho} z^n = \begin{cases} P_\rho(z)/(1-z)^{\rho+1} \\ P_{\rho-1}(z)/(1-z)^{\rho+1} \quad \text{for} \quad \alpha = 1,\ \rho = 1, 2, \cdots\ . \end{cases} \tag{12}$$

(This follows from a short induction-type proof; see also [4].)

According to (11) we write F_m as a sum $F_m^0 + \widetilde{F}_m$ (where F_m^0 is generated by c_0). It follows from (11) that $c_0(x)$ is a linear combination of functions of the type $x^{\lambda}e^{x\xi_i}$ for some $\lambda = 0, 1, \cdots$, and it follows from (12) that F_m^0 is a linear combination of terms of the type

$$e^{\tau\xi_i}P_{m+\lambda}(ze^{\xi_i})/(1-ze^{\xi_i})^{m+\lambda+1} \qquad \text{for } \tau \in (0, 1),\ m = 0, 1, \cdots\ ,$$
$$ze^{\xi_i}P_{m+\lambda-1}(ze^{\xi_i})/(1-ze^{\xi_i})^{m+\lambda+1} \qquad \text{for } \tau = 0,\ m = 1, 2, \cdots\ ,$$

and this shows that F_m^0 is defined on C^* (note that $\xi_i \leqq 0$) and that (10) is true when F_m is replaced by F_m^0.

Using a representation similar to (4) for the divided difference in (11) we have for $\tau \in (0, 1)$, $m = 0, 1, \cdots$

[1] Actually more is true, namely $\int_{+0}^{1} (|\, dh(w)\,|)/((1+\log 1/w)^{r+\eta}) < \infty$ for every $\eta > 1$.

$$\begin{aligned}\tilde{F}_m(z) &= \int_{+0}^{w_0} w^\tau \frac{P_m(wz)}{(1-wz)^{m+1}} \frac{dh(w)}{p(\log w)} \\ &+ \int_{+0}^{w_0} wdh(w) \frac{1}{2\pi i}\int_C \frac{e^{(\tau-1)\zeta}P_m(ze^\zeta)}{(1-ze^\zeta)^{m+1}} \frac{d\zeta}{p(\zeta)\,(\zeta-\log w)} \\ &+ \int_{w_0}^{1} wdh(w) \frac{1}{2\pi i}\int_{C_1} \frac{e^{(\tau-1)\zeta}P_m(ze^\zeta)}{(1-ze^\zeta)^{m+1}} \frac{d\zeta}{p(\zeta)\,(\zeta-\log w)} \\ &= \mathrm{I} + \mathrm{II} + \mathrm{III}\end{aligned}$$

(C_1 denotes a simple closed curve containing $\xi_1, \cdots, \xi_r$ and $\log w$ for $w_0 \leqq w \leqq 1$). The same representation holds for $\tau = 0$ and $m = 1, 2, \cdots$, where $P_m(z)$ contains a factor z.

It follows from this representation that $\tilde{F}_m$ is defined on C^* (choose C, C_1 suitably). Moreover, we see that for $z = -x < 0$ and $x \to \infty$

$$\mathrm{II} = O\left(\frac{1}{x}\right), \quad \mathrm{III} = O\left(\frac{1}{x}\right),$$

and that

$$\begin{aligned}|\, x^\tau \mathrm{I}\,| &= O(1)\, x^\tau \int_{+0}^{1/x} w^\tau \frac{|\,dh(w)\,|}{(\log 1/w)^r} + O(1)\, x^{\tau-1}\int_{1/x}^{w_0} w^{\tau-1}\frac{|\,dh(w)\,|}{(\log 1/w)^r} \\ &= o(1)\end{aligned}$$

by (7) and (8) ($\tau \neq 0$) or by (6) and (8) ($\tau = 0$). This proves (10).

We observe that for $\tau \in [0, 1)$

$$\frac{d}{dx}\Big((-x)^\tau F_m(x)\Big) = -(-x)^{\tau-1}F_{m+1}(x) \quad \text{for } x < 0\,, \tag{13}$$

and

$$\frac{d}{dx}\,(x^\tau F_m(x)) = x^{\tau-1}F_{m+1}(x) \quad \text{for } x > 0\,. \tag{14}$$

