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ON FOURIER MULTIPLIERS OF LIPSCHITZ CLASSES

Dedicated to Professor Gen-ichirô Sunouchi on his 60th birthday

TAKAHIRO MIZUHARA

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1. Introduction. Let E and F be any two classes of periodic functions of period 2π. We say that a two-way infinite sequence $\{\lambda(n)\}$ of numbers is a (Fourier) multiplier of type (E, F) if whenever $f(x)$ is in E, $\sum_n \lambda(n)\hat{f}(n)e^{inx}$ is the Fourier series of some function in F. We denote by (E, F) the class of all multipliers of type (E, F). The classes of functions treated here are Lipschitz class Λ_α, generalized Lipschitz class Λ_α^p, Zygmund class Λ_*, and generalized Zygmund class Λ_*^p, where $0 < \alpha \leqq 1$ and $1 \leqq p < \infty$. A. Zygmund ([4] p. 890, Theorem I. p. 894, Theorem III.) has shown that a necessary and sufficient condition for $\{\lambda(n)\}$ to belong to one of any types $(\Lambda_\alpha, \Lambda_\alpha)$, (Λ_*, Λ_*), $(\Lambda_\alpha^1, \Lambda_\alpha^1)$, $(\Lambda_*^1, \Lambda_*^1)$ is that $\sum_{n\neq 0} \{\lambda(n)/(in)\}e^{inx}$ is the Fourier series of a function in Λ_*^1.

In this paper, in general, we consider the types $(\Lambda_\alpha^p, \Lambda_\beta^p)$, $(\Lambda_\alpha^p, \Lambda_*^q)$ and $(\Lambda_*^p, \Lambda_\beta^q)$ where $0 < \alpha, \beta \leqq 1$ and $1 \leqq p, q < \infty$.

2. Notations and Preliminaries. We assume here that $0 < \alpha \leqq 1$, $0 < \beta < 2$ and $1 \leqq p < \infty$.

Let Λ_α denote the class of all continuous and periodic functions $f(x)$ satisfying a Lipschitz condition,

$$\Delta_h f(x) = f(x + h) - f(x) = O(h^\alpha) \text{ for } h > 0 \text{ and uniformly in } x .$$

Λ_α^p is the class of all L^p-functions $f(x)$ satisfying a condition,

$$\|\Delta_h f(x)\|_p = \left\{\frac{1}{2\pi}\int_0^{2\pi} |f(x + h) - f(x)|^p dx\right\}^{1/p} = O(h^\alpha) \text{ for } h > 0 .$$

A continuous function $F(x)$ is said to belong to class $\Lambda_{*,\beta}$ if

$$\delta_h^2 F(x) = F(x + h) + F(x - h) - 2F(x) = O(h^\beta) \text{ for } h > 0 \text{ and uniformly}$$

in x. In particular, the class $\Lambda_{*,1}$ is the well-known Zygmund class denoted by Λ_*. Also we say that an F in L^p is in the class Λ_*^p, if

$$\|\delta_h^2 F(x)\|_p = \left\{\frac{1}{2\pi}\int_0^{2\pi} |F(x + h) + F(x - h) - 2F(x)|^p dx\right\}^{1/p} = O(h^\beta)$$

for $h > 0$. In particular, we denote the class $\Lambda_{*,1}^p$ by Λ_*^p.

In order to prove our theorems we need several lemmas. Let $0 < \alpha < 1$ and k be any nonnegative integer from now on.

LEMMA 1. (A. Zygmund [5] p. 63, p. 64) *A necessary and sufficient condition that a continuous and periodic $f(x)$ should have a k-th derivative $f^{(k)}(x)$ in Λ_α is that $\Delta_h^{k+1}f(x) = O(h^{k+\alpha})$ for $h > 0$ and uniformly in x, where $\Delta_h^s f(x) = \Delta_h \Delta_h^{s-1} f(x)$, $\Delta_h^1 f(x) = \Delta_h f(x) = f(x+h) - f(x)$.*

