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ABSOLUTE NÖRLUND SUMMABILITY OF FOURIER SERIES

Dedicated to Professor Gen-ichirô Sunouchi on his 60th birthday

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1. Introduction. Let $\{p_n\}_0^\infty$ be a sequence of non-negative constants, $p_0 > 0$ and $P_n = \sum_0^n p_k$. A sequence $\{U_n\}_0^\infty$ will be said to be absolutely summable by the Nörlund method defined by the sequence $\{p_n\}$, or summable $|N, p_n|$, if $t_n = \sum_{\nu=0}^n (p_{n-\nu} U_\nu)/P_n$ and

$$\sum_{n-1}^{\infty} |t_n - t_{n-1}| \leqq c < \infty \, . \tag{1.1}$$

Varshney [10] showed that if $f(x)$ is a real-valued, 2π-periodic function and of bounded variation over $[0, 2\pi]$ and if

$$|f(x+h) - f(x)| \leqq A \log^{-1-\varepsilon}\left(\frac{1}{h}\right)(\varepsilon > 0,\ 0 \leqq x \leqq 2\pi,\ h > 0) \tag{1.2}$$

then $S(f)$, the Fourier series of f, is summable $|N, 1/(n+1)|$. The author [8] later proved this result under the following weaker hypothesis:

$$\sum_1^\infty \frac{1}{n}\omega\left(\frac{1}{n}\right) < \infty \, , \tag{1.3}$$

where $\omega(t, f) = \omega(t)$ denotes, as usual, the modulus of continuity of f. Recently Izumi and Izumi [3], Lal [5] and others have studied the conditions for $|N, p_n|$ summability of $S(f)$ for general $\{p_n\}$. Lal has shown that, if (i) $p_0 > 0$, (ii) $\{p_n\}$ is non-negative and non-increasing, (iii) $\lim_{n\to\infty} p_n = 0$, (iv) $\{p_n - p_{n+1}\}$ is non-increasing, and if

$$\sum_1^\infty p_n^r n^{r-2} < \infty \qquad (1 < r \leqq 2) \, , \tag{1.4$'$}$$

and

$$\sum_1^\infty \omega(n^{-1}) P_n^{-1} n^{-1/s} < \infty, \left(\frac{1}{r} + \frac{1}{s} = 1\right) , \tag{1.4}$$

then $S(f)$ is summable $|N, p_n|$. In this paper we obtain conditions for $|N, p_n|$ summability of $S(f)$ when the series in (1.4) may fail to con-

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verge. Thus our results supplement those of Lal.

In what follows we will suppose that γ is a fixed constant, $0 \leqq \gamma < 1/2$, c_1 and c_2 are fixed positive constants and $\psi(x)$ is positive on $[0, \infty)$ and slowly oscillating in the sense of Karamata (see [2], [4]). Let $\{p_n\}$ satisfy conditions (i)—(iv) and suppose that for $n \geqq 1$,

$$c_1 n^\gamma \psi(n) \leqq P_n \leqq c_2 n^\gamma \psi(n) , \tag{v}$$

These conditions are all satisfied if, for instance, we take $p_n = (n+1)^{-1+\gamma}$, $0 \leqq \gamma < 1/2$. Some further examples are given in Section 4. We prove the following

THEOREM 1. *Let $f(x)$ be a 2π-periodic function of bounded variation over $[0, 2\pi]$ and suppose that the modulus of continuity $\omega(t, f)$ satisfies* (1.3) *and*

$$\sum_{n=1}^{\infty} \frac{1}{nP_n} \omega^{1/2}\left(\frac{1}{n}\right) < \infty . \tag{1.5}$$

Then under the assumptions (i)—(v), *$S(f)$ is summable $|N, p_n|$.*

2. **Lemmas.** We shall denote by A a positive constant (possibly depending on γ, c_1, c_2) not necessarily the same at each occurence.