Let F_i (for some $i = 1, 2, \cdots$) have the zeros

$$z_\rho < z_{\rho-1} < \cdots < z_1 < 0 < z_1' < \cdots < z_\sigma' < 1\,,$$

and let $\phi_i(x) = (-x)^\tau F_i(x)$ ($x \leqq 0$), $\phi_i(x) = x^\tau F_i(x)$ ($x \geqq 0$). Then $\phi_i(x) = 0$ when $F_i(x) = 0$, and $\phi_i(0) = 0$ (note that $F_i(0) = 0$ when $\tau = 0$). Furthermore, $\phi_i(x) \to 0$ ($x \to -\infty$) by (10). It follows from (13) and (14) by Rolle's theorem that F_{i+1} has zeros $\zeta_1, \cdots, \zeta_{\rho+1}, \zeta_1', \cdots, \zeta_\sigma'$ with

$$\zeta_{\rho+1} < z_\rho < \zeta_\rho < \cdots < z_1 < \zeta_1 < 0 < \zeta_1' < z_1' < \cdots < \zeta_\sigma' < z_\sigma' < 1\,.$$

If $\tau \in (0, 1)$ and $i = 0$, then this is also true, which proves Theorem 2 for $\tau \neq 0$. If $\tau = 0$ and $i = 0$, the foregoing argument only shows that zeros

$\zeta_1, \cdots, \zeta_\rho, \zeta'_1, , \cdots, \zeta'_\sigma$ exist ($\phi_0(x) \to 0$ for $x \to -\infty$ may not be true, and $\zeta_{\rho+1}$ may be lost). But $F_i(0) = 0$ is true in this case, and this proves Theorem 2 for $\tau = 0$.

REMARKS. (i) Condition (6) is satisfied if h is absolutely continuous on $(0, \varepsilon]$ (for some $\varepsilon > 0$) and

$$wh'(w) = o((\log 1/w)^r) \quad \text{as} \quad w \to 0 . \tag{15}$$

(ii) Let the assumptions of Theorem 2 be satisfied, and let F_m (or F_0) have a zero of order λ at $z = 0$. If $\tau \in (0, 1)$, then F_m has (at least) $m + \nu + \lambda$ zeros which are $\leqq 0$ ($m + \nu$ zeros are <0), and if $\tau = 0$ and $m \geqq 1$ then F_m has (at least) $m + \nu + \lambda - 1$ zeros which are $\leqq 0$ ($m + \nu - 1$ zeros are <0).

2. In this section we shall discuss special solutions of (1), (2) under various conditions on g and for special initial conditions. These results will be needed to prove Theorems 3 and 4.

LEMMA 1. *Assume that* (1), (2) *with* $g \uparrow$ *has a solution* $\tilde{y} \in C_p\ [0, \infty)$ *for some* $p = 0, 1, \cdots$. *Then*

$$\int_{+0}^{1} \frac{dg(w)}{(1 + \log 1/w)^{k-p}} < \infty . \tag{16}$$

PROOF. It follows from (3) and for some a_0 with $\{\prod_1^k (D - x_i)\}\ a_0 = 0$ that

$$\tilde{y}(x) = a_0(x) + \int_{+0}^{1} wdg(w)\ [x_1, \cdots, x_k, \log w]_{e^{(x-1)v}} , \quad x > 0 ,$$

and by differentiation

$$\tilde{y}^{(p)}(x) = a_0^{(p)}(x) + \int_{+0}^{1} wdg(w)\ [x_1, \cdots, x_k, \log w]_{v^p e^{(x-1)v}} . \tag{17}$$

Similarly to (4) the divided difference in this integral is $(0 < w \leqq w_0)$.

$$\frac{(\log w)^p w^{x-1}}{\prod_1^k (\log w - x_i)} + O\left(\frac{1}{\log w}\right) \qquad (w \longrightarrow 0) , \tag{18}$$

where the O-term is uniform in x when x is restricted to a compact interval. The statement (16) now follows from (17) and (18) (note that $a_0^{(p)} \in C(-\infty, \infty)$, and that $g \uparrow$). In what follows we use the notation

$$a(x; x_1, \cdots, x_k;\ U(w)dg(w)) = \int_{+0}^{1} U(w)dg(w)\ [x_1, \cdots, x_k, \log w]_{e^{xv}} .$$

Our next Lemma is a kind of converse of Lemma 1.

LEMMA 2. *Let g satisfy* (2), *and assume that*

$$\int_{+0}^{1} \frac{|dg(w)|}{(1+\log 1/w)^{k-p}} < \infty \tag{19}$$

for some $p = 0, 1, \cdots, k-1$ ($k \geqq 1$). Then, for $x > 0$, $c_\nu = e^{x_\nu}$, the function $Y_{k,p}(x)$ defined by

$$Y_{k,p}(x) = \sum_{\nu=p+2}^{k} a\left(x; x_1, \cdots, x_\nu; \frac{w}{c_\nu}\left(\prod_{j=\nu+1}^{k} \frac{1-w/c_j}{\log w/c_j}\right) dg(w)\right)$$

$$+ a\left(x; x_1, \cdots, x_{p+1}; \left(\prod_{j=p+2}^{k} \frac{1-w/c_j}{\log w/c_j}\right) dg(w)\right), \qquad p \leqq k-2$$

