Tôhoku Math. Journ.
23(1971), 605-620

ROTATION OF PLANE QUASICONFORMAL MAPPINGS(1)

G. D. ANDERSON AND M. K. VAMANAMURTHY

(Received Jan. 16, 1971)

1. Introduction. 1.1. Summary of results. The main purpose of this paper is to exhibit several quasiconformal mappings in space obtained by rotating important plane quasiconformal mappings. It is well known that, in general, such mappings need not be quasiconformal in space, even if the plane mapping is conformal. As a simple example, $w=f(z)=z^2$ maps the half plane $\mathrm{Re}\,z>0$ conformally onto the w-plane minus the ray $\mathrm{Re}\ w\leqq 0$, $\mathrm{Im}\ w=0$, while the space mapping F obtained by rotating f about the real axis fails to be quasiconformal because the dilatation $K(F)=\sup_{(\mathrm{Re}\ z>0)}|z|/(\mathrm{Re}\ z)$ is infinite. A simple sufficient condition is herein provided (§ 1. 3) that a space mapping obtained by rotation from a plane quasiconformal mapping be quasiconformal, and in each of our rotation theorems this condition is shown to hold. In one case we show that the space mapping is even extremal, given a certain very natural assumption.

Next, by a *configuration* is meant a plane domain Ω bounded by m disjoint Jordan curves with n_1 distinguished interior points and n_2 distinguished boundary points. When Ω is simply-connected (i. e., $m=1$), there are three types of configuration with exactly one conformal invariant (Cf. [1], p. 88), namely, those having (i) four distinguished boundary points a_1, a_2, a_3, a_4 $(n_1=0, n_2=4)$, (ii) two distinguished interior points a_1, a_2 $(n_1=2, n_2=0)$, and (iii) one interior point a_1 and two boundary points a_2, a_3 distinguished $(n_1=1, n_2=2)$. Conformal invariants in these cases are, respectively, (i) modulus of the quadrilateral $\Omega(a_1, a_2, a_3, a_4)$, (ii) hyperbolic distance between a_1 and a_2 with respect to Ω, (iii) harmonic measure of one of the boundary arcs a_2a_3 at a_1 with respect to Ω. Now suppose Ω is a half plane, and consider two configurations of the same type. In each case we show that if the extremal quasiconformal mapping of least dilatation of the first configuration onto the second is rotated about the boundary line, then the resulting space mapping is quasiconformal.

We begin by studying f_0 of smallest dilatation which takes $\bar{R}^2$ onto itself with $f_0(a_j)=b_j$, $j=1, 2, 3, 4$, where a_j and b_j are a preassigned pair of positively ordered quadruples of points on the real axis. For a particular choice of the a_j and b_j this

---

(1) This research was supported in part by the National Science Foundation, USA, Grants GP-13022 and GP-7234. A portion of the work is included in the second author's Ph. D. dissertation written under the direction of Professor F. W. Gehring at the University of Michigan in 1969.

mapping reduces to the extremal distortion mapping which shows that the linear distortion estimates of Hersch and Pfluger ([11], Cf. [15]) are best possible. The possibility of rotating the extremal distortion mapping to obtain a quasiconformal mapping in space was first suggested by a statement without proof in a paper of Syčev [19]. We show that rotation of this mapping yields, in fact, an extremal space mapping. The proof of this result makes no use of the properties of elliptic functions—in fact, only the Riemann Mapping Theorem and the Schwarz Lemma are needed. In § 2 we observe that the rotated mapping gives some information concerning estimates of linear distortion in space(2). As a corollary of our first theorem we derive an interesting monotone property for the moduli of Grötzsch rings.

In § 3 we consider the extremal mapping of a half plane with two distinguished interior points onto another such configuration. After finding the dilatation of the space mapping obtained by rotation about the boundary line, we employ the quasiconformality of this mapping to derive inequalities for harmonic measure.

Finally, in § 4, we study the extremal mapping f_0 of a half plane with one distinguished interior point and two distinguished boundary points onto another configuration of this type. We show that the space mapping F_0 obtained by rotation about the boundary line is $K(f_0)^3$-quasiconformal.