LEMMA 1 [6]. *If $\{p_n\}$ is non-negative and non-increasing, then for $0 \leqq a \leqq b \leqq \infty$, $0 \leqq t \leqq \pi$ and any n, we have*

$$\left| \sum_{k=a}^{b} p_k e^{i(n-k)t} \right| \leqq \begin{cases} P(t^{-1}) & \text{for any } a , \\ At^{-1} p_{[a]} & \text{for } a \geqq [t^{-1}] . \end{cases} \tag{2.1}$$

Here $[x]$ denotes the integer part of x, and $P(x) = P_{[x]}$.

LEMMA 2 [6]. *If $\{p_n\}$ is non-negative and non-increasing and $\{p_n - p_{n+1}\}$ is non-increasing, then*

$$\frac{n^2(p_n - p_{n+1})}{P(n-1)} \leqq \frac{n^2(p_{n-1} - p_n)}{P(n-1)} \leqq A . \tag{2.2}$$

LEMMA 3. *If $P(x)$ satisfies* (v) *then*

$$\frac{n}{P(n-1)} \int_n^\infty \frac{P(u)du}{u^2} < A . \tag{2.3}$$

This follows from the properties of slowly oscillating functions [2]. We have $\int_n^\infty u^{\gamma-2}\psi(u)du \sim \psi(n)(n^{\gamma-1}/1-\gamma)$, and $\psi(n) \sim \psi(n-1)$ and (2.3) follows.

LEMMA 4. *Let*

$$I_n = \int_{1/\pi}^{(2^n+1)/\pi} \omega^2\Big(\frac{1}{t}\Big)dt . \tag{2.4}$$

The series in (1.3) *and the series*

$$\sum_{n=1}^{\infty} 2^{-n/2} I_n^{1/2} \tag{2.5}$$

are both convergent or both divergent.

PROOF. Since

$$I_n > \frac{1}{4} 2^n \omega^2\Big(\frac{1}{2^n}\Big)$$

the convergence of (2.5) implies the convergence of $\sum_{n=1}^{\infty} \omega(1/2^n)$ and hence that of the series in (1.3). Suppose now that the series in (1.3) is convergent. Then

$$I_n < \omega^2(\pi) + \omega^2(1) + 2\omega^2\Big(\frac{1}{2}\Big) + \cdots + 2^{n-1}\omega^2\Big(\frac{1}{2^{n-1}}\Big) ,$$

$$\begin{aligned} \sum_{n=1}^{\infty} 2^{-n/2} I_n^{1/2} &< \omega(\pi) \sum_{1}^{\infty} 2^{-n/2} + \sum_{n=1}^{\infty} 2^{-n/2} \sum_{p=1}^{n-1} 2^{p/2}\omega\Big(\frac{1}{2^p}\Big) \\ &< A + \sum_{p=1}^{\infty} 2^{p/2}\omega\Big(\frac{1}{2^p}\Big) \sum_{n=p+1}^{\infty} 2^{-n/2} \\ &< A + A \sum_{p=1}^{\infty} \omega\Big(\frac{1}{2^p}\Big) < A . \end{aligned}$$

3. Proof of Theorem 1.

Let

$$f(t) \sim \frac{1}{2}a_0 + \sum_{1}^{\infty} (a_n \cos nt + b_n \sin nt) \equiv \sum_{0}^{\infty} u_n ,$$

$$s_n = \sum_{\nu=0}^{n} u_\nu,\ t_n = \sum_{\nu=0}^{n} \frac{p_n s_{n-\nu}}{P_n} ,$$

$$\phi(t) = f(x+t) + f(x-t) - 2f(x) ,$$

$$\alpha(t) + i\beta(t) = \sum_{k=0}^{\infty} p_k e^{ikt} ,$$

$$\alpha_n = \int_0^{\pi} \phi(t)\alpha(t) \cos nt\, dt , \qquad \beta_n = \int_0^{\pi} \phi(t)\beta(t) \sin nt\, dt .$$