$$Y_{k,k-1} = a(x, x_1, \cdots, x_k; dg(w)),$$

is a solution of (1), (2). *Moreover, $Y_{k,p} \in C_p[0, \infty)$ and*

$$Y_{k,p}(0) = Y'_{k,p}(0) = \cdots = Y^{(p)}_{k,p}(0) = 0. \tag{20}$$

For $x > 0$ the general solution of (1), (2) *with $y \in C_p[0, \infty)$ and*

$$y(0) = y'(0) = \cdots = y^{(p)}(0) = 0$$

is

$$y(x) = \begin{cases} Y_{k,k-1}(x), & p = k-1, \\ \sum_{\nu=p+2}^{k} C_\nu [x_1, \cdots, x_\nu]_{e^{xv}} + Y_{k,p}, \ C_\nu \text{ constant}, & p \leqq k-2. \end{cases} \tag{21}$$

PROOF. It follows from

$$[x_1, \cdots, x_k, \log w]_{v^p e^{xv}} = \begin{cases} \dfrac{(\log w)^p w^x}{\prod_1^k (\log w - x_i)} + O\left(\dfrac{1}{\log w}\right) & \text{for } 0 < w \leqq w_0 < 1, \\ 0 \quad \text{for } w \in (0, 1], & x = 0 \end{cases}$$

(see (18), and with the same remark on the O – term) that

$$a^{(p)}(x; x_1, \cdots, x_\nu; dh(w)) \in C[0, \infty),$$

$$a^{(\mu)}(0; x_1, \cdots, x_\nu; dh(w)) = 0$$

if $\nu \geqq p+1$, $\mu = 0, 1, \cdots, p$, $h \in V[\varepsilon, 1]$ for every $\varepsilon > 0$, and

$$\int_{+0}^{1} \frac{|dh(w)|}{(1+\log 1/w)^{\nu-p}} < \infty.$$

This shows that (19) implies $Y_{k,p} \in C_p[0, \infty)$ and the conditions (20).

Now we show that the functions $Y_{k,p}$ are solutions of (1), (2). Using (5) this is obvious for $p = k-1$, and it follows for $p \leqq k-2$ from

$$\left\{\prod_1^k (D - x_i)\right\} Y_{k,p}(x)$$
$$= \sum_{\nu=p+2}^k \left\{\prod_{i=\nu+1}^k (D-x_i)\right\} a\left(x; \frac{w}{c_\nu}\left(\prod_{j=\nu+1}^k \frac{1-w/c_j}{\log w/c_j}\right) dg(w)\right)$$
$$+ \left\{\prod_{i=p+2}^k (D-x_i)\right\} a\left(x; \left(\prod_{j=p+2}^k \frac{1-w/c_j}{\log w/c_j}\right) dg(w)\right)$$
$$= \int_{+0}^1 w^x\, dg(w) \left\{ \sum_{\nu=p+2}^k \left(\prod_{j=\nu+1}^k \left(1-\frac{w}{c_j}\right) - \prod_{j=\nu}^k \left(1-\frac{w}{c_j}\right)\right) + \prod_{j=p+2}^k \left(1-\frac{w}{c_j}\right)\right\}$$
$$= \int_{+0}^1 w^x dg(w) = \varphi(x) .$$

The statement on the general solution of (1), (2) follows from

$$\left\{\prod_1^k (D-x_i)\right\}[x_1, \cdots, x_\nu]_{exv} = 0 \qquad (\nu = 1, \cdots, k)$$

(use (5)) and

$$D^q\,[x_1, \cdots, x_\nu]_{exv}\,|_{x=0} = [x_1, \cdots, x_\nu]_{vq} = \begin{cases} 0 & \text{for } q < \nu - 1 \\ 1 & \text{for } q = \nu - 1 . \end{cases}$$

(The functions $[x_1, \cdots, x_\nu]_{exv}$, $\nu = 1, \cdots, k$, represent a basis for the solutions of $\{\prod_1^k (D-x_i)\}\, y = 0$.)

