Tôhoku Math. Journ.
24 (1972), 11-19.

A REMARK ON KÄHLERIAN PINCHING

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(Received Jan. 6, 1971; Revised Sept. 23, 1971)

1. Let M be a complete Kähler manifold of complex dimension n. If P is a 2-plane in the real tangent space M_p at some point $p \in M$, we recall that the *Kählerian sectional curvature* $k^*(P)$ is defined as follows:

$$k^*(P) = 4k(P)/(1 + 3\langle X, JY\rangle^2)$$

where $k(P)$ is the Riemannian sectional curvature, $\langle\ ,\ \rangle$ is the metric, J is the almost complex structure tensor, and X, Y is an orthonormal basis for P. It is easy to check that $k^*(P)$ is independent of the choice of the orthonormal basis X, Y, and it is also well-known that for the special case in which M is complex projective space equipped with the metric of constant holomorphic curvature c, $k^*(P) = c$ for all sections.

We say that M is *δ-pinched*, where $0 < \delta \leqq 1$, if there exists a positive constant B such that $\delta B \leqq k^*(P) \leqq B$ for all sections P. Kählerian δ-pinched manifolds have been studied by do Carmo [2], Klingenberg [5], and Kobayashi [7], and their investigations have yielded the following theorem: *there exists a $\delta < 1$ such that any δ-pinched complete Kähler manifold is homotopically equivalent to complex projective space of the same dimension.* (The best estimate to date is $\delta = 9/16$, proved by Klingenberg [5, II]).

In addition, Cheeger [12] has studied a more general pinching condition for a larger class of manifolds, and his results contain, as a special case, the theorem that *a sufficiently pinched n-dimensional complete Kähler manifold is homeomorphic to complex projective n-space.* (Here the pinching may depend on the dimension). In the same paper, and also in an abstract in the AMS Notices for January, 1969, Cheeger has announced that his result may be strengthened to yield a diffeomorphism.

In this note we wish to point out that the theorem of Kobayashi may be combined with a theorem of Hirzebruch and Kodaira to obtain:

THEOREM. *There exists a sequence δ_n, with $0 < \delta_n < 1$, such that any n-dimensional δ_n-pinched complete Kähler manifold is biholomorphically equivalent to complex projective n-space.*

A key step in the proof is in establishing a relation between the curvature of a pinched Kähler manifold and its volume. In particular, we shall need the following lemma, to be proved in § 3:

LEMMA 1. *Given any* A, $0 < A < 1$, *and a positive integer* n, *there exists an* ε, $0 < \varepsilon < 1$, *with the following property: if* M *is any* n*-dimensional* ε*-pinched complete Kähler manifold whose Kählerian sectional curvature satisfies* $k^* \leqq A$ *for every section, then*

$$\text{volume}(M) > (4\pi)^n/n!\ .$$

Finally, it should be mentioned that the above theorem is an easy consequence of Cheeger's result together with the main theorem of Hirzebruch and Kodaira [4], which asserts that if M is a compact Kähler manifold diffeomorphic to complex projective space $\boldsymbol{P}^n$ whose first Chern class is not negative definite, then M is biholomorphically equivalent to $\boldsymbol{P}^n$. (As was observed by Morrow [13], the hypothesis may be weakened to homeomorphism, using the result of Novikov on invariance of the rational Pontrjagn classes). The chief purpose, then, of this paper is to present a more elementary proof of the main theorem above which does not require the use of Novikov's result or Cheeger's diffeomorphism theorem.

2. In this section we discuss the analytic aspects of the problem, and then show how the theorem is derived from lemma 1. For the moment, all that we require of M is that it be a compact n-dimensional Kähler manifold with positive definite first Chern class, whose cohomology ring is isomorphic to that of complex projective n-space, i.e., $H^*(M) \cong H^*(\boldsymbol{P}^n)$. It follows from a theorem of Kodaira that M is projective algebraic [3, theorem 18.1.2]. (We refer to [3] for basic facts concerning Chern classes, line bundles, etc.). Also, since $c(K) = -c_1(M)$ is negative definite, where $c(K)$ is the Chern class of the canonical line bundle, it follows from Kodaira's vanishing theorem [3, theorem 18.2.2] that for the sheaf $\mathcal{O}$ of germs of holomorphic sections we have $H^j(M, \mathcal{O}) = 0$, $j \geqq 1$. Thus from the sequence $0 \to \boldsymbol{Z} \to \mathcal{O} \to \mathcal{O}^* \to 0$, we obtain $H^1(M, \mathcal{O}^*) \cong H^2(M, \boldsymbol{Z})$, so that the line bundles on M are in one-to-one correspondence with their Chern classes.