We have (cf: [6], [8])

$$\pi\,|\,t_n - t_{n-1}\,| = \left|\int_0^\pi \phi(t)\sum_{k=0}^{n-1}\left(\frac{P_k}{P_n} - \frac{P_{k-1}}{P_{n-1}}\right)\cos(n-k)t\,dt\right|$$
$$\leqq \frac{1}{P_{n-1}}\left|\int_0^\pi \phi(t)\sum_{k=0}^{\infty} p_k \cos(n-k)t\,dt\right|$$
$$+ \frac{1}{P_{n-1}}\left|\int_0^{1/n} \phi(t)\sum_{k=n}^{\infty} p_k \cos(n-k)t\,dt\right|$$
$$+ \frac{p_n}{P_nP_{n-1}}\left|\int_0^{1/n} \phi(t)\sum_{k=0}^{n-1} P_k \cos(n-k)t\,dt\right|$$
$$+ \frac{1}{P_{n-1}}\left|\int_{1/n}^\pi \phi(t)\left\{\sum_{k=n}^{\infty} p_k \cos(n-k)t + \sum_{k=0}^{n-1}\frac{p_n}{P_n}P_k\cos(n-k)t\right\}dt\right|$$
$$= T_1(n) + T_2(n) + T_3(n) + T_4(n) \text{ say .}$$

We have to prove that $\sum|\,t_n - t_{n-1}\,| < \infty$. By Lemmas 1 and 3

$$T_2(n) < \frac{2}{P_{n-1}}\int_0^{1/n}\omega(t)P\left(\frac{1}{t}\right)dt < \frac{2\omega(1/n)}{P_{n-1}}\int_n^\infty \frac{P(u)du}{u^2} < A\frac{1}{n}\omega\left(\frac{1}{n}\right)$$

and by (1.3), $\sum_{n=2}^\infty T_2(n) < \infty$.

Further, since $p_n \downarrow$,

$$T_3(n) < \frac{2p_n}{P_nP_{n-1}}\omega\left(\frac{1}{n}\right)\frac{P_0 + \cdots + P_{n-1}}{n}$$
$$< \frac{2p_n}{P_nP_{n-1}}\omega\left(\frac{1}{n}\right)P_{n-1} < \frac{2}{n}\omega\left(\frac{1}{n}\right)$$

and so $\sum_{n=2}^\infty T_3(n) < \infty$.

Further

$$T_4(n) = \frac{1}{P_{n-1}}\left|\int_{1/n}^\pi \phi(t)\left[\frac{p_n}{2} + \sum_{k=n}^{\infty}(p_k - p_{k+1})\frac{\sin(n-k+(1/2))t}{2\sin(t/2)}\right.\right.$$
$$\left.\left. + \frac{p_n}{P_n}\left\{\sum_{k=0}^{n-1} p_k\frac{\sin(n-k+(1/2))t}{2\sin(t/2)} - \frac{1}{2}P_{n-1}\right\}\right]dt\right|$$
$$\leqq \frac{1}{P_{n-1}}\left|\int_{1/n}^\pi \frac{\phi(t)}{2\sin(t/2)}\left(\sum_{k=n}^{\infty}(p_k - p_{k+1})\sin\left(n-k+\frac{1}{2}\right)t\right)dt\right|$$
$$+ \frac{p_n}{P_nP_{n-1}}\left|\int_{1/n}^\pi \frac{\phi(t)}{2\sin(t/2)}\left(\sum_{k=0}^{n-1} p_k\sin\left(n-k+\frac{1}{2}\right)t\right)dt\right|$$
$$+ \frac{p_n}{2P_{n-1}}\left(1 - \frac{P_{n-1}}{P_n}\right)\left|\int_{1/n}^\pi \phi(t)dt\right|$$
$$\equiv T_{41}(n) + T_{42}(n) + T_{43}(n)\ .$$

By Lemma 1

$$T_{41}(n) \leqq \frac{A(p_n - p_{n+1})}{P_{n-1}} \int_{1/n}^{\pi} \frac{|\phi(t)|}{\sin(t/2)} t^{-1} dt$$
$$\leqq \frac{A(p_n - p_{n+1})}{P_{n-1}} \int_{1/n}^{\pi} \frac{\omega(t)}{t^2} dt$$
$$\leqq \frac{A(p_n - p_{n+1})}{P_{n-1}} \left(A + \sum_{k=2}^{n} \omega\left(\frac{1}{k}\right) \right) .$$

