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OPERATORS WITH α-CLOSED RANGE

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Let $\mathscr{H}$ denote a Hilbert space of infinite dimension h. In an earlier work we have introduced the notion of α-closed subspace, where α is a cardinal, $\aleph_0 \leqq \alpha \leqq h$ (definition 2.1 of [2]). A subspace $\mathscr{K}$ of $\mathscr{H}$ is called α-closed if there is a closed subspace $\mathscr{L}$ of $\mathscr{H}$ such that $\mathscr{L} \subset \mathscr{K}$ and such that

$$\dim(\mathscr{L}^{\perp} \cap \mathscr{K})^{-} < \alpha .$$

This notion is of interest only when $\alpha > \aleph_0$, since a subspace $\mathscr{K}$ is $\aleph_0$-closed if and only if it is closed (lemma 2.3 of [2]). This concept is important for the study of operators on nonseparable spaces, as it is used in characterizing invertibility modulo the closed two-sided ideals of the algebra $\mathscr{L}(\mathscr{H})$ of all bounded operators on $\mathscr{H}$. (Cf. definition 2.7 and theorems 2.6 and 2.8 of [2].) For each α, $\aleph_0 \leqq \alpha \leqq h$, let $\mathscr{F}_\alpha$ denote the set of operators of rank $\rho(A)$ less than α and let $\mathscr{J}_\alpha$ denote the norm closure of $\mathscr{F}_\alpha$. Then the $\mathscr{J}_\alpha$, $\aleph_0 \leqq \alpha \leqq h$ are precisely the closed two-sided ideals of $\mathscr{L}(\mathscr{H})$, and the elements of $\mathscr{J}_\alpha$ are called α-compact operators. (Cf. [5] and theorem 0 of [2].) In this terminology, the $\aleph_0$-compact operators are precisely the compact operators. Then the operators which are invertible modulo $\mathscr{J}_\alpha$ (i.e., the operators A in $\mathscr{L}(\mathscr{H})$ for which there exists an operator A' in $\mathscr{L}(\mathscr{H})$ such that $I - AA' \in \mathscr{J}_\alpha$ and $I - A'A \in \mathscr{J}_\alpha$) are precisely the α-Fredholm operators. An operator A is called α-Fredholm if its range is α-closed and its nullity $\nu(A) < \alpha$ and its corank $\rho'(A) < \alpha$. In this context we see that the notion of an operator having α-closed range is fundamental for the study of operators on a nonseparable space. We therefore examine this new concept in more detail in this paper.

We first obtain a characterization of operators with α-closed range as those which have closed range modulo the ideal $\mathscr{F}_\alpha$. More explicitly, A has α-closed range if and only if there exists an operator C in $\mathscr{F}_\alpha$ such that $A + C$ has closed range. This enables us to generalize the well-known fact that A has closed range if and only if A^* has closed range, to the case of operators with α-closed range.

We give two applications of this result. The first gives the conditions under which an operator is right invertible modulo the ideal $\mathscr{J}_\alpha$. The second gives a generalization of a definition given by T. Kato [4] of the "essential spectrum" for possibly non-normal operators. This research was supported by the National Science Foundation.

LEMMA 1. *Let T be an operator on a Hilbert space $\mathscr{H}$ such that ran T is closed. Let $\mathscr{H}_0$ be a closed subspace of $\mathscr{H}$ and let F denote the projection onto $\mathscr{H}_0 \cap (\ker T)^\perp$. Then TF has closed range.*

PROOF. Let $\{\psi_n\}$ be a sequence in $\mathscr{H}$ such that $TF\psi_n$ is convergent. Since ran T is closed, there exists $\varphi \in (\ker T)^\perp$ such that

$$\lim_n TF\psi_n = T\varphi .$$

Now T is bounded below on $(\ker T)^\perp$ (since ran T is closed, cf. problem 41 of [3]) and hence $F\psi_n$ converges to φ. Since $\mathscr{H}_0$ is closed we have $\varphi \in \mathscr{H}_0$. Thus $\varphi \in \mathscr{H}_0 \cap (\ker T)^\perp$ and $\varphi = F\varphi$. Since $F\psi_n$ converges to $F\varphi$, we have $TF\psi_n$ converges to $TF\varphi$. Thus ran TF is closed.

