Tôhoku Math. Journ.
24 (1972). 33-39.

GALOIS COHOMOLOGY IN UNRAMIFIED EXTENSIONS OF ALGEBRAIC FUNCTION FIELDS

TOYOFUMI TAKAHASHI

(Received June 28, 1971)

Let F be an algebraic function field over a finite field. It is known that the Galois group of the maximal "S-unramified" extension of F has cohomological l-dimension 2 in case $S \neq \emptyset$ and $l \neq$ the characteristic, and that there are duality theorems in Galois cohomology (Takahashi [6], Tate [7] and Uchida [8]). In this paper we shall study the maximal unramified extension of F (i.e., $S = \emptyset$). The author should point out that Milne [4] has found a duality theorem which is one of the results obtained here by more elementary means.

0. Notations. Let Z, Q, Z_l and Q_l denote the ring of integers, the field of rational numbers, the ring of l-adic integers and the field of l-adic numbers for a prime number l, respectively. By m we shall understand a power of the prime number l in question. We put $A^* = \mathrm{Hom}(A, Q/Z)$, $A_m = \{a \in A \mid ma = 0\}$, ${}_mA = A/mA$ and $A(l) = A \otimes Z_l$ for a module A. If A is a G-module, we let A^G denote the subgroup of all G-invariant elements of A; $A^G = H^0(G, A)$. Throughout this paper we assume that the constant field of the algebraic function field F is finite and of characteristic p, and that the genus of F is not zero. We use following notations;

μ: the group of roots of unity,
U: the group of unit ideles,
V: the group of unit idele classes,
C: the group of idele classes,
Cl: the group of divisor classes,
Cl^0: the group of divisor classes of degree 0, i.e., the torsion part of Cl.

Then we have exact sequences

$$(1) \qquad 0 \longrightarrow \mu \longrightarrow U \longrightarrow V \longrightarrow 0\,,$$

$$(2) \qquad 0 \longrightarrow V \longrightarrow C \longrightarrow Cl \longrightarrow 0$$

and

$$(3) \qquad 0 \longrightarrow Cl^0 \longrightarrow Cl \xrightarrow{\deg} Q$$

where the map deg means $(f/[K: F])\deg_K$ on Cl_K with the degree f of the constant field extension in a finite extension K/F. It is well known (by the exact sequences (1) and (2)) that there is an exact sequence

$$(4)\qquad 0 \longrightarrow H^1(G(K/F), \mu_K) \longrightarrow Cl_F \longrightarrow Cl_K^{G(K/F)} \longrightarrow H^2(G(K/F), \mu_K) \longrightarrow 0$$

for an unramified Galois extension K of F (possibly of infinite degree).

1. **The maximal unramified extension.** Let Ω be an unramified Galois extension of F with Galois group G satisfying the following three conditions for a fixed prime number $l \neq p$:

(A) Every proper l-extension of Ω ramifies.

(B) $l^\infty \mid [\Omega: \Omega^0]$, where Ω^0 is the maximal constant field extension of F contained in Ω.

(C) $\Omega \supset \mu_l$.

Of course, when Ω is the maximal unramified extension of F, it satisfies the above three conditions.

For each $c \in (Cl_\Omega)_l$, there exists a finite extension K of F contained in Ω and there exists a divisor D of K representing c such that $lD = (f)$ is a divisor of a function f of K. Since the field $K(f^{1/l})$ is an unramified l-extension of K if we choose K containing μ_l, we have $f^{1/l} \in \Omega$ by the condition (A) hence $c = 0$. This shows that Cl_Ω has no l-primary torsion part. Since Cl_Ω/Cl_Ω^0 is l-divisible by the exact sequence (3) and by the condition (B), Cl_Ω is uniquely l-divisible. Using the exact sequence (4), we have an isomorphism

$$(5)\qquad H^1(G, \mu(l)) \cong Cl_F^0(l)\ .$$

Consider a commutative exact diagram

$$\begin{array}{ccccccc}
 & & & & 0 & & 0 \\
 & & & & \uparrow & & \uparrow \\
 & & & & H^2(G, \mu) & \dashrightarrow & Q/Z \\
 & & & & \uparrow & & \uparrow \\
0 & \longrightarrow & Cl_\Omega^{0G} & \longrightarrow & Cl_\Omega^G & \longrightarrow & Q \\
 & & \uparrow & & \uparrow & & \uparrow \\
0 & \longrightarrow & Cl_F^0 & \longrightarrow & Cl_F & \longrightarrow & Z & \longrightarrow & 0\ .
\end{array}$$