We choose an isomorphism $H^2(M, \boldsymbol{Z}) \cong \boldsymbol{Z}$ such that the generator $e \in H^2(M, \boldsymbol{Z})$ corresponding to 1 is represented by a positive definite form, and we shall denote line bundles and cohomology classes with the corresponding integer under this isomorphism. The theorem of Hirzebruch and Kodaira [4, theorem 6] referred to earlier, implies the following: *if* $\dim H^0(M, q) = \dim H^0(\boldsymbol{P}^n, q)$ *for* $q = 0, 1, 2, \cdots$, *then* M *is biholomorphically equivalent to* $\boldsymbol{P}^n$. Using this, we prove:

LEMMA 2. *Let M be a compact n-dimensional Kähler manifold with positive definite first Chern class, $c_1 > 0$, and assume $H^*(M, \boldsymbol{Z}) \cong H^*(\boldsymbol{P}^n, \boldsymbol{Z})$. If $c_1 > n-1$, then M is biholomorphically equivalent to $\boldsymbol{P}^n$.*

PROOF. By Kodaira's vanishing theorem, $\dim H^j(M,q) = \dim H^j(\boldsymbol{P}^n,q) = 0$ for $j \geqq 1$ and $q \geqq 0$, so that

$$\dim H^0(M, q) = \sum_{j=0}^{n} (-1)^j \dim H^j(M, q) = \chi(M, q)$$

for $q \geqq 0$, and the same is true for $\dim H^0(\boldsymbol{P}^n, q)$. From the Hirzebruch-Riemann-Roch theorem for line bundles [3, theorem 20.3.2], $\chi(M, q)$ and $\chi(\boldsymbol{P}^n, q)$ are both polynomials of degree n in q (with rational coefficients), and they have the same leading coefficient. To show that they agree for all q, it suffices to show that they agree for any n distinct values of q.

We first observe that $\chi(M, 0) = \chi(\boldsymbol{P}^n, 0) = 1$. Next, if $-1 \geqq q \geqq -(n-1)$, Kodaira's vanishing theorem implies

$$\dim H^j(M, q) = \dim H^j(\boldsymbol{P}^n, q) = 0$$

for $0 \leqq j \leqq n-1$. Applying Serre's duality theorem [3, theorem 15.4.3], and observing that for the canonical bundles $K(M)$ and $K(\boldsymbol{P}^n)$ we have $c(K(\boldsymbol{P}^n)) = -(n+1)$ and $c(K(M)) = -c_1 \leqq -n$, we obtain

$$\dim H^n(\boldsymbol{P}^n, q) = \dim H^0(\boldsymbol{P}^n, -(n+1+q)) = 0$$

and

$$\dim H^n(M, q) = \dim H^0(M, -(c_1 + q)) = 0 .$$

Therefore, $\chi(M, q) = \chi(\boldsymbol{P}^n, q)$ for $q = 0, -1, \cdots, -(n-1)$, and the lemma follows from the Hirzebruch-Kodaira theorem.

We are now in a position to prove the theorem, assuming lemma 1. We first recall that the curvature assumptions imply that M is compact, by a theorem of Myers [9]. Next we define on M two exterior forms of type (1, 1): the *fundamental form*

$$\omega(X, Y) = \frac{1}{2}\langle JX, Y\rangle$$

and the *Ricci form*

$$\sigma(X, Y) = S(JX, Y) ,$$

where S is the Ricci tensor on M. We shall call the metric on M a *generating metric* if the form $\omega/2\pi$ generates $H^2(M, \boldsymbol{Z})$. (Note that in the case $M = \boldsymbol{P}^n$ the metric of constant holomorphic curvature 1 is a generating metric, and $\sigma = (n+1)\omega$).