LEMMA 2. *A necessary and sufficient condition that an L^p-function $f(x)$ should have a k-th derivative $f^{(k)}(x)$ almost everywhere in Λ_α^p is that*

$$\| \Delta_h^{k+1} f(x) \|_p = \left\{ \frac{1}{2\pi} \int_0^{2\pi} | \Delta_h^{k+1} f(x) |^p dx \right\}^{1/p} = O(h^{k+\alpha}) \qquad \text{for } h > 0 .$$

In the preceeding two lemmas, k on the left side may be replaced by any $l \geqq k$. These results are simple applications of the theory of best approximation. For the proof of Lemma 1, see G. G. Lorentz ([3] p. 56, Theorem 2, p. 59, Theorem 6, 7, p. 62, Theorem 9). Lemma 2 is its analogue in the space L^p. For an integrable and periodic function $f(x)$, $f_\alpha(x)$, $f^\alpha(x)$ will denote the α-th fractional integration and differentiation in the sense of A. Zygmund [6].

LEMMA 3. (i) (G. H. Hardy and J. E. Littlewood. A. Zygmund [5] p. 53, Theorem 11.) *Let $0 < \alpha < 1$, $0 < \beta$ and $f(x)$ be in Λ_α. Then f_β is in $\Lambda_{\alpha+\beta}$ if $\alpha + \beta < 1$, and f_β is in Λ_* if $\alpha + \beta = 1$, and f_β is in $\Lambda_{*,\alpha+\beta}$, or, equivalently, $d/dx\, f_\beta(x)$ is in $\Lambda_{\alpha+\beta-1}$ if $1 < \alpha + \beta < 2$.*

(ii) (G. H. Hardy and J. E. Littlewood) *Let $0 < \beta < \alpha < 1$ and $f(x)$ be in Λ_α. Then f^β is in $\Lambda_{\alpha-\beta}$.*

(iii) (A. Zygmund [5] p. 53, Theorem 12.) *Let $0 < \alpha < 1$. If $f(x)$ is in Λ_*, then f^α is in $\Lambda_{1-\alpha}$ and f_α has a continuous derivative $d/dx\, f_\alpha(x)$ in Λ_α.*

LEMMA 4. (i) (G. H. Hardy and J. E. Littlewood. A. Zygmund [5] p. 69, Remark (b).) *Let $0 < \alpha < 1$, $0 < \beta$ and $f(x)$ be in Λ_α^p. Then f_β is in $\Lambda_{\alpha+\beta}^p$ if $\alpha + \beta < 1$, and f_β is in Λ_*^p if $\alpha + \beta = 1$, and f_β is in $\Lambda_{*,\alpha+\beta}^p$, or, equivalently, $d/dx\, f_\beta(x)$ exists almost everywhere and belongs to $\Lambda_{\alpha+\beta-1}^p$ if $1 < \alpha + \beta < 2$.*

(ii) (G. H. Hardy and J. E. Littlewood) *Let $0 < \beta < \alpha < 1$ and $f(x)$ be in Λ_α^p. Then $f_{1-\beta}$ is continuous and $f^\beta(x)$ exists almost everywhere and belongs to $\Lambda_{\alpha-\beta}^p$.*

(iii) (A. Zygmund [5] p. 69, Remarks (b).) *Let $0 < \alpha < 1$. If $f(x)$ is in Λ_*^p, then $f_{1-\alpha}(x)$ is continuous and $f^\alpha(x)$ exists almost everywhere and belongs to $\Lambda_{1-\alpha}^p$.*

LEMMA 5. (G. H. Hardy and J. E. Littlewood) *A necessary and suf-*

ficient condition for a function $f(x)$ to be in Λ_1^p, $1 < p < \infty$, is that $f(x)$ should be equivalent to the indefinite integral of a function of L^p. Similarly, a necessary and sufficient condition for a function $f(x)$ to be in Λ_1 is that $f(x)$ should be equivalent to the indefinite integral of a function of L^∞.

3. Type $(\Lambda_\alpha^p, \Lambda_\alpha^p)$ where $0 < \alpha < 1$ and $1 < p < \infty$. Our first aim is to prove the following.