Both the kernel and the cokernel of the induced homomorphism of $H^2(G, \mu)$ into Q/Z have no l-primary torsion part, and the image of that is l-divisible, hence we have

$$(6)\qquad H^2(G, \mu(l)) \cong Q_l/Z_l\ .$$

Since U_Ω and Cl_Ω are cohomological l-trivial modules in the exact sequences (1) and (2), we get isomorphisms

$$H^r(G, \mu(l)) \cong H^{r-1}(G, V_\Omega)(l) \cong H^{r-1}(G, C_\Omega)(l)$$

for $r \geqq 3$. Hence we have

$$H^3(G, \mu(l)) = Q_l/Z_l \tag{7}$$

and

$$H^r(G, \mu(l)) = 0 \qquad (r \geqq 4) \tag{8}$$

cf. [6; § 3, Lemma 1]. Now, it is easy to determine the cohomology groups of m-th roots of unity, using the exact sequence

$$0 \longrightarrow \mu_m \longrightarrow \mu(l) \xrightarrow{m} \mu(l) \longrightarrow 0$$

and the isomorphisms (5), (6), (7) and (8):

$$0 \longrightarrow {}_m\mu_F \longrightarrow H^1(G, \mu_m) \longrightarrow (Cl_F^0)_m \longrightarrow 0 \qquad \text{(exact)}, \tag{9}$$

$$H^2(G, \mu_m) \cong {}_mCl_F , \tag{10}$$

$$H^3(G, \mu_m) \cong Z/mZ \tag{11}$$

and

$$H^r(G, \mu_m) = 0 \qquad (r \geqq 4) . \tag{12}$$

By a G-module we shall always understand a discrete G-module. For a G-module M, we put

$$D_r(M) = \varinjlim_K H^r(G(\Omega/K), M)(l)^* ,$$
$$E_r = D_r(Z) ,$$

the limit being taken over the extensions of F contained in Ω of finite degree, and with respect to cores*, and put

$$E_r' = \varinjlim_m D_r(Z/mZ) .$$

Then Tate showed the following theorems (I) and (I)′ (cf. Serre [5; Chap. I, Annexe]):

(I) $H^r(G, \mathrm{Hom}(M, E_n)) \cong H^{n-r}(G, M)(l)^*$ for all r and for all G-modules M of finite type over Z if and only if $\mathrm{scd}_l\, G = n$, E_n is divisible and $D_r(Z) = 0$ for $r < n$.

(I)′ $H^r(G, \mathrm{Hom}(M, E_n')) \cong H^{n-r}(G, M)^*$ for all r and for all finite l-primary G-modules M if and only if $\mathrm{cd}_l\, G = n$ and $D_r(Z/lZ) = 0$ for $r < n$.

For any unramifield Galois extension Ω of F with group G, we have, by class field theory,

$$D_1(Z/mZ) \cong \varinjlim_K {}_mCl_K \cong {}_mCl_\Omega$$

and

$$D_2(Z) \cong \varinjlim_K H^1(G(\Omega/K),\ Q_l/Z_l)^* \cong \varinjlim_K Cl_K(l) \cong Cl_\Omega(l)\ .$$

THEOREM 1. *Let l be a prime number $\neq p$ and let Ω be an unramified Galois extension of F with Galois group G satisfying the three conditions* (A), (B) *and* (C). *Then we have*

(i) $\mathrm{cd}_l G = \mathrm{scd}_l G = 3$,

(ii) $H^{3-r}(G, M)^* \cong H^r(G, \mathrm{Hom}(M, \mu(l))$

for all r and for all finite l-primary G-modules M,

(iii) $H^3(G, M)(l)^* \cong \mathrm{Hom}_G(M, \mu(l))$

for all G-modules M of finite type over Z.

PROOF. (i): Let H be a l-Sylow subgroup of G and L be its invariant field. Then we have $L \supset \mu_l$ and

$$H^4(H, Z/lZ) \cong H^4(H, \mu_l) \cong \varinjlim_{K\subset L} H^4(G(\Omega/K),\ \mu_l)\ .$$

We get $H^4(H, Z/lZ) = 0$ by (12) and we get $\mathrm{cd}_l G = 3$. Using the isomorphism (11): $H^3(G, \mu_m) \cong Z/mZ$ and $\mu(l) \cong Q_l/Z_l$ as abelian groups, the dualizing module E_3' must be isomorphic to the module $\mu(l)$ as G-modules by the same way as the proof of Th. 1 in Chap. II, section 5 of Serre [5]. Since $\mu_K(l)$ are finite for all extensions K of F of finite degree, we get $\mathrm{scd}_l\, G = 3$.

