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ON THE BOUNDEDNESS OF SOLUTIONS OF A SYSTEM OF DIFFERENTIAL EQUATIONS

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Introduction. In this paper, starting from the papers of T. Yoshizawa [14] – [18] and V. A. Pliss [13], we prove that in the case of the finite dimensional spaces, every ω-periodic ultimately bounded system is uniformly bounded (Theorem 2.2), is uniformly ultimately bounded (Theorem 2.3) and has at least one periodic solution of period ω (Theorem 2.4). However, for almost periodic systems, ultimate boundedness does not necessarily imply uniform boundedness.

We give also some necessary and sufficient conditions in order that the systems (1.1) and (2.1) be uniformly bounded (Theorem 3.1 and Theorem 2.1 with Remark 2.1, respectively). Consequences of Theorem 2.2 are a result of V. A. Pliss and the fact that some conditions of Proposition 5 in [14] and of Corollary 1 in [14, p. 116] are superfluous (Remark 2.3). Theorem 2.2 of V. A. Pliss [13] and Theorem 9.3 of T. Yoshizawa [16] follow from Theorem 2.3, and in addition it follows that some conditions of Proposition 6 in [14] and of Theorem 3 in [18] are superfluous. Moreover, form Theorem 2.3 it follows that in the case of the periodic system, the concepts of ultimate boundedness, equiultimate boundedness and uniformly ultimate boundedness are equivalent. So far, it was known only the fact that the concepts of equiultimate boundedness and uniformly ultimate boundedness are equivalent in the case of the periodic systems (see Theorem 9.3 in [16].)

In the last section, a necessary and sufficient condition in order that the equation (4.1) (considered on a real Banach space X, with its dual space X^* strictly convex) be strictly uniformly bounded is given (Theorem 4.1). Theorem 4.2 is similar to a result of Gerstein and Krasnoselskii [6].

We mention that the ultimately bounded systems defined by T. Yoshizawa (see e.g. [14], [16]) are identical with the dissipative systems of Levinson [9]. From Corollary 2.1 in [13] it follows that every dissipative system (ultimately bounded system) periodic in t of period ω (ω-periodic) has at least one periodic solution of period $k\omega$, for some in-integer $k \geq 1$. From Theorem 2.4 it follows that such a system has a periodic solution of period precisely ω. A similar result for the equiultimately bounded systems is Theorem 29.3 of T. Yoshizawa [16].

The idea to prove Theorem 2.2 by using the Liapunov's function has been suggested to me by Professor C. Corduneanu. We mention that Theorem 2.2 may be proved with the method of Liapunov's function, but only in the case $f(t, x) \in C^1(t, x)$ (see Section 2).

On this opportunity the author wishes to thank Professors T. Yoshizawa, C. Corduneanu, D. Petrovanu and Gh. Bantas for their useful suggestions during the preparation of this paper.

1. **Definitions.** Let us consider the initial value problem

$$\frac{dx}{dt} = f(t, x) \tag{1.1}$$

$$x(t_0) = x_0 , \tag{i}$$

where $f: R_+ \times R^n \to R^n$ is continuous, $R_+ = \{t \in R, t \geqslant 0\}$ and R^n denotes Euclidean n-space. Assume that the initial value problem (1.1) and (i) has a unique solution $x(t, t_0, x_0)$ for every $(t, t_0, x_0) \in R_+ \times R_+ \times R^n$ [3].

We say that $f \in C^1(t, x)$, if it is continuously differentiable with respect to (t, x). For the definitions of $C_0(t, x)$ and of different concepts of boundedness of the solutions of (1.1), see Yoshizawa [14], [15], [17].

Roughly speaking, we will say that (1.1) is bounded or uniformly bounded, if the solutions of (1.1) are bounded or uniformly bounded respectively. In the same way, we define ultimately bounded systems, equiultimately bounded systems and uniformly ultimately bounded systems.

We denote:

$$\begin{aligned} \|x\| &= \text{the Euclidean norm of } x; \\ S_\alpha &= \{x \in R^n,\ \|x\| \le \alpha,\ \alpha > 0\},\ S_\alpha^* = \{x \in R^n, \|x\| \ge \alpha\} ; \\ \Delta_\alpha &= \text{the product space } R_+ \times S_\alpha ; \\ \Delta_\alpha^* &= \text{the product space } R_+ \times S_\alpha^* . \end{aligned}$$

We say that the system (1.1) is ω-periodic (resp. almost periodic) if $f(t, x)$ is periodic in t of period ω (resp. almost periodic in t).