Now for a given n, take $A = n/(n+1)$ in lemma 1, and let ε_n denote the corresponding ε. Let $\delta_n = \max[9/16, \varepsilon_n, (n-1)/n]$, and suppose that M is δ_n-pinched. By normalizing, we may assume that the metric on M is a generating metric. Corresponding to this metric there is some $B > 0$ such that

$$\delta_n B \leqq k^* \leqq B .$$

We claim that $B > n/(n+1)$; for otherwise, by lemma 1 and our choice of δ_n, we would have

$$\text{volume}(M) > (4\pi)^n/n! .$$

However, the volume form on M is easily seen to be the form $(2\omega)^n/n!$ and since the metric on M is a generating metric and $H^*(M, \boldsymbol{Z}) \cong H^*(\boldsymbol{P}^n, \boldsymbol{Z})$, we have:

$$\text{volume}(M) = \int_M \frac{(2\omega)^n}{n!} = \frac{(4\pi)^n}{n!}\int_M \left(\frac{\omega}{2\pi}\right)^n = \frac{(4\pi)^n}{n!} .$$

Let X be a unit vector in the tangent space at some $p \in M$, and extend X to an orthonormal basis of the form $X = X_1, JX_1, X_2, JX_2, \cdots, X_n, JX_n$. Then

$$\begin{aligned}
\sigma(X, JX) &= S(X, X) \\
&= k(X_1 \wedge JX_1) + \sum_{j=2}^{n} \{k(X_1 \wedge X_j) + k(X_1 \wedge JX_j)\} \\
&= k^*(X_1 \wedge JX_1) + \frac{1}{4}\sum_{j=2}^{n} \{k^*(X_1 \wedge X_j) + k^*(X_1 \wedge JX_j)\} \\
&\geqq \left(\frac{n+1}{2}\right)\delta_n B \\
&> \frac{n}{2}\delta_n \\
&\geqq \frac{(n-1)}{2} \\
&= (n-1)\omega(X, JX) .
\end{aligned}$$

Since the first Chern class of M is represented by the form $\sigma/2\pi$ (see, for example [1, § 7–8]), and since $\omega/2\pi$ generates $H^2(M, \boldsymbol{Z})$, it is evident that $c_1 > (n-1)$, and the theorem now follows from lemma 2.

3. The proof of lemma 1 may be achieved in either of two ways: by applying the techniques and estimates of do Carmo [2] or Cheeger [12] or by using the circle bundle construction of Kobayashi [7]. The latter approach is presented below.

For convenience, we reformulate lemma 1 in the following way, which is easily seen to be equivalent:

LEMMA 1′. *Given any* λ, $0 < \lambda < 1$, *and a positive integer* n, *there exists an* ε, $0 < \varepsilon < 1$, *such that if* M *is any* n*-dimensional Kähler manifold whose Kählerian sectional curvature satisfies* $\varepsilon \leqq k^* \leqq 1$ *for every section, then*

$$\text{volume}(M) > \lambda(4\pi)^n/n!\ .$$

Next, we prove the following Riemannian version:

LEMMA 3. *Let* P *be a simply-connected* m*-dimensional Riemannian manifold whose sectional curvature satisfies* $c/4 \leqq k \leqq c$ *for every section, where* c *is some positive constant. Then*

$$\text{volume}(P) \geqq \text{volume}(S^m(c))$$

where $S^m(c)$ *is the* m*-sphere of constant curvature* c.

PROOF. Fix points $p \in P$ and $p^* \in S^m(c)$, and choose an isometry $F: P_p \to S^m(c)_{p^*}$ between the tangent spaces. Let $E: P_p \to P$ and $E^*: S^m(c)_{p^*} \to S^m(c)$ be the exponential maps. Since $k \leqq c$, a theorem of Klingenberg [6] asserts that E is a diffeomorphism between the open ball of radius $\pi/\sqrt{c}$ in P_p and the open ball of the same radius in P. A similar statement holds for $S^m(c)$. If we let U (resp. U^*) denote the open ball of radius $\pi/\sqrt{c}$ around p (resp. p^*) in M (resp. $S^m(c)$), it follows that the map $\Phi = E^* \circ F \circ E^{-1} | U$ is a diffeomorphism of U onto U^*.

From the Rauch comparison theorem [10], it follows easily that the differential $d\Phi$ has norm at most 1 at each point of U. Since moreover U^* is a dense open subset of S^m, we obtain:

$$\begin{aligned} \text{volume}(M) \geqq \text{volume}(U) &\geqq \text{volume}(U^*) \\ &= \text{volume}(S^m(c))\ , \end{aligned}$$

which completes the proof.

We now consider a Kähler manifold of complex dimension n whose Kählerian sectional curvature satisfies $\varepsilon \leqq k^* \leqq 1$ for some $\varepsilon > 0$. We choose ε sufficiently large so that M is homotopically equivalent to $\boldsymbol{P}^n$. Let g denote the Kähler metric and ω the fundamental form on M. Since $b_2(M) = 1$, there is a positive real number b such that $4\omega/b$ generates $H^2(M, \boldsymbol{Z})$. We may assume that $b < 8\pi$, for otherwise

$$\text{volume}(M) = \int_M \frac{(2\omega)^n}{n!} = \frac{b^n}{2^n n!}\int_M \left(\frac{4\omega}{b}\right)^n \geqq \frac{(4\pi)^n}{n!},$$

and lemma 1′ would be proved.