THEOREM I. *Let $0 < \alpha < 1, 1 < p < \infty$. Then a necessary and sufficient condition for $\{\lambda(n)\}$ to be of type $(\Lambda_\alpha^p, \Lambda_\alpha^p)$ is that*

$$\{(2i \sin nh/2)^2/(inh)\}\lambda(n),\ n \neq 0,\ \textit{is of type } (L^p, L^p) \textit{ for } h > 0\ ,$$

uniformly.

PROOF. Necessity. In view of (i) and (iii) of Lemma 4, it is observed that, for periodic functions with mean value 0, the class Λ_*^p is identical with the class of $(1-\alpha)$-th fractional integrals of functions in Λ_α^p, so that the types $(\Lambda_\alpha^p, \Lambda_\alpha^p)$ and $(\Lambda_*^p, \Lambda_*^p)$ are necessarily the same. We assume that $\{\lambda(n)\}$ is of type $(\Lambda_\alpha^p, \Lambda_\alpha^p)$, then $\{\lambda(n)\}$ is *a fortiori* of type $(\Lambda_*^p, \Lambda_*^p)$ so that it is of type $(\Lambda_1^p, \Lambda_*^p)$ since Λ_1^p is contained in Λ_*^p.

Now let $G(x) = \sum_{n\neq 0} \{\lambda(n)/(in)\}e^{inx}$. By assumption and Lemma 5, whenever $f(x)$ is in L^p, the convolution $f * G(x)$ is in Λ_*^p.
Since $\delta_h^2(f*G)(x) = \{(\delta_h^2 G)*f\}(x)$,

$$\| \delta_h^2(f*G)(x)\|_p = \|\{(\delta_h^2 G)*f\}(x)\|_p = O(h) \text{ for } h > 0$$

and for any $f(x)$ in L^p. This means that, for any $f(x)$ in L^p,

$$\|\{(\delta_h^2 G)/h * f\}(x)\|_p = O(1) \text{ for } h > 0\ .$$

That is, $\{(2i \sin nh/2)^2/(inh)\}\lambda(n)$, $n \neq 0$, is of type (L^p, L^p) for $h > 0$, uniformly, and the necessity of the condition is established.

Sufficiency. Conversely, we assume that $\{(2i \sin nh/2)^2/(inh)\}\lambda(n)$ is of type (L^p, L^p) for $h > 0$, uniformly. That is, whenever $f(x) \sim \sum_n \hat{f}(n)e^{inx}$ is in L^p, $\sum_{n\neq 0}\{(2i\sin nh/2)^2/(inh)\}\lambda(n)\hat{f}(n)e^{inx}$ is in L^p for $h > 0$, uniformly, or, equivalently, whenever $f(x)$ is in L^p, the convolution $\{(\delta_h^2 G)/h * f\}(x)$ is in L^p for $h > 0$, uniformly, where $G(x) = \sum_{n\neq 0}\{\lambda(n)/(in)\}e^{inx}$. Let $g(x) \sim \sum_n \hat{g}(n)e^{inx}$ be in Λ_α^p. Then

$$\|\delta_h g(x)\|_p = \|g(x+h/2) - g(x-h/2)\|_p = \|\Delta_h g(x)\|_p = O(h^\alpha)$$

for $h > 0$, that is, $\|\{\delta_h g(x)\}/h^\alpha\|_p = O(1)$ for $h > 0$, that is, $(\delta_h g)/h^\alpha$ is in L^p for $h > 0$, uniformly.

Let $H(x) = G * g(x)$ for any $g(x)$ in Λ_α^p. Then

$$\|\{\Delta_h^3 H(x)\}/h^{1+\alpha}\|_p = \|\{\delta_h^3 H(x)\}/h^{1+\alpha}\|_p = \|(\delta_h^2 G)/h * (\delta_h g)/h^\alpha\|_p = O(1)$$

for $h > 0$, that is, $\|\Delta_h^3 H(x)\|_p = O(h^{1+\alpha})$ for $h > 0$.

This implies, by Lemma 2, that $d/dxH(x)$ exists almost everywhere and belongs to Λ_α^p and the sufficiency of the condition is established.