Lemma 2 now shows that

$$\sum_{n=2}^{\infty} T_{41}(n) \leqq A \sum_{n=2}^{\infty} \frac{1}{n^2} \left(A + \sum_{k=2}^{n} \omega\left(\frac{1}{k}\right) \right) < A + A \sum_{k=2}^{\infty} \omega\left(\frac{1}{k}\right) \sum_{n=k}^{\infty} \frac{1}{n^2}$$
$$< A + A \sum_{k=2}^{\infty} \frac{1}{k} \omega\left(\frac{1}{k}\right) < \infty .$$

Further $\int_{1/n}^{\pi} |\phi(t)| \, dt < A$ and so

$$\sum_{n=2}^{\infty} T_{43}(n) < A \sum_{2}^{\infty} \frac{p_n^2}{P_n P_{n-1}} < A \sum_{2}^{\infty} \frac{p_n^2}{(n+1) p_n n p_{n-1}} < A \sum_{2}^{\infty} \frac{1}{n^2} < \infty .$$

By Lemma 1

$$T_{42}(n) \leqq \frac{A p_n}{P_n P_{n-1}} \int_{1/n}^{\pi} \frac{|\phi(t)|}{t} P\left(\frac{1}{t}\right) dt \leqq \frac{A p_n}{P_n P_{n-1}} \int_{1/\pi}^{n} \omega\left(\frac{1}{t}\right) P(t) \frac{dt}{t} ,$$

and

$$\sum_{n=2}^{\infty} T_{42}(n) \leqq A \sum_{n=2}^{\infty} \frac{p_n}{P_n P_{n-1}} \left(A + \sum_{1}^{n} \omega\left(\frac{1}{k}\right) \frac{P(k)}{k} \right)$$
$$< A + A \sum_{n=2}^{\infty} \frac{p_n}{P_n P_{n-1}} \sum_{k=1}^{n} \omega\left(\frac{1}{k}\right) \frac{P(k)}{k}$$
$$= A + A \sum_{k=1}^{\infty} \omega\left(\frac{1}{k}\right) \frac{P(k)}{k} \sum_{n=k}^{\infty} \frac{p_n}{P_n P_{n-1}}$$
$$< A + A \sum_{k=1}^{\infty} \omega\left(\frac{1}{k}\right) \frac{P(k)}{k} \frac{1}{P(k-1)}$$
$$< A + A \sum_{1}^{\infty} \omega\left(\frac{1}{k}\right) \frac{1}{k} < \infty .$$

We now consider $T_1(n) \leqq (|\alpha_n| + |\beta_n|)/P_{n-1}$.

Let $\Psi(t) = \phi(t+h)\alpha(t+h) - \phi(t-h)\alpha(t-h)$. Then $\Psi(t)$ is even and $\alpha(t) \in L^2$, for by Lemma 1,

$$\int_0^{\pi} \alpha^2(t) dt \leqq \int_0^{\pi} P^2\left(\frac{1}{t}\right) dt < A \int_0^{\pi} \psi^2\left(\frac{1}{t}\right) t^{-2\gamma} dt < \infty .$$

By Bessel's inequality we have for $0 < h \leqq \pi/4$,

$$\begin{aligned}\sum_1^\infty |\alpha_n^2 \sin^2 nh| &< A\int_0^\pi |\Psi^2(t)|\, dt \\ &< A\Big[\int_0^\pi \alpha^2(t+h)\,|\{\phi(t+h)-\phi(t-h)\}|^2 dt \\ &\quad + \int_{-h}^{h} |\phi^2(t)|\,\alpha^2(t+2h)dt + \int_{-h}^{h} |\phi^2(t)|\,\alpha^2(t)dt \\ &\quad + \int_h^\pi |\phi^2(t)|\,\{\alpha(t+2h)-\alpha(t)\}^2 dt\Big] \\ &\equiv A[I_1(h)+I_2(h)+I_3(h)+I_4(h)]\,.\end{aligned}$$