2. **Periodic bounded system.** Let us consider the following ω-periodic system

$$\frac{dx}{dt} = f(t, x) \tag{2.1}$$

under the same hypotheses as (1.1).

In this section, using Liapunov's function method, we will discuss relationships between uniform boundedness and ultimate boundedness of (2.1).

THEOREM 2.1. 1) *Assume that there exists a continuous function* $V(t, x)$ *which is defined in* Δ_α^* ($\alpha > 0$, *arbitrary*) *and satisfies locally a Lipschitz condition with respect to* x. *If in addition,*

2) $V(t_p, x_p) \to +\infty$ *as* $p \to +\infty$, *for every* $(t_p, x_p) \in R_+ \times R^n$ *with* $\|x_p\| \to +\infty$ *as* $p \to +\infty$,

3) $V'_{(2.1)}(t, x) \leqq 0$ *for* $(t, x) \in \Delta_\alpha^*$,

then (2.1) *is uniformly bounded.*

PROOF. Suppose that (2.1) is not uniformly bounded. Then there exist

$$r_0 > \alpha,\ t_0^p,\ x_0^p \in S_{r_0},\ t^p > t_0^p,\ R_p > r_0,\ p = 1, 2, \cdots,$$

$R_p \to \infty$ as $p \to \infty$, such that

$$\|x(t^p, t_0^p, x_0^p)\| = R_p,\ p = 1, 2, \cdots . \tag{2.1}'$$

Let $\theta_p \in [t_0^p, t^p)$ be such that

$$\|x(\theta_p, t_0^p, x_0^p)\| = r_0\,, \qquad p = 1, 2, \cdots , \tag{2.2}$$

and

$$r_0 < \|x(t, t_0^p, x_0^p)\| \text{ for } \theta_p < t \leq t^p\,, \qquad p = 1, 2, \cdots . \tag{2.3}$$

If we set

$$\theta_p = m_p\omega + \bar\theta_p,\ 0 \leq \bar\theta_p \leq \omega,\ m_p \in N\,, \tag{2.4}$$

where N is the set of natural numbers, and set

$$\bar t^p = t^p - m_p\omega\,, \qquad \bar x_0^p = x(\theta_p, t_0^p, x_0^p) \qquad p = 1, 2, \cdots, \tag{2.5}$$

we have

$$x(t, \bar\theta_p, \bar x_0^p) = x(t + m_p\omega, t_0^p, x_0^p) \qquad p = 1, 2, \cdots \tag{2.6}$$

since (2.1) is ω-periodic. Taking into account (2.1)′–(2.6), we obtain

$$\|\bar x_0^p\| = r_0\,, \qquad \|x(\bar t^p, \bar\theta_p, \bar x_0^p)\| = R_p \tag{2.7}$$

$$r_0 < \|x(t, \bar\theta_p, \bar x_0^p)\| \text{ for } \bar\theta_p < t \leq \bar t^p\,, \qquad p = 1, 2, \cdots . \tag{2.8}$$

From the hypothesis 3) it follows that

$$V'(t, x(t, \bar\theta_p, \bar x_0^p)) \leq 0 \text{ for } \bar\theta_p < t \leq \bar t^p\,. \tag{2.9}$$

Therefore

$$V(\bar{t}^{\,p}, x(\bar{t}^{\,p}, \bar{\theta}_p, \bar{x}_0^p)) \le V(\bar{\theta}_p, \bar{x}_0^p), \qquad p = 1, 2, \cdots . \tag{2.10}$$

Since $0 \le \bar{\theta}_p \le \omega$, $\|\bar{x}_0^p\| = r_0$ and $V(t, x)$ is a continuous function on Δ_α^*, there exists a constant M independent of p such that

$$V(\bar{t}^{\,p}, x(\bar{t}^{\,p}, \bar{\theta}_p, \bar{x}_0^p)) \le M , \qquad p = 1, 2, \cdots . \tag{2.11}$$

The contradiction obtained from (2.7), (2.11) and the hypothesis 2) proves the theorem.