1. 2. Definitions and notation. Suppose that f is a diffeomorphism of an n-space domain Ω_1 onto Ω_2. Then for $P \in \Omega_1$ the differential $df = df(P)$ is affine and maps the unit ball onto an ellipsoid E. The lengths of the n semiaxes of E are called the *stretchings* of f at P. Let $L_n = L_n(P)$ and $l_n = l_n(P)$ denote the maximum and minimum stretchings of f at P. Then we define the dilatation of f by

$$K(f) = \sup_{P \in \Omega_1} \frac{L_n(P)}{l_n(P)}. \tag{1}$$

If this dilatation is finite we say that f is a *differentiable quasiconformal mapping* of Ω_1 onto Ω_2. If $K(f) \leqq K < \infty$, then we say that f is a *differentiable K-quasiconformal mapping* of Ω_1 onto Ω_2.

This definition may be generalized in the following way to include an arbitrary homeomorphism f of Ω_1 onto Ω_2. If f is differentiable with Jacobian $J > 0$ a. e. in Ω_1 and if f is *absolutely continuous on lines* (Cf. [9]), then we define $K(f)$ by taking the essential supremum in (1); otherwise we let $K(f) = \infty$. If this dilatation, as redefined, is finite, the mapping is said to be *quasiconformal*, and if $K(f) \leqq K < \infty$ it is called *K-quasiconformal*. The quasiconformal mappings

(2) Other distortion theorems making use of results of this paper appear in a paper of M. Virsu [20].

considered in this paper will be differentiable except at a finite number of points or on a finite number of smooth curves.

Next, we shall find it convenient in some of our proofs to make use of the hyperbolic density [17]. Suppose that G is a simply-connected domain in the z-plane, $z=x+iy$, with nondegenerate boundary, and that $w=g(z)$, $w=u+iv$, is a conformal mapping of G onto the half plane $v>0$. Then the *hyperbolic density* of G is defined by

$$(2) \qquad \rho(z)=\rho(z,G)=\frac{|g'(z)|}{2v},$$

and the density ρ is independent of the conformal mapping g. The hyperbolic density satisfies the transformation law

$$(3) \qquad \rho(z,G)=\rho(w,G')\left|\frac{dw}{dz}\right|$$

if $w=w(z)$ is a conformal mapping of G onto G'. If G_1 and G_2 are two simply-connected domains with nondegenerate boundaries such that $G_1<G_2$ (i.e., G_1 is a proper subset of G_2), then it follows by the Schwarz Lemma that

$$(4) \qquad \rho(z,G_1)>\rho(z,G_2) \text{ for } z\in G_1 .$$

If G is such a domain and if a_1 and a_2 are points in G, then the *hyperbolic distance* from a_1 to a_2, with respect to G, is

$$h=h(a_1,\ a_2)=\inf_{\gamma}\int_{\gamma}\rho(z)|dz|,$$

where the infimum is taken over all arcs joining a_1 and a_2 in G.

Finally, let G be a Jordan domain and γ a boundary arc. By the *harmonic measure* of γ at z with respect to G is meant the unique function $\omega=\omega(\gamma,z)$ which is bounded and harmonic in G, and which has boundary values 1 at all interior points of γ, and 0 at all points which are interior to the complementary arc.

1.3. Rotation of plane mappings. Suppose that G_1 and G_2 are domains in the x_1y_1- and x_2y_2-planes which are symmetric with respect to the x_1- and x_2-axes, respectively. Let $x_2+iy_2=f(x_1+iy_1)$ be a differentiable K-quasiconformal mapping of G_1 onto G_2 such that $f(x_1-iy_1)=\overline{f(x_1+iy_1)}$ for each point $x_1+iy_1\in G_1$. We may assume that $y_2>0$ for $y_1>0$. Let $P=x_1+iy_1$, $y_1\geqslant 0$, be a point in G_1, and let $L_2=L_2(P)$ and $l_2=l_2(P)$ be the maximum and minimum stretchings of f at P. Then $L_2/l_2 \leqq K$.

Next, let Ω_1 and Ω_2 be the domains in space obtained by rotating G_1 and G_2 about the x_1- and x_2-axes, respectively, and let (r, θ, x_1) and (s, ϕ, x_2) be cylindrical coordinates about the x_1- and x_2-axes, respectively. Then the mapping given by $F(r, \theta, x_1)=(s, \phi, x_2)$, where

$$x_2 + is = f(x_1 + ir), \ \phi = \theta,$$

is a diffeomorphism of Ω_1 onto Ω_2.