By (v) and Lemma 1,

$$\begin{aligned}I_2(h) &< A\int_{-h}^{h} \omega^2(t)P^2\Big(\frac{1}{t+2h}\Big)dt \\ &\leqq A\omega^2(h)\int_h^{3h} P^2\Big(\frac{1}{t}\Big)dt < Ah\omega^2(h)P^2\Big(\frac{1}{h}\Big)\,,\end{aligned}$$

and

$$\begin{aligned}I_3(h) &= \int_{-h}^{h} |\phi^2(t)|\,\alpha^2(t)dt < A\omega^2(h)\int_{-h}^{h} \alpha^2(t)dt \\ &< A\omega^2(h)\int_0^h P^2\Big(\frac{1}{t}\Big)dt < A\omega^2(h)hP^2\Big(\frac{1}{h}\Big)\,.\end{aligned}$$

Since [6]

$$|\alpha(t+2h)-\alpha(t)| \leqq Aht^{-1}P(h^{-1})\,,$$

$$I_4(h) < Ah^2P^2\Big(\frac{1}{h}\Big)\int_h^\pi \frac{\omega^2(t)}{t^2}dt < Ah^2P^2\Big(\frac{1}{h}\Big)\int_{1/\pi}^{1/h} \omega^2\Big(\frac{1}{t}\Big)dt\,.$$

We now estimate I_1. Since f is of bounded variation over $[0, 2\pi]$ we have

$$\sum_{k=1}^{2N} \alpha^2\Big(t+\frac{k\pi}{N}\Big)\Big|\Big\{\phi\Big(t+\frac{k\pi}{N}\Big)-\phi\Big(t+(k-1)\frac{\pi}{N}\Big)\Big\}^2\Big| < A\omega\Big(\frac{\pi}{N}\Big)P^2\Big(\frac{1}{t}\Big)\,.$$

Integrating from 0 to π (cf. [8; p. 241-2]) we get

$$2NI_1\Big(\frac{\pi}{2N}\Big) < A\omega\Big(\frac{\pi}{N}\Big)\int_0^\pi P^2\Big(\frac{1}{t}\Big)dt < A\omega\Big(\frac{\pi}{N}\Big)\int_{1/\pi}^\infty \frac{P^2(t)dt}{t^2} < A\omega\Big(\frac{\pi}{N}\Big)\,.$$

Taking $h = \pi/(2N)$ we get

$$\sum_1^\infty \left| \alpha_n^2 \sin^2\left(\frac{n\pi}{2N}\right)\right| < A\left\{\frac{1}{N}\omega\left(\frac{\pi}{N}\right) + \frac{1}{N}\omega^2\left(\frac{\pi}{2N}\right)P^2\left(\frac{2N}{\pi}\right) + \frac{1}{N^2}P^2\left(\frac{2N}{\pi}\right)\int_{1/\pi}^{(2N)/\pi}\omega^2\left(\frac{1}{t}\right)dt\right\}$$

$$< A\left\{\frac{1}{N}\omega\left(\frac{\pi}{N}\right) + \frac{1}{N^2}P^2\left(\frac{2N}{\pi}\right)\int_{1/\pi}^{(2N)/\pi}\omega^2\left(\frac{1}{t}\right)dt\right\}$$

Letting $N = 2^\nu$ we have

$$\left\{\sum_{2^{\nu-1}+1}^{2^\nu} |\alpha_n^2|\right\}^{1/2} < A\left\{\sum_1^\infty |\alpha_n^2| \sin^2\left(\frac{n\pi}{2^{\nu+1}}\right)\right\}^{1/2}$$

$$< A\left\{\frac{1}{2^\nu}\omega\left(\frac{\pi}{2^\nu}\right) + \frac{1}{2^{2\nu}}P^2\left(\frac{2^{\nu+1}}{\pi}\right)\int_{1/\pi}^{2^{\nu+1}/\pi}\omega^2\left(\frac{1}{t}\right)dt\right\}^{1/2}$$

$$< A\left\{\frac{1}{2^{\nu/2}}\omega^{1/2}\left(\frac{\pi}{2^\nu}\right) + \frac{1}{2^\nu}P\left(\frac{2^{\nu+1}}{\pi}\right)\left(\int_{1/\pi}^{2^{\nu+1}/\pi}\omega^2\left(\frac{1}{t}\right)dt\right)^{1/2}\right\}.$$