REMARK 2.1. If $f(t, x) \in C_0(t, x)$, from Theorem 5 in [14] it follows that the conditions of Theorem 2.1 are also necessary.

COROLLARY 2.1. *Assume that the hypotheses* 1) *and* 3) *of Theorem* 2.1 *are satisfied. If in addition*

2^0) $b(\|x\|) \le V(t, x)$, $(t, x) \in \Delta_\alpha^*$ *where* $b(r)$ *is a positive function such that* $b(r) \to +\infty$ *as* $r \to +\infty$, *then* (2.1) *is uniformly bounded.*

PROOF. Obviously, 2^0) implies the condition 2) of Theorem 2.1, and hence Corollary 2.1 follows from Theorem 2.1.

The following theorem which has been reported in Boll. U.M.I. [11] is important, because many results of the present paper are closely related to this theorem.

THEOREM. 2.2. *Any ultimately bounded periodic system is uniformly bounded.*

PROOF. Assume that (2.1) is ultimately bounded for bound R, but not uniformly bounded. Then there exists $r_0 > 0$ such that for every $\bar{R} > 0$, there exist $t_0^{\bar{R}} > 0$, $x_0^{\bar{R}} \in S_{r_0}$ and $t^{\bar{R}} \ge t_0^{\bar{R}}$ such that $\|x(t^{\bar{R}}, t_0^{\bar{R}}, x_0^{\bar{R}})\| \ge \bar{R}$. Let $\alpha > 0$ be such that

$$R < r_0 + \alpha . \tag{2.12}$$

Consider a sequence $\{R_p\}$ such that $R_p \to +\infty$ as $p \to +\infty$ and

$$R_p > r_0 + \alpha , \qquad p = 1, 2, \cdots . \tag{2.13}$$

Then there exist t_0^p, x_0^p, t^p such that

$$x_0^p \in S_{r_0},\ t^p > t_0^p,\ \|x(t^p, t_0^p, x_0^p)\| > R_p > r_0 + \alpha, \qquad p = 1, 2, \cdots . \tag{2.14}$$

Let us consider $x(t) = x(t, t_0^p, x_0^p)$. Since $\|x(t_0^p)\| = \|x_0^p\| \le r_0 < r_0 + \alpha$ and $\|x(t^p)\| > R_p > r_0 + \alpha$, denoting θ_p the largest of all $\theta \in (t_0^p, t^p)$ with the property $\|x(\theta, t_0^p, x_0^p)\| = r_0 + \alpha$, we have

$$\|x(\theta_p, t_0^p, x_0^p)\| = r_0 + \alpha \tag{2.15}$$

and

$$(2.16)\qquad \|x(t, t_0^p, x_0^p)\| > r_0 + \alpha\ , \quad \text{for} \quad \theta_p < t \leq t^p\ , \qquad p = 1, 2, \cdots .$$

Using the relations (2.4) and (2.5) we derive

$$(2.17)\qquad x(t, \bar{\theta}_p, \bar{x}_0^p) = x(t + m_p\omega, t_0^p, x_0^p)\ , \qquad p = 1, 2, \cdots .$$

Taking into account (2.4), (2.5), (2.14) – (2.17), we obtain

$$(2.18)\qquad \|x(\bar{t}^p, \bar{\theta}_p, \bar{x}_0^p)\| \geq R_p > r_0 + \alpha$$

and

$$(2.19)\qquad \|x(t, \bar{\theta}_p, \bar{x}_0^p)\| > r_0 + \alpha \quad \text{for} \quad \bar{\theta}_p < t \leq \bar{t}^p\ , \qquad p = 1, 2, \cdots .$$

Since $0 \leq \bar{\theta}_p \leq \omega$, and $\|\bar{x}_0^p\| = r_0 + \alpha$, $p = 1, 2, \cdots$, we can assume (without the loss of generality) that the sequences $\{\theta_p\}_1^\infty$ and $\{\bar{x}_0^p\}_1^\infty$ are convergent. Let θ_0 (resp. x_0) be their limits.