Q.E.D.

By the similar method we can prove the followings.

THEOREM I′. *Let* $0 < \alpha < 1$ *and* $1 < p, q < \infty$. *Then* $\{\lambda(n)\}$ *is of type* $(\Lambda_\alpha^p, \Lambda_\alpha^q)$ *if and only if* $\{(2i \sin nh/2)^2/(inh)\}\lambda(n)$, $n \neq 0$, *is of type* (L^p, L^q) *for* $h > 0$, *uniformly.*

THEOREM I″. *Let* $0 < \alpha < 1, 1 < p < \infty$ *and* $1/p + 1/p' = 1$. *Then* $\{\lambda(n)\}$ *is of any of the types* $(\Lambda_\alpha^p, \Lambda_\alpha)$, $(\Lambda_\alpha^1, \Lambda_\alpha^{p'})$, (Λ_*^p, Λ_*), $(\Lambda_*^1, \Lambda_*^{p'})$ *if and only if* $\sum_{n\neq 0} \{\lambda(n)/(in)\}e^{inx}$ *is the Fourier series of a function in* $\Lambda_*^{p'}$.

4. Type $(\Lambda_\alpha, \Lambda_\alpha^p)$ where $0 < \alpha < 1$ and $1 < p < \infty$. S. Kaczmarz ([2] p. 40. Theorem 1, p. 42, Theorem 2, Theorem 3 and p. 45, Theorem 7.) gave a necessary and sufficient condition in order that a bounded sequence $\{\lambda(n)\}$ should belong to some multiplier classes by using class V_p, $1 \leqq p < \infty$. A periodic L^p-function $f(x)$ is of class $V_p(1 \leqq p < \infty)$ if there is a constant K such that, if $(a_1, b_1), (a_2, b_2), \cdots, (a_n, b_n)$ is any set of non-overlapping intervals in $(0, 2\pi)$, then

$$\left\{\frac{1}{2\pi}\int_0^{2\pi}\left|\sum_{i=1}^{n}[f(x-b_i)-f(x-a_i)]\right|^p dx\right\}^{1/p} < K .$$

Now we introduce a new class of functions. $V_p^*(1 \leqq p < \infty)$ will denote the class of V_p-functions $f(x)$ satisfying a condition, there is a constant K such that $\|\sum_{i=1}^n [\delta_h^2 f(x-b_i) - \delta_h^2 f(x-a_i)]\|_p \leqq Kh$ for $h > 0$ and for any set of non-overlapping intervals $\{(a_i, b_i)\}_{i=1}^n$ in $(0, 2\pi)$.

Our second aim is to prove the following.

THEOREM II. *Let* $0 < \alpha < 1$ *and* $1 < p < \infty$. *Then a necessary and sufficient condition for* $\{\lambda(n)\}$ *to be of any of the types* $(\Lambda_\alpha, \Lambda_\alpha^p)$, $(\Lambda_\alpha^{p'}, \Lambda_\alpha^1)$, (Λ_*, Λ_*^p), $(\Lambda_*^{p'}, \Lambda_*^1)$ *is that* $G^*(x)$ *is in* V_p^*, *where the function* $G^*(x)$ *is defined by* $G^*(x) = \sum_{n\neq 0}\{\lambda(n)/(in)^2\}e^{inx}$, *and the series converging absolutely.*

PROOF. It is observed that the types $(\Lambda_\alpha, \Lambda_\alpha^p)$, $(\Lambda_\alpha^{p'}, \Lambda_\alpha^1)$, (Λ_*, Λ_*^p) and $(\Lambda_*^{p'}, \Lambda_*^1)$ are the same. See A. Zygmund ([4] p. 894, Theorem IV). We confine our attention to the type $(\Lambda_\alpha, \Lambda_\alpha^p)$. By the same method of Theorem I′, it is seen that, if $0 < \alpha < 1$ and $1 < p < \infty$, a necessary and sufficient condition for $\{\lambda(n)\}$ to be of type $(\Lambda_\alpha, \Lambda_\alpha^p)$ is that $\{(2i \sin nh/2)^2/(inh)\}\lambda(n)$, $n \neq 0$, is of type (L^∞, L^p) for $h > 0$, uniformly.