(ii): $D_0(Z/lZ) = 0$ by $l \mid [\Omega\colon F]$ and $D_1(Z/lZ) \cong {}_lCl_\Omega = 0$, for Cl_Ω is l-divisible. Using the isomorphism (10), we have

$$D_2(Z/lZ) \cong \varinjlim_K H^2(G(\Omega/K),\ \mu_l)^* \cong \varinjlim_K ({}_lCl_K)^* \cong (\varprojlim_K {}_lCl_K)^*\ ,$$

the projective limit being taken with respect to the norm map. Let L be the unramified class field over K for the subgroup lCl_K of Cl_K, then the norm map of ${}_lCl_L$ into ${}_lCl_K$ is the null map. Hence we have $D_2(Z/lZ) = 0$. By the Tate's duality theorem (I)′ we get the isomorphisms (ii).

(iii): We have

$$E_3 \cong \varinjlim_K H^2(G(\Omega/K),\ Q_l/Z_l)^* \cong \varinjlim_K \varprojlim_m H^2(G(\Omega/K),\ Z/ZmZ)^*$$
$$\cong \varinjlim_K \varprojlim_m H^1(G(\Omega/K),\ \mu_m)\ ,$$

the last isomorphism is given by the isomorphisms (ii). Consider a commutative exact diagram (cf. (9))

$$\begin{array}{ccccccccc} 0 & \longrightarrow & {}_m\mu_K & \longrightarrow & H^1(G(\Omega/K),\ \mu_m) & \longrightarrow & (Cl_K^0)_m & \longrightarrow & 0 \\ & & \uparrow 1 & & \uparrow l & & \uparrow l & & \\ 0 & \longrightarrow & {}_{ml}\mu_K & \longrightarrow & H^1(G(\Omega/K),\ \mu_{ml}) & \longrightarrow & (Cl_K^0)_{ml} & \longrightarrow & 0\ . \end{array}$$

Since Cl_K^0 is finite, $\varprojlim_m (Cl_K^0)_m = 0$. Hence we have

$$\varprojlim_m H^1(G(\Omega/K),\ \mu_m) \cong \varprojlim_m {}_m\mu_K \cong \mu_K(l)$$

and

$$E_3 \cong \varinjlim_K \mu_K(l) \cong \mu(l)\ . \qquad \text{Q.E.D.}$$

2. **The maximal unramified l-extension.** Let Ω_l be the maximal unramified l-extension of F. It is easy to see that Ω_l is a constant field extension of F if and only if the class number h_F of F (i.e., the order of Cl_F^0) is prime to l. When $l \mid h_F$, we have $l^\infty \mid [\Omega\colon \Omega^0]$ (the condition (B)) where Ω^0 is the maximal constant field extension of F contained in Ω, because the l-class field tower of F is infinite by Madan [3].

THEOREM 2. *Let Ω be an unramified Galois extension of F with Galois group G satisfying the condition* (A) *and* (B). *If $\Omega \not\supset \mu_l$ or $l = p$, then we have*

(i) *$Cl(l)$ is a formation for the extension Ω/F, that is,*

$$Cl_K(l) \cong H^0(G(L/K),\ Cl_L(l))$$

for each Galois extension L/K of finite degree such that $\Omega \supset L \supset K \supset F$.

(ii) $\mathrm{cd}_l G = \mathrm{scd}_l G = 2$.

(iii) $H^{2-r}(G,\ M)(l)^* \cong H^r(G,\ \mathrm{Hom}(M,\ Cl_\Omega(l))$

for all r and for all G-modules M of finite type over Z.

PROOF. (i): Consider the exact sequence (4):

$$0 \longrightarrow H^1(G(L/K),\ \mu_L) \longrightarrow Cl_K \longrightarrow Cl_L^{G(L/K)} \longrightarrow H^2(G(L/K),\ \mu_L) \longrightarrow 0\ .$$

Since $H^r(G(L/K),\ \mu_L)(l) = 0$ by the assumption $\Omega \not\supset \mu_l$, we have

$$Cl_K(l) \cong Cl_L^{G(L/K)}(l) \cong Cl_L(l)^{G(L/K)}\ .$$

(ii): Let ω_L denote the norm residue map of the idele class group C_L into the Galois group $G(\Omega/L)^{ab}$ of the maximal abelian extension of L contained in Ω. Since Ω contains the maximal unramified abelian l-extension of L, $(\mathrm{Ker}\,\omega_L)/V_L$ and Coker ω_L are uniquely l-divisible. In the exact sequence (1), μ_L is uniquely l-divisible and U_L is cohomologically trivial, hence $\mathrm{Ker}\,\omega_L$ is cohomologically l-trivial. By Brumer [2], we get $\mathrm{scd}_l\, G \leqq 2$. And we have $\mathrm{cd}_l G = \mathrm{scd}_l G = 2$, since the torsion part $Cl^0_\Omega(l)$ of the dualizing module $E_2 = Cl_\Omega(l)$ is not zero.