Since (2.1) is supposed to be ultimately bounded for bound R, there is $\tau_0 > \theta_0$ such that

$$(2.20)\qquad \|x(\tau_0, \theta_0, x_0)\| < R < r_0 + \alpha\ .$$

Taking into account that the solution $x(t, \theta_0, x_0)$ depends continuously on θ_0, x_0, and using (2.18), (2.19) and (2.20), it follows that for p_0 sufficiently large, the cases $\bar{t}^{p_0} > \tau_0$ and $\bar{t}^{p_0} \leq \tau_0$ are impossible (see also [13] p. 31). This contradiction proves the theorem.

REMARK 2.2. 1^0) $f(t, x) \in C^1(t, x)$, Theorem 2.2 may be proved by using the Liapunov's function.

Indeed, from Theorem 2.6 in [13] it follows that there exists a function $V(t, x) \in C^1(t, x)$ with the following properties:

a) $V(t + \omega, x) = V(t, x)$, $(t, x) \in \Delta_\alpha^*$.
b) $V(t, x) > 0$ on Δ_α^*.
c) $V(t, x) \to +\infty$ as $\|x\| \to +\infty$ uniformly with respect to $t \in [0, \omega]$.
d) $\partial V/\partial t + \sum_{i=1}^n \partial V/\partial x_i f_i < 0$, $(t, x) \in \Delta_\alpha^*$.

Set $b_1(\|x\|) = \inf_{\substack{0 \leq t \leq \omega \\ \|y\| = \|x\|}} V(t, y)$ for each $x \in S_\alpha^*$.
Obviously we have

$$(2.21)\qquad b_1(\|x\|) \leq V(t, x)\ , \qquad (t, x) \in \Delta_\alpha^*\ .$$

There exists $t_0^x \in [0, \omega]$ such that

$$(2.22)\qquad b_1(\|x\|) = V(t_0^x, y)\ , \qquad \|y\| = \|x\|\ .$$

Taking into account b) and c), it follows that $V(t_0^x, y) \to +\infty$ as $\|x\| + \infty$, and therefore $b_1(r) \to +\infty$ as $r \to +\infty$ so that Theorem 2.2 is a consequence of Corollary 2.1.

2°) In the case of almost periodic systems, the ultimate boundedness does not necessarily imply the uniform boundedness.

Indeed, by Theorem 1 in [14] for the linear system the ultimate boundedness is equivalent to the asymptotic stability, and the uniform boundedness is equivalent to the uniform stability. But there exists a linear almost periodic equation for which the zero solution is asymptotically stable, but not uniformly stable (see C. C. Conley and R. K. Miller [4]).

REMARK 2.3. From Theorem 2.2 it follows that the conditions "$\|x(t; x_0, 0)\| \le \kappa$ for $x_0 \in E_B$, $t \ge 0$" and "the solutions issuing from $\Pi(0)$ are equibounded" of Proposition 5 in [14] and of Corollary 1 in [14], respectively, are superfluous.

THEOREM 2.3. *Ultimate boundedness of the solutions of* (2.1), *implies uniformly ultimate boundedness.*

PROOF. This theorem follows from Theorem 2.2 and Proposition 5 in [14]. However, we mention the following simple proof.

a) First of all, it is easy to see that it is sufficient to prove only that the solutions from M_ω, where $M_\omega = \{x(t, t_0, x_0)$, with $(t_0, x_0) \in [0, \omega] \times R^n\}$ are uniformly ultimately bounded.

b) Since ultimate boundedness of (2.1) implies uniform boundedness (Theorem 2.2), for proving that the solutions belonging to M_ω are uniformly ultimately bounded, it is sufficient to prove only that the solutions issuing from $\Pi(\omega)$ are quasi-equiultimately bounded.

c) Therefore, let us consider the solutions issuing from $\Pi(\omega)$. If $y \in S_\alpha$ ($\alpha > 0$, arbitrary), there exists t_y such that

$$\|x(\omega + t_y, \omega, y)\| < b \,, \tag{2.23}$$

(where b is the bound of ultimate boundedness of (2.1)). There exists a neighborhood V_y of y such that

$$\|x(\omega + t_y, \omega, x)\| < b \qquad \text{for every } x \in V_y \,. \tag{2.24}$$