The construction in [7] now proceeds as follows. There exists a principal circle bundle P over M, and a connection 1-form γ on P such that $d\gamma = \rho^*(4\omega/b)$, where $\rho: P \to M$ is the projection. If we define a metric $h = \rho^*(g) + (ab\gamma)^2$ on P, where $a = \sqrt{\varepsilon}/2$, then the sectional curvatures on P satisfy [7, section 4]

$$a^2 \leqq k \leqq 1 - 3a^2 . \tag{3.1}$$

We next prove two lemmas.

LEMMA 4. volume $(P) = ab$ volume (M).

PROOF. As in [7], we identify S^1 with $\boldsymbol{R}/\boldsymbol{Z}$, and let x_0 be the coordinate on $\boldsymbol{R}$. Around a point $p \in P$ we may choose coordinates $x_0, x_1, \cdots, x_n$, where $x_1, \cdots, x_n$ are coordinates around $\rho(p)$ in M. Then

$$\gamma = dx_0 + \sum_{j=1}^n \gamma_j dx_j ,$$

so that the metric takes the matrix form

$$h = \left[\begin{array}{c|c} a^2b^2 & a^2b^2\gamma_1 \cdots a^2b^2\gamma_n \\ \hline a^2b^2\gamma_1 & \\ \vdots & g_{ij} + a^2b^2\gamma_i\gamma_j \\ a^2b^2\gamma_n & \end{array}\right] .$$

It follows easily that det $h = a^2b^2$ det g, and the lemma follows from Fubini's theorem.

LEMMA 5. *P is simply-connected.*

PROOF. Let $\{U_\alpha\}$ be a covering of M such that $P|U_\alpha$ is trivial. Choose coordinates $x_0^\alpha, x_1^\alpha, \cdots, x_n^\alpha$ for $P|U_\alpha$ as in lemma 4, where x_0^α is determined mod 1. Then P is determined by maps $f_{\alpha\beta}: U_\alpha \cap U_\beta \to S^1$, and if we choose $\{U_\alpha\}$ so that $U_\alpha \cap U_\beta$ is simply connected, then we can lift $f_{\alpha\beta}$ to $\hat{f}_{\alpha\beta}: U_\alpha \cap U_\beta \to \boldsymbol{R}$ such that $x_0^\alpha = x_0^\beta + \hat{f}_{\alpha\beta}(x_1^\beta, \cdots, x_n^\beta)$ mod 1. As was pointed out in [7], the exact sequence of sheaves $0 \to \boldsymbol{Z} \to \tilde{R} \to \tilde{S}^1 \to 0$ (where the bar denotes germs of differentiable sections) yields an isomorphism $H^1(M, \tilde{S}^1) \cong H^2(M, \boldsymbol{Z}) = \boldsymbol{Z}$. The principal S^1-bundles over M may be identified with $H^1(M, \tilde{S}^1)$, and P corresponds under the above isomorphism to a generator of Z. The bundle mP, then, is determined by the cocycle $\{mf_{\alpha\beta}\}$. With these identifications, we may, for $m \neq 0$, construct an $|m|$-fold covering of mP by P as follows: $P|U_\alpha$ and $mP|U_\alpha$ are both trival, and we map $(x_0^\alpha, x_1^\alpha, \cdots, x_n^\alpha) \in P$ into $(mx_0^\alpha, x_0^\alpha, \cdots, x_n^\alpha) \in mP$.

Now let $\tilde{P}$ be the universal covering space of P with covering map $\tau: \tilde{P} \to P$, and let F be a fibre of P. It follows from (3.1) and Meyers' theorem that $\tilde{P}$ is compact. From the homotopy sequence

$$\cdots \to \pi_2(M) \to \pi_1(F) \to \pi_1(P) \to \pi_1(M) = 0$$

we see that $\pi_1(P)$ is finite cyclic and generated by the inclusion map $S^1 \approx F \subsetneqq P$. We may therefore conclude that $\tau^{-1}(F)$ is connected. For if $\tilde{p}_1$ and $\tilde{p}_2$ are two points of $\tilde{P}$ lying above the point $p \in F$, and if σ is a path from $\tilde{p}_1$ to $\tilde{p}_2$, then $\tau(\sigma)$ is homotopic to a path lying in F. Thus, by the homotopy lifting property, $\tilde{p}_1$ and $\tilde{p}_2$ are connected by a path in $\tau^{-1}(F)$.