THEOREM 2. *Let A be a bounded linear operator on a Hilbert space $\mathscr{H}$ of infinite dimension h. Let α be a cardinal number, $\aleph_0 \leqq \alpha \leqq h$. Then the following conditions are equivalent.*

(i) *There exists $C \in \mathscr{J}_\alpha$ such that $ran(A + C)$ is closed.*
(ii) *A has α-closed range.*
(iii) *A^* has α-closed range.*

PROOF. We first show i) implies ii). Let E denote the projection onto

$$\ker C \cap [\ker (A + C)]^\perp .$$

Let $\mathscr{L} = \text{ran}((A + C)E)$. By lemma 1, $\mathscr{L}$ is closed. Since the range of E is contained in $\ker C$, we have $(A + C)E = AE$. Thus

$$\mathscr{L} = \text{ran}(AE) \subset \text{ran}\, A .$$

Thus it remains to show that

$$\dim(\mathscr{L}^\perp \cap \text{ran}\, A) < \alpha .$$

Note that

$$\text{ran}\, A = \text{ran}\, AE + \text{ran}\, A(I - E) = \mathscr{L} + \text{ran}\, A(I - E) .$$

Also ran A $= \mathscr{L} \oplus (\mathscr{L}^\perp \cap \text{ran}\, A)$ since $\mathscr{L} \subset \text{ran}\, A$. Thus

$$\dim (\mathscr{L}^\perp \cap \text{ran}\, A) \leqq \dim \text{ran}\, A(I - E) = \rho(A(I - E)) .$$

The range of $(I - E)$ is

$$((\ker C) \cap (\ker(A + C))^{\perp})^{\perp} = (\ker C)^{\perp} \vee (\ker(A + C)) .$$

Now $\dim(\ker C)^{\perp} = \rho(C) < \alpha$ and hence $\dim A((\ker C)^{\perp}) < \alpha$. Further note that if $f \in \ker(A + C)$, then $Af = C(-f) \in \operatorname{ran} C$. Hence

$$\dim A(\ker(A + C)) \leqq \rho(C) < \alpha .$$

Hence $\rho(A(I - E)) < \alpha + \alpha = \alpha$. Thus $\dim(\mathscr{L}^{\perp} \cap \operatorname{ran} A) < \alpha$.

We next show ii) implies i). Since A has α-closed range, there exists a closed subspace $\mathscr{L}$ of $\mathscr{H}$ such that $\mathscr{L} \subset \operatorname{ran} A$ and

$$\dim(\mathscr{L}^{\perp} \cap \operatorname{ran} A) = \dim(\mathscr{L}^{\perp} \cap (\operatorname{ran} A)^{-}) < \alpha .$$

(Cf. lemma 2.2 of [2], which asserts that $(\mathscr{L}^{\perp} \cap \operatorname{ran} A)^{-} = \mathscr{L}^{\perp} \cap (\operatorname{ran} A)^{-}$.) Let E denote the projection onto $\mathscr{L}^{\perp} \cap (\operatorname{ran} A)^{-}$. Then $E \in \mathscr{I}_{\alpha}$ since $\rho(E) < \alpha$. Since $\mathscr{I}_{\alpha}$ is a two-sided ideal, we have $C = -EA \in \mathscr{I}_{\alpha}$. We assert that $\operatorname{ran}(A + C) = \mathscr{L}$ and hence that $A + C$ has closed range.

Suppose $f \in \mathscr{L}$. Then there exists $g \in \mathscr{H}$ such that $f = Ag$, since $\mathscr{L} \subset \operatorname{ran} A$. Since $f \in \mathscr{L}$ we have $Ef = 0$. Hence $(A - EA)g = f - Ef = f$. Thus $\mathscr{L} \subset \operatorname{ran}(A + C)$.

Next suppose f is an arbitrary element of $\mathscr{H}$. Let $g = Af$. Then we may write $g = g_1 + g_2$, where $g_1 \in \mathscr{L}$ and $g_2 \in \mathscr{L}^{\perp}$. Since $\mathscr{L} \subset \operatorname{ran} A$, there exists $f_1 \in \mathscr{H}$ such that $Af_1 = g_1$. Let $f_2 = f - f_1$. Then $Af = Af_1 + Af_2 = g_1 + g_2 \in \mathscr{L} \oplus ((\operatorname{ran} A) \cap \mathscr{L}^{\perp})$. Hence

$$\begin{aligned}(A + C)f &= (A - EA)f \\ &= Af_1 + Af_2 - E(Af_1 + Af_2) \\ &= Af_1 \in \mathscr{L} .\end{aligned}$$

Thus $\operatorname{ran}(A + C) \subset \mathscr{L}$.

We next show ii) implies iii). Since A is α-closed there exists $C \in \mathscr{I}_{\alpha}$ such that $\operatorname{ran}(A + C)$ is closed. By theorem 4, p. 488 of [1], $\operatorname{ran}(A^* + C^*)$ is closed. Since $\mathscr{I}_{\alpha}$ is self-adjoint we have $\operatorname{ran} A^*$ is α-closed. Applying the same argument to A^* gives the equivalence of conditions ii) and iii).

PROPOSITION 3. *The only α-compact operators with α-closed range are those of rank less than α.*

PROOF. Let A be an α-compact operator and suppose $\rho(A) = \dim \operatorname{ran} A \geqq \alpha$. If $\mathscr{K}$ is a closed subspace and if $\mathscr{K} \subset \operatorname{ran} A$, then by theorem 5.1 of [5], $\dim \mathscr{K} < \alpha$. Thus

$$\dim(\mathscr{K}^{\perp} \cap \operatorname{ran} A) \geqq \alpha .$$

Hence A does not have α-closed range.