But $\{V_y\}_{y \in S_\alpha}$ covers S_α and hence there exist $x_1, \cdots, x_p \in S_\alpha$ such that S_α be covered by $\{V_{x_i}\}_{1 \le i \le p}$ and

$$\|x(\omega + t_i, \omega, x)\| < b \qquad \text{for } x \in V_{x_i} \,. \tag{2.25}$$

Let us consider now $T(\alpha) = \max(t_1, \cdots, t_p)$. By Theorem 2.2 there exists $\beta(b)$ such that

$$\|x(t, t_0, x_0)\| < \beta \quad \text{for} \quad (t_0, x_0) \in R_+ \times S_b \quad \text{and} \quad t \ge t_0 \,. \tag{2.26}$$

Now, if $x_0 \in S_\alpha$, there exists i, $1 \le i \le p$, such that $x_0 \in V_{x_i}$. But

$$\|x(t, \omega, x_0)\| = \|x(t, \omega + t_i, x(\omega + t_i, \omega, x_0))\| .$$

From (2.25), (2.26) we obtain

$$\|x(t, \omega, x_0)\| < \beta \quad \text{for} \quad t > \omega + T(\alpha) , \tag{2.27}$$

and hence the solutions issuing from $\Pi(\omega)$ are quasi-equiultimately bounded of bound $\beta(b)$. The proof is completed.

From Theorem 2.3 it follows:

COROLLARY 2.2. *In the case of the periodic systems, the concepts of ultimate boundedness, equiultimate boundedness and uniformly ultimate boundedness are equivalent.*

REMARK 2.4. The Theorem 9.3 in [16] follows from Corollary 2.2. Taking into account the Corollary 2.2 we easily see that the condition: "the solutions issuing from $\Pi(0)$ are quasi-equiultimately bounded" of Proposition 6 in [14] is superfluous.

THEOREM 2.4. *If* (2.1) *is ultimately bounded, then it has at least a periodic solution* $x(t, 0, \bar{x}_0)$ *of period* ω *and in addition*

$$\|x(t, 0, \bar{x}_0)\| < b \tag{$*$}$$

for all $t \geq 0$, *where* b *is the bound of uniformly ultimate boundedness.*

The proof is immediate, using a fixed point theorem of Browder [2], in a similar way as in the proof of Theorem 29.3 in [16], so that we omit it.

The relation $(*)$ is obviously necessary.

Theorem 2.4 specifies Corollary 2.1 in [13], which shows that (2.1) has at least one periodic solution of periodic $k\omega$ for some integer $k \geq 1$.

REMARK 2.5. From Theorem 2.2 and Theorem 2.3 we shall derive two results of V. A. Pliss [13].

If (2.1) *is ultimately bounded, then there is* $h > 0$ *such that for every* $a > 0$, *there exists a number* $k(a)$ *such that*

1^0) $T^k S_a \subset S_h$ *for every* $k \geq k(a)$, *where* T *is defined as usually:* $T x_0 = x(\omega, 0, x_0)$, [1], [16].

Since ultimate boundedness of (2.1) implies uniformly ultimately boundedness, we have

$$\|x(t, 0, x_0)\| < b \quad \text{for} \quad t \geq T(a) , \qquad x_0 \in S_a \tag{2.28}$$

where b is the bound of uniformly ultimate boundedness. Let $k_1(a)$ be the first natural number such that $k_1(a)\omega \geq T(a)$. Obviously, letting $k(a) = k_1(a)$ and $h = b$, Theorem 2.2 in [13] can be obtained.

In the proof of the result mentioned above, Pliss used the following proposition [13, p. 31].

Assuming that (2.1) *is ultimately bounded for bound* b, *there is a number* $h > 0$ *such that*

$$T^k S_b \subset S_h \qquad k = 1, 2, \cdots . \tag{2.29}$$

From Theorem 2.2 there exists $\bar{b}$ such that

$$\| x(t, 0, x_0) \| < \bar{b} \quad \text{for} \quad t \geq 0 \quad \text{and} \quad x_0 \in S_b , \tag{2.30}$$

so that (2.29) follows from (2.30) with $h = \bar{b}$.

3. **Generalizations.** Let us consider again the system (1.1). In this section we shall give a slight extension of Theorem 10.2 in [16].