Necessity. We assume that $\{\lambda(n)\}$ is of type $(\Lambda_\alpha, \Lambda_\alpha^p)$, then $\{(2i \sin nh/2)^2/(inh)\}\lambda(n)$, $n \neq 0$, is of type (L^∞, L^p) for $h > 0$, uniformly, or, equivalently, whenever $f(x)$ is in L^∞, the convolution $\{(\delta_h^2 G)/h * f\}(x)$ is in L^p for $h > 0$, uniformly, where $G(x) = \sum_{n\neq 0}\{\lambda(n)/(in)\}e^{inx}$. It is known that $\{\phi(n)\}$ is of type (L^∞, L^p) if and only if $\sum_{n\neq 0}\{\phi(n)/(in)\}e^{inx}$ is the Fourier series of a function in V_p (S. Kaczmarz [2] p. 45, Theorem 7). Hence there is a constant K such that, if $\{(a_i, b_i)\}_{i=1}^n$ is any set of non-overlapping intervals in $(0, 2\pi)$, $\|\sum_{i=1}^n [\delta_h^2 G^*(x - b_i) - \delta_h^2 G^*(x - a_i)]\|_p \leqq Kh$ for $h > 0$, where $G^*(x)$ is the function defined by $G^*(x) = \sum_{n\neq 0}\{\lambda(n)/(in)^2\}e^{inx}$. This implies that $G^*(x)$ is in V_p^*, and the necessity of the condition is established.

Sufficiency. We assume that $G^*(x)$ is in V_p^*, where $G^*(x)$ is the function defined above. From the definition of the class V_p^*, $(\delta_h^2 G^*)/h$ is in V_p for $h > 0$, uniformly. Therefore $\{(2i \sin nh/2)^2/(inh)\}\lambda(n)$, $n \neq 0$, is of type (L^∞, L^p) for $h > 0$, uniformly. This implies that $\{\lambda(n)\}$ is of type $(\Lambda_\alpha, \Lambda_\alpha^p)$, and the sufficiency of the condition is established. Q.E.D.

5. Type $(\Lambda_\alpha, \Lambda_\beta)$ where $0 < \alpha < \beta < 1$. When $0 < \alpha < \beta < 1$ the following result is known. Here we shall give a proof.

THEOREM A. *Let $0 < \alpha < \beta < 1$. Then a necessary and sufficient condition for $\{\lambda(n)\}$ to be of type $(\Lambda_\alpha, \Lambda_\beta)$ is that $\sum_n \lambda(n)e^{inx}$ is the Fourier series of a function in $\Lambda_{\beta-\alpha}^1$.*

PROOF. Sufficiency. Write $\lambda(x) \sim \sum_n \lambda(n)e^{inx}$ for $\lambda(n)$ in $(\Lambda_\alpha, \Lambda_\beta)$ and $H(x) \sim \sum_n \lambda(n)\hat{f}(n)e^{inx}$ for $f(x)$ in Λ_α. Then $H(x) = 1/2\pi\int_0^{2\pi} f(t)\lambda(x - t)dt$, so that $\delta_h^2 H(x) = 1/2\pi\int_0^{2\pi} \delta_h f(x)\delta_h\lambda(x - t)dt$ for a fixed $h > 0$. Then, by assumption, $|\delta_h^2 H(x)| = O(h^\alpha)\int_0^{2\pi} |\delta_h\lambda(x - t)|dt = O(h^\alpha)\int_0^{2}|\delta_h\lambda(t)|dt = O(h^\beta)$ for $h > 0$ and uniformly in x. This implies, by Lemma 1, that

$$H(x) \sim \sum_n \lambda(n)\hat{f}(n)e^{inx}$$

is Λ_β and the sufficiency of the condition is established.