We remark that proposition 3 may be used to show that one cannot replace the ideal $\mathscr{I}_\alpha$ of operators of rank less than α, by its closure $\mathscr{J}_\alpha$, in condition i) of theorem 2. Indeed if $\mathscr{I}_\alpha \neq \mathscr{J}_\alpha$ and $A \in \mathscr{J}_\alpha - \mathscr{I}_\alpha$ then ran A is not α-closed by proposition 3, yet clearly there is an operator $C(=-A)$ in $\mathscr{J}_\alpha$ such that ran $(A + C)$ $(=\{0\})$ is closed.

In an earlier paper (cf. theorem 2.6 of [2]) we characterized the operators which are left invertible modulo the ideals $\mathscr{I}_\alpha$ and $\mathscr{J}_\alpha$. As a corollary of this result and theorem 2 we obtain the conditions characterizing right invertibility modulo these ideals. Here $\rho'(A)$ denotes the corank of A, i.e., $\rho'(A) = \dim(\operatorname{ran} A)^\perp$.

THEOREM 4. *Let $\mathscr{H}$ be a Hibert space of infinite dimension h. Let $A \in \mathscr{L}(\mathscr{H})$ and let α be a cardinal number, $\aleph_0 \leqq \alpha \leqq h$. Then the following conditions are equivalent.*

(i) *A is right-invertible modulo $\mathscr{I}_\alpha$.*

(ii) *A is right-invertible modulo $\mathscr{J}_\alpha$.*

(iii) *ran A is α-closed and $\rho'(A) < \alpha$.*

PROOF. Clearly i) implies ii). We next show ii) implies iii). If A is right invertible modulo $\mathscr{J}_\alpha$, then A^* is left invertible modulo $\mathscr{J}_\alpha$. By theorem 2.6 of [2], ran A^* is α-closed and $\nu(A^*) < \alpha$. By theorem 2 ran A is α-closed and $\rho'(A) = \nu(A^*) < \alpha$.

We next show iii) implies i). By the previous theorem ran A^* is α-closed and $\nu(A^*) = \rho'(A) < \alpha$. Thus by theorem 2.6 of [2], A^* is left invertible modulo $\mathscr{I}_\alpha$. Hence A is right invertible modulo $\mathscr{I}_\alpha$.

T. Kato [4] has defined a notion of essential spectrum (which we shall refer to as the Kato essential spectrum) for nonnormal operators. For an operator A, the Kato essential spectrum $\Sigma_e(A)$ is the set of complex numbers λ such that either $\nu(A - \lambda I) \geqq \aleph_0$ and $\rho'(A - \lambda I) \geqq \aleph_0$, or ran $(A - \lambda I)$ is not closed. We next give a generalization of this notion.

DEFINITION 5. If h is the dimension of the Hilbert space and $\aleph_0 \leqq \alpha \leqq h$, we define the *Kato essential spectrum of A of weight α*, denoted $\Sigma_\alpha(A)$, to be the set of complex numbers λ such that either $\nu(A - \lambda I) \geqq \alpha$ and $\rho'(A - \lambda I) \geqq \alpha$ or ran $(A - \lambda I)$ is not α-closed.

Recall (definition 3.1 and theorem 3.2 of [2]) that the approximate point spectrum of A, of weight α, $\Pi_\alpha(A)$, is the set of complex numbers λ such that $\nu(A - \lambda I) \geqq \alpha$ or the range of $(A - \lambda I)$ is not α-closed. Using the fact (theorem 2) that A has α-closed range if and only if A^* has α-closed range, we obtain the following characterization of the Kato spectrum of weight α.

PROPOSITION 6. *Let A be a bounded operator on a Hilbert space of dimension h. Then*

$$\Sigma_\alpha(A) = \Pi_\alpha(A) \cap \Pi_\alpha(A^*)^*$$

for $\aleph_0 \leqq \alpha \leqq h$.

COROLLARY 7. *The Kato essential spectrum $\Sigma_\alpha(A)$ of weight α is a compact set which is invariant under perturbations by α-compact operators* (i.e., *by elements of the ideal $\mathscr{J}_\alpha$.*) *Further $\Sigma_e(A) = \Sigma_{\aleph_0}(A)$.*

PROOF. $\Sigma_\alpha(A)$ is compact, since $\Pi_\alpha(A)$ and $\Pi_\alpha(A^*)^*$ are compact (cf. remark 3.8 of [2].). Similarly $\Pi_\alpha(A)$ and $\Pi_\alpha(A^*)^*$ are invariant under perturbations by α-compact operators, by theorem 4.5 of [2]. Finally $\Sigma_e(A) = \Sigma_{\aleph_0}(A)$ since ran $(A-\lambda I)$ is $\aleph_0$-closed if and only if ran $(A-\lambda I)$ is closed, by lemma 2.3 of [2].

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