It follows that every fibre of $\rho\tau: \tilde{P} \to M$ is homeomorphic to S^1, and the action of S^1 on P lifts to an action on $\tilde{P}$. It is easy, then, to verify that $\tilde{P}$ is a principal S^1-bundle over M, hence $\tilde{P} = mP$ for some $m \neq 0$. Thus P is a covering space of $\tilde{P}$, from which it follows that they are homeomorphic, and the lemma is proved.

Combining lemmas 3 through 5, and using (3.1) we obtain the inequalities:

$$\begin{aligned} 4\pi\sqrt{\varepsilon}\,\mathrm{volume}(M) &> \mathrm{volume}(P) \\ &> S^{2n+1}\left(\frac{4-3\varepsilon}{4}\right) \\ &= \frac{2^{n+1}\pi^{n+1}\left(\dfrac{2}{\sqrt{4-3\varepsilon}}\right)^{2n+1}}{2n(2n-2)\cdots 2} \end{aligned} \tag{3.2}$$

and hence

$$\mathrm{volume}(M) > \frac{(4\pi)^n}{n!\sqrt{\varepsilon}\,(\sqrt{4-3\varepsilon})^{2n+1}} \tag{3.3}$$

where we take $4/7 < \varepsilon < 1$. For a fixed n and a given $\lambda < 1$, then, we choose ε sufficiently close to 1 so that $\sqrt{\varepsilon}\,(\sqrt{4-3\varepsilon})^{2n+1} < 1/\lambda$. This completes the proof of lemma 1′.

4. Some comments:

(i) For $n = 3$ or 5 we may actually take $\delta_n = 9/16$ or any other constant which insures homotopy equivalence. In fact, it is possible to prove the following: *If M is a compact Kähler manifold of positive sectional curvature, where $n = 3$ or 4, and if $H^*(M, Z) \cong H^*(P^n, Z)$, then M is biholomorphically equivalent to P^n.*

The idea of the proof is as follows: identifying the Chern classes of M with integers, we have for $n = 4$, $c_4 = 5$. Also, from the Hirzebruch-Riemann-Roch theorem,

$$1 = \chi(M, \mathcal{O}) = \frac{1}{720}(-c_4 + c_3c_1 + 3c_2^2 + 4c_2c_1^2 - c_1^4)$$

and from the Hirzebruch index theorem,

$$1 = \text{index}(M) = \frac{1}{45}(14c_4 - 14c_3c_1 + 3c_2^2 + 4c_2c_1^2 - c_1^4) .$$

Using these equations, and the fact that $c_1 > 0$, one may compute that $c_1 \geqq 5$, so that lemma 2 may be applied.

For $n = 3$, we have $c_3 = 4$, and

$$1 = \chi(M, \mathcal{O}) = c_1c_2/24 .$$

The index theorem is vacuous in this dimension, but a recent theorem of Kobayashi and Ochiai [8] asserts that $H^j(M, \Theta) = 0$ for $j \geqq 1$, where Θ is the sheaf of germs of holomorphic vector fields, so, from the Hirzebruch-Riemann-Roch theorem, we have:

$$0 \leqq \chi(M, \Theta) = \frac{c_1^3}{2} + \frac{c_3}{2} - 19 .$$

As before, one may compute from these that $c_1 = 4$ and apply lemma 2.

(ii) It would seem likely that lemma 1 can be considerably strengthened. In fact, the proof of lemma 1 is the result of an attempt to prove the following conjecture: if M^n is a compact Kähler manifold such that $0 < k^* < 1$ for all sections, then volume $(M) > (4\pi)^n/n!$.

There is some evidence for this conjecture. First there is the Riemannian analogy provided by lemma 3 above. (In the even-dimensional case the lower bound on the curvature can be replaced by zero). Secondly, if it is true, as has been conjectured, that any compact Kähler M^n with $k^* > 0$ for all sections is biholomorphically equivalent to P^n, then the above conjecture is true as well. To see this, assume volume$(M) \leqq (4\pi)^n/n!$. Then we may obtain a generating metric by multiplying the original metric by some constant $a \geqq 1$. In this new metric the curvature still satisfies $0 < k^* < 1$, and as in §2, we may conclude that $c_1 < (n + 1)$, a contradiction if $M^n = P^n$.

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