Necessity. Let $G(x) = \sum_{n\neq 0}\{\lambda(n)/(in)\}e^{inx}$. We assume that whenever $f(x) \sim \sum_n \hat{f}(n)e^{inx}$ is in Λ_α, $\sum_n \lambda(n)\hat{f}(n)e^{inx}$ is in Λ_β, or, equivalently, the convolution $f * G(x)$ has a derivative in Λ_β.

If $\phi(x)$ is bounded and has mean value 0, (iii) of Lemma 3 implies that $\phi_\alpha(x)$ is in Λ_α. Therefore $\phi_\alpha * G(x) = \phi * G_\alpha(x)$ has a derivative in Λ_β, so that, by (i) of Lemma 3, $\phi * (G_\alpha)_{1-\alpha}(x) = \phi * G_1(x)$ has a derivative in $\Lambda_{*,1+(\beta-\alpha)}$, that is, $\phi * G(x)$ is in $\Lambda_{*,1+(\beta-\alpha)}$.

Since the class $\Lambda_{*,1+(\beta-\alpha)}$ is a Banach space when equipped with the norm

$$\|G\|_{\Lambda_{*,1+(\beta-\alpha)}} = \sup_x .|G(x)| + \sup_{\substack{x\\h>0}} .|\delta_h^2 G(x)/h^{1+(\beta-\alpha)}| ,$$

we associate with a continuous linear operator T from the Banach space L^∞ to the Banach space $\Lambda_{*,1+(\beta-\alpha)}$, by putting $Tg = G*g$ for g in L^∞. The continuity of T is an immediate consequence of the closed graph theorem. Therefore $\{\lambda(n)/(in)\}$ is of type $(L^\infty, \Lambda_{*,1+(\beta-\alpha)})$. That is, $\|Tg\|_{\Lambda_{*},1+(\beta-\alpha)} \leqq \|T\|\cdot\|g\|_\infty$. The definition of $\|Tg\|_{\Lambda_{*},1+(\beta-\alpha)}$ in $\Lambda_{*,1+(\beta-\alpha)}$ implies that

$$\left|\frac{1}{2\pi}\int_0^{2\pi}[G(x-y+h)+G(x-y-h)-2G(x-y)]g(y)dy\right| \leqq \|T\|\cdot\|g\|_\infty\cdot h^{1+(\beta-\alpha)} .$$

Then $(1/2\pi)\int_0^{2\pi}|G(x-y+h)+G(x-y-h)-2G(x-y)|dy \leqq \|T\|\cdot h^{1+(\beta-\alpha)}$. See J. Caveny ([1] p. 347, the proof of Theorem 1.).

That is, $(1/2\pi)\int_0^{2\pi}|G(x+h)+G(x-h)-2G(x)|dy \leqq \|T\|\cdot h^{1+(\beta-\alpha)}$. This implies that $G(x) = \sum_{n\neq 0}\{\lambda(n)/(in)\}e^{inx}$ is in $\Lambda^1_{*,1+(\beta-\alpha)}$, that is, $\sum_n \lambda(n)e^{inx}$ is in $\Lambda^1_{\beta-\alpha}$ and the necessity of the condition is established. Q.E.D.

In the case $1 \leqq \beta < 2$, we have the following analogue.

THEOREM A′. *Let* $0<\alpha<1\leqq\beta<2$. *Then* $\{\lambda(n)\}$ *is of type* $(\Lambda_\alpha, \Lambda_{*,\beta})$ *if and only if* $\sum_n \lambda(n)e^{inx}$ *is the Fourier series of a function in* $\Lambda^1_{\beta-\alpha}$.

When $0<\beta<\alpha<1$, as the corollary of Theorem A′, we obtain the following result.

THEOREM A″. *Let* $0<\beta<\alpha<1$. *Then* $\{\lambda(n)\}$ *is of type* $(\Lambda_\alpha, \Lambda_\beta)$ *if and only if* $\sum_{n\neq 0}\{\lambda(n)/(in)\}e^{inx}$ *is the Fourier series of a function in* $\Lambda^1_{1-(\alpha-\beta)}$.

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MATHEMATICAL INSTITUTE
AKITA UNIVERSITY
AKITA, JAPAN