LEMMA 3. *Let g satisfy* (2), *and assume that* (19) *holds for some* $p = 0, 1, \cdots, k - 1$ $(k \geqq 1)$. *If y_p is a solution of* (1), (2) *with $x_i \leqq 0$, and if $y_p \in C_p\,[0, \infty)$ and $y_p(0) = y_p'(0) = \cdots = y_p^{(p)}(0) = 0$* (i.e. *$y_p$ is one of the solutions* (21)) *then*

$$\left\{\prod_{i=p+2}^k (D-x_i)\right\} \frac{y_p(x)}{x^{p+1}} = \int_0^1 t^x\, H(t)\, \frac{dt}{t}\, , \quad x > 0\, , \tag{22}$$

where

$$\begin{cases} \dfrac{1}{t}\, H(t) \in L\,[\varepsilon, 1] \textit{ for every } \varepsilon > 0, \textit{ and} \\ H(t) = o\,((\log 1/t)^{k-p-1}) \quad \text{as } t \longrightarrow 0 . \end{cases} \tag{23}$$

PROOF. We may without loss of generality assume that $g \uparrow$. We discuss first the case $p = k - 1$, i.e. we show that (2) and

$$\int_{+0}^1 \frac{|\, dg(w)\,|}{(1+\log 1/w)} < \infty \qquad \text{imply}$$

$$\frac{Y_{k,k-1}(x)}{x^k} = \int_0^1 t^x H(t)\, \frac{dt}{t}\, , \quad x > 0\, , \tag{24}$$

where $1/t\, H(t) \in L[\varepsilon, 1]$ for every $\varepsilon > 0$ and $H(t) \to 0$ as $t \to 0$. Let (without loss of generality) $x_1 \leqq x_2 \leqq \cdots \leqq x_k \leqq 0$, and define $d_i \in (0, 1]$ by $\prod_i^k d_\nu = e^{x_i}$. Then

$$[x_1, \cdots, x_k, \log w]_{e^{xv}} = (d_1 \cdots d_k)^x \int_{0 \leqq t_k \leqq \cdots \leqq t_1 \leqq x} \cdots \int w^{t_k} \prod_{i=1}^k d_i^{-t_i}\, dt_i ,$$

since both sides satisfy the differential equation (1) with $\varphi(x) = w^x$ (this follows for the right side from a short calculation, and from (5) for the left side) and initial conditions $y(0) = y'(0) = \cdots = y^{(k-1)} = 0$. Hence, $Y_{k,k-1}$ can be written in the form

$$Y_{k,k-1}(x) = x^k \int_{+0}^1 dg(w) \int_{0 \leqq t_k \leqq \cdots \leqq t_1 \leqq 1} \cdots \int w^{x t_k} \prod_{i=1}^k d_i^{x(1-t_i)}\, dt_i .$$

Denote the region $0 \leqq t_{k-1} \leqq \cdots \leqq t_1 \leqq 1$ by Δ, and let

$$\rho\,(t_1, \cdots, t_{k-1}) = d_1^{1-t_1} d_2^{1-t_2} \cdots d_{k-1}^{1-t_{k-1}} d_k .$$

The following computations simplify for $k = 1$ ($t_{k-1} = 1$). We have

$$\frac{Y_{k,k-1}(x)}{x^k} = \int_\Delta \cdots \int dt_1 \cdots dt_{k-1}\, \rho^x \int_{+0}^1 dg(w) \int_0^{t_{k-1}} \left(\frac{w}{d_k}\right)^{x t_k} dt_k .$$

But

$$\begin{aligned}
\rho^x \int_{+0}^1 dg(w) \int_0^{t_{k-1}} \left(\frac{w}{d_k}\right)^{x t_k} dt_k &= \int_{+0}^1 \frac{dg(w)}{\log w/d_k}\, \frac{\rho^x (w/d_k)^{x t_{k-1}} - \rho^x}{x} \\
&= \int_{+0}^1 \frac{dg(w)}{\log w/d_k} \int_\rho^{\rho (w/d_k)^{t_{k-1}}} t^{x-1}\, dt \\
&= \int_0^\rho t^{x-1}\, dt \int_{+0}^{d_k (t/\rho)^{1/t_{k-1}}} \frac{dg(w)}{\log d_k/w} \\
&\quad + \int_\rho^{\rho (1/d_k)^{t_{k-1}}} t^{x-1}\, dt \int_{d_k (t/\rho)^{1/t_{k-1}}}^1 \frac{dg(w)}{\log w/d_k} \\
&= \int_0^1 t^{x-1}\, h_{d_k}\!\left(d_k \left(\frac{t}{\rho}\right)^{1/t_{k-1}}\right) dt
\end{aligned}$$

where (for $0 < d \leqq 1$)

$$h_d(y) = \begin{cases} \displaystyle\int_{+0}^y \frac{dg(w)}{\log d/w} & \text{for} \quad 0 < y < d \\ \displaystyle\int_y^1 \frac{dg(w)}{\log w/d} & \text{for} \quad d < y \leqq 1 \\ 0 & \text{for} \quad y > 1 \end{cases}$$

(note that $\rho \leqq d_k^{t_{k-1}}$).