THEOREM 3.1. 1) *Assume that there exists a Liapunov function* $V(t, x)$ *defined in* Δ^*_α ($\alpha > 0$, *arbitrary*) *such that*

2) $V(t_p, x_p) \to +\infty$ *as* $p \to +\infty$ *for every* $(t_p, x_p) \in R_+ \times R^n$ *with* $\| x_p \| \to +\infty$ *as* $p \to +\infty$,

3) $V(t, x) \leq b(\| x \|)$, *where* $b(r)$ *is a continuous positive function,*

4) $V'_{(1.1)}(t, x) \leq 0$.

Then, (1.1) *is uniformly bounded.*

PROOF. Suppose that (1.1) is not uniformly bounded. Following the proof of Theorem 2.1 (Section 2), there exist $r_0 > \alpha$, t_0^n, $x_0^n \in S_{r_0}$, $t^n > t_0^n$, $t_0^n \leq \theta_n < t^n$, $R_n > r_0$ with $R_n \to +\infty$ as $n \to +\infty$, such that

$$\| x(\theta_n, t_0^n, x_0^n) \| = r_0, \quad \| x_0^n \| \leq r_0 , \qquad n = 1, 2, \cdots \tag{3.1}$$

$$\| x(t^n, t_0^n, x_0^n) \| = R_n > r_0 , \qquad n = 1, 2, \cdots \tag{3.2}$$

$$r_0 < \| x(t, t_0^n, x_0^n) \| \quad \text{for} \quad \theta_n < t < t^n , \qquad n = 1, 2, \cdots . \tag{3.3}$$

Taking into account 3), 4), (3.1) and (3.3), we derive

$$V(t^n, x(t^n, t_0^n, x_0^n)) \leq V(\theta_n, x(\theta_n, t_0^n, x_0^n)) \leq M \tag{3.4}$$

where $n = 1, 2, \cdots$ and M is a constant independent of n. The contradiction between (3.4) and the hypothesis 2) proves the theorem.

REMARK 3.1. The hypothesis 3) of Corollary 2.1 (or the hypothesis (ii) of Theorem 10.2 in [16]) may be weakened by using a method in [10].

4. **Some remarks in the case of a Banach space.** We consider now the initial vulue problem

$$\frac{dx}{dt} = f(t, x) , \qquad (\mathrm{i}) \quad x(t_0) = x_0 \tag{4.1}$$

with $f: R_+ \times X \to X$, where X is a real Banach space and f is a continuous function. Assume that the initial value problem (4.1) and (i) has a unique solution for every $(t_0, x_0) \in R_+ \times X$. In addition, we assume that the norm of the solutions of (4.1) is differentiable almost everywhere (a.e.).

We denote by (,) the pairing between X and X^* (X^* being the dual space of X), and by F the duality mapping of X into X^* (see e.g. [7], [8]).

DEFINITION 4.1. *A mapping $A: X \to X$ is said to be dissipative if for any $x, y \in D(A)$ and $f \in F(x - y)$, we have*

$$(Ax - Ay, f) \leq 0 . \tag{4.2}$$

DEFINITION 4.2. *We say that* (4.1) *is strictly uniformly bounded* (s.u.b) *if the norm of the solutions is a nonincreasing function.*

Obviously, the strictly uniform boundedness is a particular case of uniform boundedness. The following result will be necessary.

KATO'S LEMMA. *Let $x(t)$ be a X-valued function defined on R_+. Suppose that $x(t)$ and $\|x(t)\|$ are differentiable at $t = s$. Then*

$$\|x(s)\| \frac{d}{dt} \|x(t)\| |_{t=s} = (x'(s), f) , \qquad f \in F(x(s)) . \tag{4.3}$$

THEOREM 4.1. *Suppose that X^* is strictly convex. Then, the equation* (4.1) *is* s.u.b. *if and only if*

$$(f(t, x), F(x)) \leq 0 , \qquad \text{for any } (t, x) \in R_+ \times X . \tag{4.4}$$

PROOF. Sufficiency. Let $(t_0, x_0) \in R_+ \times X$. From (4.3) we have

$$\frac{1}{2} \frac{d}{dt} \|x(t, t_0, x_0)\|^2 = (f(t, x(t, t_0, x_0)), F(x(t, t_0, x_0))) \qquad \text{a.e.} \tag{4.5}$$

and hence

$$\frac{d}{dt} \|x(t, t_0, x_0)\| \leq 0 \qquad \text{a.e.} \tag{4.6}$$

Integrating (4.6) over $[t_0, t]$, we obtain

$$\|x(t, t_0, x_0)\| \leq \|x_0\| \qquad \text{for } t \geq t_0 . \tag{4.7}$$

From (4.7) (see also [12]) it follows that (4.1) is s.u.b.