Now let P be a point in Ω_1. By symmetry we may assume that P lies in G_1 and that $P = x_1 + iy_1, y_1 \geqslant 0$. The maximum and minimum stretchings of F at P are given by

$$(5) \qquad \begin{cases} L_3 = \max(L_2, y_2/y_1), \ l_3 = \min(l_2, y_2/y_1) \ \text{for } y_1 > 0, \\ L_3 = L_2 \text{ and } l_3 = l_2 \ \text{for } y_1 = 0. \end{cases}$$

Thus a sufficient condition for F to be a differentiable quasiconformal mapping is that there exist $m \leqq 1 \leqq M$ such that

$$(6) \qquad ml_2 \leqq y_2/y_1 \leqq ML_2 \ \text{for } y_1 > 0,$$

in which case

$$(7) \qquad K(f) \leqq K(F) \leqq \frac{M}{m} K(f).$$

Finally, suppose that $m=1=M$ in (6). Then it follows from (7) that $K(F)=K(f)$. If we make the natural assumption that an extremal mapping of Ω_1 onto Ω_2 must take a plane section of Ω_1 containing the real axis onto a plane section of Ω_2 containing the real axis, then by (5) we may conclude that F is extremal. This is the case in Theorem 1 of this paper.

2. Rotation of the extremal distortion mapping. For $j=1,2,3,4$, let a_j and b_j denote a pair of positively ordered quadruples of points on the real axis in $\bar{R}^2$. Let $H_1 = H(a_1, a_2, a_3, a_4)$ and $H_2 = H(b_1, b_2, b_3, b_4)$ be the quadrilaterals formed by the upper half plane H with the vertices a_j and b_j, respectively. If f is any quasiconformal mapping of H_1 onto H_2 with vertices corresponding, then $\text{mod } H_2 \leqq K(f) \text{ mod } H_1$ (See [15]). If we take $K=(\text{mod } H_2)/(\text{mod } H_1)$, then there exists a unique extremal mapping f_0 of $\bar{R}^2$ onto itself such that $f_0(a_j)=b_j$, $j=1,2,3,4$, and $K(f_0)=K$.

We now briefly describe f_0. First, by performing preliminary Möbius transformations of $\bar{R}^2$ onto itself we may assume that the given pair of quadruples are 0, k_1, $1/k_1$, ∞ and 0, k_2, $1/k_2$, ∞. Since $K(f)=K(f^{-1})$ we may assume that $0<k_1 \leqq k_2<1$. If $k_1=k_2$ there will be nothing to prove; hence we take $0<k_1<k_2<1$. Then for $j=1,2$, let $z_j=x_j+iy_j$ and $w_j=u_j+iv_j$, and let

$$K_j = K(k_j),\ K_j' = K'(k_j) = K(k_j'),$$

where, for $0<k<1$, $K(k)$ and $K'(k)$ are the complete elliptic integrals[3] defined by

$$K = K(k) = \int_0^1 [(1-t^2)(1-k^2t^2)]^{-\frac{1}{2}}\,dt, \tag{8}$$

$$K' = K'(k) = K(k'),\ k' = (1-k^2)^{\frac{1}{2}}.$$

Next, let $z_j=g_j(w_j)$ map the rectangle R_j: $0<u_j<K_j$, $0<v_j<K_j'$ conformally onto the quadrilateral H_j with vertices corresponding[4]. Finally, the affine mapping

$$u_2+iv_2 = \varphi(u_1+iv_1) = \frac{K_2}{K_1}u_1 + i\frac{K_2'}{K_1'}v_1$$

carries R_1 onto R_2. Then $f_0=g_2\circ\varphi\circ g_1^{-1}$ is the required extremal mapping, after being extended by reflection in the real axis. This mapping is differentiable at all points of $\bar{R}^2$ but a_j, $j=1,2,3,4$.

To calculate the dilatation of f_0, let P_1 be any point in the upper half plane $\operatorname{Im} z_1>0$, and let $P_2=f_0(P_1)$, $Q_1=g_1^{-1}(P_1)$, and $Q_2=g_2^{-1}(P_2)$. The maximum and minimum stretchings $L_2(P_1)$ and $l_2(P_1)$ of f_0 at P_1 are

$$L_2 = \frac{K_2}{K_1}\frac{|g_2'(Q_2)|}{|g_1'(Q_1)|},\ l_2 = \frac{K_2'}{K_1'}\frac{|g_2'(Q_2)|}{|g_1'(Q_1)|}, \tag{9}$$

whence, because of symmetry and the removability of analytic arcs for quasiconformal mappings [15],

$$K = K(f_0) = \frac{L_2}{l_2} = \frac{K_1'K_2}{K_1K_2'}. \tag{10}$$

---

(3) Although K is being used in this paper to denote either the dilatation of a quasiconformal mapping or the value of an elliptic integral, the context will always make clear the meaning of K.