It follows that

$$(25)\quad \begin{cases} \dfrac{Y_{k,k-1}(x)}{x^k} = \displaystyle\int_0^1 t^{x-1}dt \int \cdots \int_\Delta h_{d_k}\left(d_k\left(\frac{t}{\rho}\right)^{1/t_{k-1}}\right) dt_1 \cdots dt_{k-1} \\ \qquad = \displaystyle\int_0^1 t^x H(t)\frac{dt}{t}, \quad H(t) = \int \cdots \int_\Delta h_{d_k}\left(d_k\left(\frac{t}{\rho}\right)^{1/t_{k-1}}\right)dt_1 \cdots dt_{k-1} . \end{cases}$$

Obviously $H(t)/t \in L[\varepsilon, 1]$ for every $\varepsilon > 0$. Let $t < d_1 \cdots d_k$, then $t < \rho$ and

$$d_k\left(\frac{t}{\rho}\right)^{1/t_{k-1}} \leqq d_k \frac{t}{\rho} \leqq \frac{t}{d_1 \cdots d_{k-1}} ,$$

hence

$$| H(t) | \leqq \int \cdots \int_\Delta dt_1 \cdots dt_{k-1} \int_{+0}^{t/d_1\cdots d_{k-1}} \frac{| dg(w) |}{\log d_k/w} \leqq \int_{+0}^{t/d_1\cdots d_{k-1}} \frac{| dg(w) |}{\log d_k/w} .$$

It follows that $H(t) \to 0$ as $t \to 0$, and this proves the case $k = p - 1$ of Lemma 3.

We mention a special case of (24) which will be needed later on. Let $g(w) = 0$ for $w < e^\delta$, $g(w) = 1$ for $e^\delta \leqq w \leqq 1$ $(\delta \leqq 0)$. Then

$$Y_{k,k-1}(x) = [x_1, \cdots, x_k, \delta]_{e^{xv}} ,$$

and it follows from (24) that

$$(26)\qquad \frac{[x_1, \cdots, x_k, \delta]_{e^{vx}}}{x^k} = \int_0^1 t^x H(t) \frac{dt}{t}$$

where $H(t)/t \in L[\varepsilon, 1]$ for every $\varepsilon > 0$ and $H(t) \to 0$ as $t \to 0$ (even more is true, namely $H(t) = 0$ in a neighborhood of $t = 0$). We now turn to the case $p \leqq k - 2$, and we will use the relation

$$(27)\qquad \left\{\prod_{i=1}^{\rho} (D-\eta_i)\right\}a(x)b(x) = \sum_{\mu=0}^{\rho-1} (D^\mu a) \sum_{1\leqq i_1<\cdots<i_{\rho-\mu}\leqq\rho} (D-\eta_{i_1}) \cdots (D-\eta_{i_{\rho-\mu}})b + (D^\rho a)b \ \ (\rho \geqq 0)$$

which follows from a short induction-type proof. Let

$$y_p(x) = \sum_{\nu=p+2}^{k} C_\nu[x_1, \cdots, x_\nu]_{e^{xv}} + Y_{k,p} , \quad Y_{k,p} = \sum_{\nu=p+2}^{k} A_\nu + A ,$$

where A, A_ν denote the functions occurring in $Y_{k,p}$. It follows from (5) and (27) that

$$\left\{\prod_{i=p+2}^{\nu-1} (D-x_i)\right\} \frac{[x_1, \cdots, x_\nu]_{e^{xv}}}{x^{p+1}}$$

is a linear combination of terms of the type

$$\frac{[x_1, \cdots, x_{p+1}, x_{i_1}, \cdots, x_{i_{\mu+1}}]_{exv}}{x^{p+1+\mu}}, \quad \mu = 0, 1, \cdots, \nu - p - 2 .$$

Hence we obtain from (26) a representation

$$\left\{\prod_{i=p+2}^{\nu-1} (D-x_i)\right\}\frac{[x_1, \cdots, x_\nu]_{exv}}{x^{p+1}} = \int_0^1 t^{x-1}H_\nu(t)dt ,$$

$H_\nu(t) \to 0$ as $t \to 0$, and it follows that

$$\begin{aligned}\left\{\prod_{i=p+2}^{k} (D-x_i)\right\} \sum_{\nu=p+2}^{k} C_\nu \frac{[x_1, \cdots, x_\nu]_{exv}}{x^{p+1}} &= \sum_{\nu=p+2}^{k} C_\nu\left\{\prod_{i=\nu}^{k} (D-x_i)\right\}\int_0^1 t^{x-1}H_\nu(t)dt \\ &= \sum_{\nu=p+2}^{k} C_\nu\int_0^1 t^{x-1}\left(\prod_{i=\nu}^{k} (\log t-x_i)\right)H_\nu(t)dt \\ &= \int_0^1 t^{x-1}\hat{H}(t)dt\end{aligned}$$

where $\hat{H}(t) = o\,((\log 1/t)^{k-p-1})$ as $t \to 0$.