(iii): To show this duality it suffices to show that $E_2 = Cl_\Omega(l)$ is divisible. Let K be an extension of F of finite degree contained in Ω, and let Ω_1 be the maximal constant field extension of K. We abbreviate $Cl^0_{\Omega_0}$ by Cl^0_0 and $Cl^0_{\Omega_1}$ by Cl^0_1 where $\Omega_0 = \Omega_1 \cap \Omega$. We put $H = G(\Omega_1/\Omega_0)$. Then the "Jacobian variety" Cl^0_1 of K is divisible and $l \nmid [\Omega_1 \colon \Omega_0]$. Using the exact sequence

$$0 \longrightarrow (Cl^0_1)_l \longrightarrow Cl^0_1 \xrightarrow{\ l\ } Cl^0_1 \longrightarrow 0$$

and by $H^0(H, Cl^0_1) = Cl^0_0$, we get an exact sequence

$$Cl^0_0 \xrightarrow{\ l\ } Cl^0_0 \longrightarrow H^1(H, (Cl^0_1)_l)\ .$$

Since $(Cl^0_1)_l$ is an l-primary torsion group and $l \nmid (H\colon 1)$, we have $H^1(H, (Cl^0_1)_l) = 0$. Hence Cl^0_0 and $Cl^0_\Omega \cong \varinjlim_K Cl^0_0$ are l-divisible. Consequently, $Cl_\Omega(l)$ is l-divisible, for $Cl_\Omega(l)/Cl^0_\Omega(l)$ is isomorphic to Q_l by the condition (B). Q.E.D.

3. **Remarks.** Let Ω and Ω_l denote the maximal unramified Galois extension of F and the maximal unramified l-extension of F respectively. Put $G = G(\Omega/F)$ and $G(l) = G(\Omega_l/F)$.

REMARK 1. There is an isomorphism

$$H^3(G, M)^* \cong \mathrm{Hom}_G(M, \mu)$$

for each G-module M by Th. 1 and Th. 2.

REMARK 2. For the Galois group N of the extension Ω over Ω_l, we have $H^r(N, Z/mZ) = H^r(N, Z)(l) = 0$ for $r \geqq 1$. Hence we have

$$H^r(G(l), M)(l) \cong H^r(G, M)(l)$$

for all r and for all $G(l)$-modules M.

REMARK 3. Let q be the number of elements of the constant field of F. Then F contains the l-th roots of unity if and only if $q \equiv 1 \bmod l$. We see by Th. 1 that if $q \equiv 1 \bmod l$ and $h_F \equiv 0 \bmod l$, then the Galois group $G(l)$ of the maximal unramified l-extension over F is a Poincaré pro-l-group of dimension 3, cf. Serre [5; Chap.I, n^0 4.5].

REMARK 4. Let M be a finite G-module. It can be proved by the method of Serre [5; Chap. II, n^0 5.7] that the "Euler-Poincaré characteristic" of M has the value one.

$$\chi_F(M) = \frac{|H^0(G, M)|\,|H^2(G, M)|}{|H^1(G, M)|\,|H^3(G, M)|} = 1 .$$

REFERENCES

[1] A. BRUMER, Galois groups of extensinos of algebraic number fields with given ramification, Michigan Math. J., 13 (1966), 33-40.

[2] A. BRUMER, Pseudocompact algebras, profinite groups and class formations, J. of Alg., 4 (1966), 442-470.

[3] M. MADAN, Class number relations in fields of algebraic functions, J. reine u. angew. Math., 238 (1969), 59-92.

[4] J. MILNE, The Tate-Safarevic group of a constant abelian variety, Inventions math., 6 (1968), 91-105.

[5] J.—P. SERRE, Cohomologie galoisienne, 1965, Springer.

[6] T. Takahashi, Galois cohomology of finitely generated modules, Tôhoku Math. J., 21 (1969), 102-111.

[7] J. TATE, Duality theorems in Galois cohomology over number fields, Proc. Intern. Congress Math., 1962, 288-295.

[8] K. UCHIDA, On Tate's duality theorms in Galois cohomology, Tôhoku Math. J., 21(1969), 92-101.

DEPARTMENT OF MATHEMATICS
YAMAGATA UNIVERSITY
YAMAGATA, JAPAN