(4) This mapping is $g_j(w_j)=k_j sn^2(w_j, k_j)$, where sn denotes Jacobi's elliptic sine function [6; 14], but in our proofs that fact is not needed.

Next, for $0<k<1$, let $R(G, n, k)$ denote the Grötzsch ring in $\bar{R}^n$ whose boundary components are the segment $[0, k]$ of the x_1-axis and the sphere S^{n-1}. Let

$$m(n, k) = \text{mod } R(G, n, k). \tag{11}$$

It is well known [15] that

$$m(2, k) = \frac{\pi K'(k)}{2 K(k)}, \tag{12}$$

where $K(k)$ and $K'(k)$ are the elliptic integrals defined in (8). If $n=2$, then by (10) and (11) obviously $K = m(2, k_1)/m(2, k_2)$. Then f_0 is the *extremal distortion mapping* of $R(G, 2, k_1)$ onto $R(G, 2, k_2)$. That is,

$$\max \{|f(z)| : |z| = k_1\} \leqq k_2,$$

where f is any K-quasiconformal mapping of the unit disk onto itself with $f(0)=0$, and f_0 is the unique mapping of this class such that $f_0(k_1, 0) = (k_2, 0)$ and $K(f_0) = K$ (See [15]).

In order to prove that the space mapping F_0 obtained by rotating f_0 about the real axis is also quasiconformal, we shall employ the following lemma, which is an application of the Schwarz Lemma.

LEMMA 1. *For $j=1, 2$, let R_j be the rectangle $0<u_j<p_j$, $0<v_j<q_j$, where $0<p_1 \leqq p_2 < \infty$, $0<q_2 \leqq q_1 < \infty$, and let*

$$w_2 = u_2 + iv_2 = \varphi(w_1) = \frac{p_2}{p_1} u_1 + i \frac{q_2}{q_1} v_1$$

be the natural affine mapping of R_1 onto R_2. If $Q_1 \in R_1$ and $Q_2 = \varphi(Q_1) \in R_2$, and if $\rho(Q_j, R_j)$ is the hyperbolic density of R_j at Q_j, then

$$\frac{p_1}{p_2} \leqq \frac{\rho(Q_2, R_2)}{\rho(Q_1, R_1)} \leqq \frac{q_1}{q_2}, \tag{13}$$

with equality if and only if R_1 and R_2 are similar.

PROOF. We prove only the left side of (13), the right side being proved similarly. Because of the transformation law (3), the equality is obvious when the rectangles are similar. Otherwise, because of (3), it is sufficient to prove the lemma under the assumption that $p_1 = p_2 = 1$ and $0 < q_2 < q_1 < \infty$. Then the left side of (13) reduces to

$$\rho(Q_1, R_1) < \rho(Q_2, R_2). \tag{14}$$

To prove (14), identify the w_1- and w_2-planes so that the corresponding axes coincide. Then clearly $R_2 < R_1$ and Q_1 lies vertically above Q_2. Since

$$|Q_1 - Q_2| = v_1\left(1 - \frac{q_2}{q_1}\right) < q_1\left(1 - \frac{q_2}{q_1}\right) = q_1 - q_2,$$

it is clear that $R_2' < R_1$, where R_2' denotes the translate of R_2 by an amount $|Q_1 - Q_2|$. Hence by the Schwarz Lemma in terms of hyperbolic densities (4), we have

$$\rho(Q_1, R_1) < \rho(Q_1, R_2') = \rho(Q_2, R_2).$$

We now prove that the space mapping F_0 is quasiconformal. Under the assumption stated at the end of § 1.3 it is extremal.