Similarly it follows from (5) and (24) that $\{\prod_{i=p+2}^{\nu} (D-x_i)\}A_\nu/x^{p+1}$ is a linear combination of terms of the type

$$\frac{a(x; x_1, \cdots, x_{p+1}, x_{i_1}, \cdots, x_{i_\mu}; dg_\nu(w))}{x^{p+\mu+1}}, \quad \mu = 0, \cdots, \nu - p - 1$$

$$dg_\nu(w) = \frac{w}{c_v}\prod_{j=\nu+1}^{k} \frac{1 - w/c_j}{\log w/c_j} dg(w) .$$

Hence, it follows like in (24) $(k=p+\mu+1)$ that

$$\left\{\prod_{i=p+2}^{\nu} (D-x_i)\right\}\frac{A_\nu}{x^{p+1}} = \int_0^1 t^{x-1}\tilde{H}_\nu(t)dt , \quad \tilde{H}_\nu(t) \to 0 \quad \text{as} \quad t \to 0 ,$$

$$\left\{\prod_{i=p+2}^{k} (D-x_i)\right\}\frac{A_\nu}{x^{p+1}} = \int_0^1 t^{x-1}\left(\prod_{i=\nu+1}^{k} (\log t-x_i)\right) \tilde{H}_\nu(t)dt$$

where $(\prod_{i=\nu+1}^{k}(\log t-x_i))\ \tilde{H}_\nu(t) = o\,((\log 1/t)^{k-\nu}) = o\,((\log 1/t)^{k-p-1})$ $\left(\text{note that } \int_{+0}^1 |dg_\nu(w)|/(1+\log 1/w) < \infty\right)$.

Finally, it follows like in (24) $(k=p+1)$ that

$$\frac{A(x)}{x^{p+1}} = \int_0^1 t^{x-1}\tilde{H}(t)dt , \quad \tilde{H}(t) \to 0 \quad \text{as} \quad t \to 0 ,$$

$$\left\{\prod_{i=p+2}^{k} (D-x_i)\right\}\frac{A(x)}{x^{p+1}} = \int_0^1 t^{x-1}\left(\prod_{i=p+2}^{k} (\log t-x_i)\right) \tilde{H}(t)dt ,$$

where

$$\Big(\prod_{i=p+2}^{k}(\log t-x_i)\Big)\tilde{H}(t) = o\Big((\log 1/t)^{k-p-1}\Big) \quad \text{as} \quad t\to 0\,.$$

$$\left(\text{note that} \int_{+0}^{1}\left|\prod_{j=p+2}^{k}\frac{1-w/c_j}{\log w/c_j}\,\frac{dg(w)}{1+\log 1/w}\right| < \infty\right).$$

This proves the case $p \leqq k-2$ of Lemma 3.

3. THEOREM 3. *Let $a \in C\,[0, \infty)$ be a real solution of the differential equation*

$$\Big\{\prod_{1}^{k}(D-x_i)\Big\}a = \varphi(x),\quad x>0,\quad x_i \leqq 0\,, \tag{28}$$

$$\varphi \text{ completely monotone for } x>0 \ (k \geqq 0)\,.$$

Then $f(z) = \sum_0^\infty a(n)z^n$ defines on C^* (uniquely) a holomorphic function which has at most k zeros unless $f \equiv 0$.

PROOF. It follows from (3) that

$$a(x) = a_0(x) + \int_{+0}^{1} wdg(w)\,[x_1, \cdots, x_k, \log w]_{e^{(x-1)v}},\quad \Big\{\prod_{1}^{k}(D-x_i)\Big\}a_0 = 0\,,$$

and Lemma 1 shows that

$$\int_{+0}^{1}\frac{dg(w)}{(1+\log 1/w)^k} < \infty\,. \tag{29}$$

A short calculation shows that $(\prod_1^k \Delta(e^{x_i}))\ a_0(n+k) = 0$, and writing $a_n = a(n)$ we find

$$b_n = \Big(\prod_{1}^{k}\Delta(e^{x_i})\Big)\,a_{n+k} = \int_{+0}^{1} wdg(w)\,\frac{1}{2\pi i}\int_{C_w}\frac{e^{(n-1)z}\prod_{1}^{k}(e^z-e^{x_i})\,dz}{(z-x_1)\cdots(z-x_k)\,(z-\log w)}$$
$$= \int_{+0}^{1} w^n\Big(\prod_{1}^{k}\frac{w-e^{x_i}}{\log w - x_i}\Big)dg(w)$$

(C_w denotes a simple closed curve containing $x_1, \cdots, x_k$ and $\log w$ in its interior). This shows that b_n is completely monotone (observe (29)), and Theorem 3 follows from Theorem 1.