Necessity. Assume that (4.6) is not true, therefore assume that there exists $(t_0, x_0) \in R_+ \times X$, such that

$$(f(t_0, x_0), F(x_0)) > 0 \ . \tag{4.8}$$

Set $\varphi(t) = (f(t, x(t, t_0, x_0)), F(x(t, t_0, x_0))), \ t \geq t_0$.

Since the duality mapping is demicontinuous (because X^* is strictly convex), it follows that $\varphi(t)$ is a continuous function for $t \geq t_0$. But $\varphi(t_0) = (f(t_0, x_0), F(x_0))$, and therefore there is a $\delta > 0$, such that

$$\varphi(t) > 0 \qquad \text{for any} \qquad t \in [t_0, t_0 + \delta] \ . \tag{4.9}$$

From (4.5) and (4.9) we obtain

$$\frac{d}{dt} \| x(t, t_0, x_0) \| > 0 \qquad \text{a.e.} \qquad \text{on } [t_0, t_0 + \delta] \ . \tag{4.10}$$

But (4.10) implies

$$\| x(t, t_0, x_0) \| > \| x_0 \| \qquad \text{for} \qquad t \in [t_0, t_0 + \delta] \ . \tag{4.11}$$

Since (4.1) is supposed to be s.u.b., the inequality (4.11) is a contradiction, so that the theorem is proved.

COROLLARY 4.1. *If $f(t, x)$ is a linear mapping from X to X for every $t \in R_+$, then* (4.1) *is* s.u.b. *if and only if $f(t, x)$ is a dissipative mapping.*

PROOF. Indeed, in this case, (4.4) is the definition of dissipativeness of $f(t, x)$ ([19], ch. IX).

REMARK 4.1. Let us consider

$$\frac{dx}{dt} = Ax \ , \tag{E}$$

where A is a nonlinear operator which generates the nonlinear semigroups T_t of contractions [8]. Set $N(A) = \{x, x \in D(A), \ A \ x = 0\}$. If $N(A) \neq \varnothing$ ($\varnothing$ being the empty set), then the equation (E) is uniformly bounded.

Indeed, $x(t, t_0, x_0) = T_{t-t_0}x_0, \ t \geq t_0, \ x_0 \in D(A)$. Let us consider $e \in N(A)$. Then we have $x(t, t_0, e) = T_{t-t_0}e = e, \ t \geq t_0$. But

$$\| x(t, t_0, x_0) \| = \| T_{t-t_0}x_0 - T_{t-t_0}e \| + \| e \| \leq \| x_0 \| + 2 \| e \| \ . \tag{4.12}$$

Therefore

$$\| x(t, t_0, x_0) \| < r + 2 \| e \| \quad \text{for} \quad \| x_0 \| \leq r \ , \qquad t \geq t_0 \tag{4.13}$$

i.e. (E) is uniformly bounded with $\beta(r) = r + 2 \| e \|$ ([16], Definition 9.3).

Conversely (if X is a Hilbert space), *from a result of Browder mentioned in* [5] (Section 5, Theorem 5.1) if the solutions of (E) are uniformly bounded (or only bounded), then $N(A) \neq \emptyset$.

Finally, let us consider (4.1) again. Using a fixed point theorem of Browder [2], it is easy to prove the following result.

THEOREM 4.2. *Assume that* (4.1) *is periodic in* t *of period* ω, *and the mapping* T *defined by* $Tx_0 = x(\omega, 0, x_0)$ *is a compact mapping.*

If in addition (4.1) *is uniformly bounded and uniformly ultimately bounded, then there exists a periodic solution of period* ω.

Theorem 4.2 is similar to Theorem 2.4 and to a result of Gerstein and Krasnoselski [6].

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