By (v) we have

$$\sum_{2^{\nu-1}+1}^{2^\nu} \frac{1}{P_{n-1}^2} < A\frac{2^\nu}{P^2(2^\nu)}$$

and an application of Schwarz inequality gives

$$\sum_{2^{\nu-1}+1}^{2^\nu} \frac{|\alpha_n|}{P_{n-1}} \leqq A\frac{2^{\nu/2}}{P(2^\nu)}\left\{\frac{1}{2^{\nu/2}}\omega^{1/2}\left(\frac{\pi}{2^\nu}\right) + \frac{1}{2^\nu}P\left(\frac{2^{\nu+1}}{\pi}\right)\left(\int_{1/\pi}^{2^{\nu+1}/\pi}\omega^2\left(\frac{1}{t}\right)dt\right)^{1/2}\right\}.$$

By (1.5),

$$\sum \frac{1}{P(2^\nu)}\omega^{1/2}\left(\frac{\pi}{2^\nu}\right) < A\ ,$$

and by Lemma 4,

$$\sum \frac{P(2^{\nu+1/\pi})}{P(2^\nu)}\,\frac{1}{2^{\nu/2}}\left(\int_{1/\pi}^{2^{\nu+1/\pi}}\omega^2\left(\frac{1}{t}\right)dt\right)^{1/2} < A\ .$$

Hence $\sum_{n=1}^\infty |\alpha_n|/P_{n-1} < \infty$. Similarly $\sum_{n=1}^\infty |\beta_n|/P_{n-1} < \infty$ and so $\sum_2^\infty |t_n - t_{n-1}| < A < \infty$ and the proof is complete.

4. Remarks and Examples.

(a) If (1.3) holds and $\sum 1/(nP^2(n)) < \infty$, then an application of Schwarz inequality shows that (1.5) holds.

(b) The condition (1.5) implies that

$$\omega\left(\frac{1}{t}\right) < A(P^2(t))/\log^2 t, \qquad t \geqq 2\ .$$

Consequently

$$\sum \omega\left(\frac{1}{2^n}\right) < A \sum (P^2(2^n))/n^2 \, .$$

Hence if (1.5) holds and

$$\sum (P^2(2^n))/n^2 < \infty \, , \tag{4.1}$$

then the series in (1.3) is convergent.

If we take, for instance, $p_n = (n+a)^{-1}(\log(n+a))^{-1}$, $a \geqq 3$, then by considering $y(x) = (x+a)^{-1}(\log(x+a))^{-1}$ we see that p_n satisfies the conditions (i)–(iv). Further $P_n \sim \log\log n$ and so (4.1) and (v) are satisfied (with $\gamma = 0$).

(c) Zygmund [11; 241–2] proved that if $f(x)$ is of bounded variation and

$$\sum n^{-1}\omega^{1/2}(n^{-1}) < \infty \, , \tag{4.2}$$

then $S(f)$ is absolutely convergent. Our theorem gives the following analogue of Zygmund's result:

If $f(x)$ is of bounded variation and if (4.1) holds, the then convergence of the series in (1.5) implies the absolute summability $|N, p_n|$ of $S(f)$.

Note that if we take $p_0 = 1$ and $p_n = 0$ $(n > 0)$ then (1.5) is the same as (4.2) and the summability $|N, p_n|$ is the same as the absolute convergence.