REMARKS. (i) Let $\tau \geqq 0$, and let $a(x)$ satisfy the assumptions of Theorem 3. Then $\sum_0^\infty a(n+\tau)z^n$ is defined on C^* and has at most k zeros (unless it is $\equiv 0$). This follows immediately from Theorem 3 when $a(x)$ is replaced by $a^*(x) = a(x+\tau)$.

(ii) Replace in the assumptions of Theorem 3 $C[0, \infty)$ and $x>0$ by $C[\nu, \infty)$ and $x>\nu(\nu=1, 2, \cdots)$. Then it follows that $b_n = (\prod_1^k(e^{x_i}))a_{n+k+\nu}$ is completely monotone, and Theorem 1 shows that $P_{\nu-1}(z) + \sum_{n=\nu}^\infty a(n)z^n$,

$P_{\nu-1}(z)$ any real polynomial of degree $\leqq \nu - 1$ is (uniquely) defined on C^* and has at most $k + \nu$ zeros (unless it is $\equiv 0$).

THEOREM 4. *Let $a \in C_p[0, \infty)$ for some $p = 0, 1, \cdots, k-1$ ($k \geqq 1$) be a real solution of the differential equation* (28). *Moreover, let*

$$a(0) = a'(0) = \cdots = a^{(p)}(0) = 0 .^{2}$$

Then $f(z) = \sum_0^\infty a(n+\tau)z^n$, $\tau \in [0, 1)$, defines on C^*(uniquely) a holomorphic function which has at most k zeros (unless $f \equiv 0$) and at least $p + 1$ different zeros which are $\leqq 0$.

PROOF. On account of Theorem 3 it remains to prove the lower estimate for the number of zeros.

It follows from Lemma 1 that (19) holds, and Lemma 3 shows that

$$\left\{\prod_{i=p+2}^{k}(D-x_i)\right\}\frac{a(x)}{x^{p+1}} = \int_0^1 t^{x-1}H(t)dt ,$$

where H satisfies (23). Theorem 2 and Remark (i) after Theorem 2

$$(c(x) = a(x)/x^{p+1},\ h'(t) = H(t)/t,\ m = p+1,\ r = k-p-1,\ \nu = \mu = 0)$$

now show that f has at least $p + 1$ different zeros which are $\leqq 0$.

REMARK. The example $\sum_0^\infty (n+1)^2 z^n = (1+z)/(1-z)^3$ shows that Theorem 4 cannot be extended to $\tau = 1$ ($k=2$, $p=1$).

4. Applications of Theorem 4.

(i) Let $a(x) = x^\kappa$, $x \geqq 0$, $k-1 < \kappa \leqq k$, $k = 1, 2, \cdots$. Here $a \in C_{k-1}[0, \infty)$, and $D^k a = Cx^{\kappa-k}$ is completely monotone for $x > 0$; moreover,

$$a(0) = a'(0) = \cdots = a^{(k-1)}(0) = 0 .$$

It follows from Theorem 4 that $f_\kappa(z) = \sum_0^\infty (n+\tau)^\kappa z^n$, $\tau \in [0, 1)$, has exactly k zeros in C^*, and these are simple and $\leqq 0$.

(ii) Let $a(x) = (1-c^x)^\kappa$, $x \geqq 0$, $0 < c < 1$, $k-1 < \kappa \leqq k$. The relation $(D-\kappa \log c)\, a(x) = \kappa \log 1/c\,(1-c^x)^{\kappa-1}$ shows that $a(x)$ satisfies a differential equation (28) ($\varphi(x) = C(1-c^x)^{\kappa-k}$), and we have $a \in C_{k-1}[0, \infty)$ and $a(0) = a'(0) = \cdots = a^{(k-1)}(0) = 0$. *It follows from Theorem 4 that $g_\kappa(z) = \sum_0^\infty (1-c^{n+\tau})^\kappa z^n$, $\tau \in [0, 1)$, has exactly k zeros in C^*, and these zeros are different and $\leqq 0$.*

(iii) Let $a(x)$ be the incomplete Γ-function

$$\int_0^x t^{\kappa-1}e^{-t}dt,\ x \geqq 0,\ k-1 < \kappa \leqq k .$$

[2] This is equivalent to $a(0) = 0$, $\{\prod_1^\nu (D - x_i)\}a|_{x=0} = 0$, $\nu = 1, \cdots, p$.

Then $Da = x^{\kappa-1}e^{-x}$, $(D+1)\,Da = (\kappa-1)x^{\kappa-2}e^{-x}$. *It follows from Theorem* 4 *that* $\sum_0^\infty a(n+\tau)z^n$, $\tau \in [0, 1)$, *has exactly* k *zeros in* C^*, *and these are simple and* $\leqq 0$.

(iv) Let $a(x) = x^\kappa \log x$, $x \geqq 0$, $k-1 < \kappa \leqq k$. We have

$$D^q a = q\,!\binom{\kappa}{q} x^{\kappa-q}(\log x + A_q)\ , \qquad A_q \text{ constant}\ ,$$

for $\kappa \neq 1, 2, \cdots$ or $q \leqq k$. This shows that $a \in C_{k-1}[0, \infty)$ and

$$a(0) = a'(0) = \cdots = a^{(k-1)}(0) = 0\ .$$

Let $\kappa = k$. Then $D^{k+1}a = k!/x$ is completely monotone.

Let $\kappa \neq k$. Then it follows from

$$\frac{1}{x^\alpha} = \frac{1}{\Gamma(\alpha)}\int_0^1 t^{x-1}\left(\log\frac{1}{t}\right)^{\alpha-1} dt$$

$$\frac{\log x}{x^\alpha} = \frac{1}{\Gamma(\alpha)}\int_0^1 t^{x-1}\left(\log\frac{1}{t}\right)^{\alpha-1}\left\{\frac{\Gamma'(\alpha)}{\Gamma(\alpha)} - \log\log\frac{1}{t}\right\}dt \quad (\alpha > 0,\ x > 0)$$

(differentiate with respect to α to obtain the second formula from the first) that

$$(D-\xi)D^k a$$
$$= k\,!\binom{\kappa}{k}\frac{1}{\Gamma(k-\kappa)}\int_0^1 t^{x-1}\left(\log\frac{1}{t}\right)^{k-\kappa-1}(\log t - \xi)\left\{\frac{\Gamma'(k-\kappa)}{\Gamma(k-\kappa)} - \log\ \log\frac{1}{t} + A_k\right\}dt$$

is completely monotone for a suitable $\xi < 0$. It follows from Theorem 4 that $F_\kappa(z) = \sum_0^\infty (n+\tau)^\kappa \log(n+\tau)\, z^n$ has at most $k+1$ zeros and at least k zeros which are different and $\leqq 0$.

Let $\tau = 0$. *Then* F_κ *has a zero of order* 2 *at* $z = 0$, *so that* F_κ *actually has* $k+1$ *zeros* (*and all zeros are* $\leqq 0$). *Let* $\tau \in (0, 1)$. *Then* $F_\kappa(0) < 0$, $F_\kappa(x) \to \infty$ *as* $x \to 1$, *and it follows that* F_κ *has at least one zero which is* > 0, *hence* F_κ *has again exactly* $k+1$ *zeros. These are simple, and* k *zeros are* < 0, *one zero is* > 0. (A lower estimate for the number of zeros *of* F_κ follows from Wirsing [7].)

Subbotin [5] has shown that $\sum_0^\infty z^n \int_0^1 f(t)(n+t)^{2k}dt$ ($f \geqq 0$, $k=0,1,\cdots$) has exactly $2k$ zeros (unless $\equiv 0$) in C^*, and these are simple and negative. This result is not a consequence of Theorem 4; however, using our results on the zeros of $\sum (n+\tau)^\kappa z^n$ and the fact that these zeros are monotone functions of τ (which was observed by Subbotin for $\kappa = 2k$) we are in a position to discuss the zeros of the more general functions

$$\sum z^n \int_0^1 f(t)(n+t)^\kappa dt\ .$$

REFERENCES

[1] A. EDREI, Proof of a conjecture of Schoenberg on the generating function of a totally positive sequence. Canad. J. Math. 5 (1953), 86–94.
[2] G. G. LORENTZ, Bernstein Polynomials. Univ. of Toronto Press, Toronto 1953.
[3] A. PEYERIMHOFF, On the zeros of power series. Mich. Math. J. 13 (1966), 193-214.
[4] G. POLYA UND G. SZEGÖ, Aufgaben und Lehrsätze aus der Analysis. Springer, Berlin, 1954.
[5] JU. N. SUBBOTIN, Funktionelle Interpolation im Mittel mit kleinster n-ter Ableitung (Russian). Trudy mat. Inst. Steklov. 88 (1967), 30-60.
[6] D. V. WIDDER, The Laplace Transform. Princeton 1946.
[7] E. WIRSING, On the monotonicity of the zeros of two power series. Mich. Math. J. 13 (1966), 215-218.

W.B. JURKAT, SYRACUSE UNIVERSITY,
DEPT. OF MATHEMATICS, SYRACUSE, N.Y. U.S.A.
A. PEYERIMHOFF, UNIVERSITÄT ULM DONAU ABT.
FÜR MATHEMATIK, GERMANY