Example. Let

$$p_n = \frac{c\log(n+c)}{(n+c)\log c}\, , \qquad \log c \geqq 2 \, .$$

Then $p_n > 0$, $\{p_n\}\downarrow$, $\{p_n - p_{n+1}\}\downarrow$ (cf: [6]). $P_n \sim A(\log n)^2$. Hence condition (v) is satisfied (with $\gamma = 0$) and $\sum 1/(nP_n^2) < \infty$. (This implies that (1.5) is satisfied if (1.3) is.) By considering $y'(x)$ where

$$y(x) = \frac{(x+c)}{(x+1+c)}\frac{\log(x+1+c)}{\log(x+c)}\, ,$$

we see that $p_{n+1}/p_n \uparrow$ and so by a known inclusion theorem [6], $|N, p_n| \subset |C, 1|$.

5. **Weighted Arithmetic Means.** We now consider the weighted arithmetic mean ([7; pp. 16–17], [9; p. 32]) of the series $\sum_0^\infty u_n$. Let $S_k = \sum_0^k u_n$. Let $p_n \geqq 0$, $P_n > 0$ and $\sigma_n = 1/P_n \sum_{k=0}^n p_k S_k$. To avoid trivial cases we shall suppose that $p_n > 0$ for an infinity of n. The sequence $\{S_k\}$ is said to be absolutely summable by the weighted arithmetic mean method, defined by the sequence $\{p_n\}$, or briefly summable $|M, p_n|$, if

$$\sum_{n=1}^{\infty} |\sigma_n - \sigma_{n-1}| \leqq A < \infty .$$

Let $f \in C_{2\pi}$ (continuous and 2π-periodic) and let

$$\omega_2(\delta, f) = \sup_{0 \leqq h \leqq \delta} |f(x+h) + f(x-h) - 2f(x)| \ (x \in [0, 2\pi])$$

denote the modulus of smoothness of f.

THEOREM 2. *Let $p_n \geqq 0$, $P_n = \sum_0^n p_j > 0$, $P_n \to \infty$ and $f \in C_{2\pi}$. If*

$$\sum_{n=1}^{\infty} \frac{p_n}{P_n} \log n\omega_2\left(\frac{1}{n}\right) < \infty ,$$

then $S(f)$ is summable $|M, p_n|$.

PROOF. We have [1; p. 300, p. 533]

$$|S_n(t) - f(t)| < C\omega_2((n+1)^{-1}) \max(1, \log n)$$

where C is an absolute constant. Hence for $n \geqq 1$,

$$|\sigma_n(t) - \sigma_{n-1}(t)| = \left| \frac{1}{P_n} \sum_0^n p_k(S_k(t) - f(t)) - \frac{1}{P_{n-1}} \sum_0^{n-1} p_k(S_k(t) - f(t)) \right|$$

$$= \left| \left(\frac{1}{P_n} - \frac{1}{P_{n-1}}\right) \sum_0^{n-1} p_k(S_k(t) - f(t)) + \left(\frac{1}{P_n}\right) p_n(S_n(t) - f(t)) \right| .$$

Thus for $0 \leqq t \leqq 2\pi$,

$$\sum_{n=1}^{\infty} |\sigma_n(t) - \sigma_{n-1}(t)| \leqq C \sum_{n=1}^{\infty} \left(\frac{1}{P_{n-1}} - \frac{1}{P_n}\right) \sum_{k=0}^{n-1} p_k\omega_2\left(\frac{1}{k+1}\right) \max(1, \log k)$$

$$+ C \sum_{n=1}^{\infty} \frac{p_n}{P_n} \omega_2\left(\frac{1}{n+1}\right) \max(1, \log n)$$

$$\leqq 2C\left\{\sum_{k=0}^{\infty} \frac{p_k}{P_k} \omega_2\left(\frac{1}{k+1}\right) \max(1, \log k)\right\} ,$$

and our hypothesis shows that the last series is convergent. The proof is complete.

COROLLARY. *If $f \in C_{2\pi}$ and $\sum_1^{\infty} (\log n/(n+1))\omega_2(1/n) < \infty$, then $S(f)$ is summable $|C, 1|$.*

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