THEOREM 1. *For $j = 1, 2, 3, 4$, let a_j and b_j be a pair of positively ordered quadruples of points on the real axis in $\bar{R}^2$. Let f_0 be the extremal mapping of least dilatation from $\bar{R}^2$ onto itself with $f_0(a_j) = b_j$, and let $K(f_0) = K$. If F_0 is the mapping of $\bar{R}^3$ onto itself obtained by rotating f_0 about the real axis in $\bar{R}^2$, then F_0 is an extremal quasiconformal mapping of $\bar{R}^3$ onto itself with $K(F_0) = K$.*

PROOF. As already remarked, we may assume that the given pair of quadruples are $0, k_1, 1/k_1, \infty$ and $0, k_2, 1/k_2, \infty$, with $0 < k_1 < k_2 < 1$. Since the mappings obtained by rotating the preliminary Möbius transformations about the real axis are again Möbius in space, this normalization does not affect the dilatations of the space mapping F_0.

Now let P_1 be any point in $\bar{R}^3 - \{0, k_1, 1/k_1, \infty\}$. By symmetry we may assume that $P_1 = (x_1, y_1, 0)$, $y_1 \geqq 0$. The three strecthings of F_0 at P_1 are easily seen to be

$$\begin{cases} L_2,\ l_2,\ y_2/y_1 \text{ for } y_1 > 0, \\ L_2,\ l_2,\ l_2 \text{ for } y_1 = 0;\ 0 < x_1 < k_1 \text{ or } x_1 > 1/k_1, \\ L_2,\ L_2,\ l_2 \text{ for } y_1 = 0;\ x_1 < 0 \text{ or } k_1 < x_1 < 1/k_1, \end{cases}$$

and it will follow that $L_3 = L_2$ and $l_3 = l_2$ at P_1 if

$$l_2 < \frac{y_2}{y_1} < L_2 \text{ for } y_1 > 0 .$$

But this follows immediately from Lemma 1, in view of (2) and (9).

From (9) and (10) it now follows that $L_3/l_3 = K$ at each point of $\bar{R}^3 - \{0, k_1, 1/k_1, \infty\}$. We conclude that F_0 is a differentiable quasiconformal mapping of $\bar{R}^3 - \{0, k_1, 1/k_1, \infty\}$ onto $\bar{R}^3 - \{0, k_2, 1/k_2, \infty\}$ with $K(F_0) = K$, and, by removing the singularities, that F_0 is a (generalized) quasiconformal mapping of $\bar{R}^3$ onto itself with the same dilatation(5). Under the assumption at the end of § 1. 3 we may conclude that F_0 is extremal.

REMARK. Suppose that f is a K-quasiconformal mapping of the unit ball onto itself with $f(0) = 0$. Then it is known ([7], [12], Cf. [18]) that

(15) $$|f(P)| \leqq c^{1-1/K} |P|^{1/K} \text{ for } |P| < 1,$$

where c is a constant, $4 \leqq c \leqq 4 \cdot 2^{1/2} e^{\pi/4}$. Now the mapping F_0 in Theorem 1 is a K-quasiconformal mapping of the unit ball onto itself with $F_0(0) = 0$ and $F_0(k_1, 0, 0) = (k_2, 0, 0)$. Since, as stated in [15],

$$\lim_{k_1 \to 0} \frac{k_2}{k_1^{1/K}} = 4^{1-1/K}$$

it follows that the constant c in (15) cannot be replaced by a number less than 4.

Next, let $m(n, k)$ be as in (11). The following result on the monotoneity of $m(3, k)/m(2, k)$ is an immediate consequence of Theorem 1.

COROLLARY 1. *For* $0 < k < 1$, $m(3, k)/m(2, k)$ *is a monotone increasing function of* k.

PROOF. Let $0 < k_1 < k_2 < 1$ and $K = m(2, k_1)/m(2, k_2)$. Then $1 < K < \infty$. Let f_0 and F_0 be the corresponding mappings in Theorem 1. Since F_0 is K-quasiconformal and maps the Grötzsch ring $R(G, 3, k_1)$ onto the ring $R(G, 3, k_2)$, it follows from [8] (Cf. also (41) in [3]) that

$$m(3, k_1) \leqq K\, m(3, k_2),$$

whence

(5) It is also easy to see that $K_I(F_0) = K_O(F_0) = K$, where $K_I(F_0)$ and $K_O(F_0)$ are the inner and outer dilatations of F_0 as defined in [9].

$$m(3, k_1)/m(2, k_1) \leqq m(3, k_2)/m(2, k_2)$$

as asserted.

3. Rotation of the extremal mapping for the case of two interior points. For $j=1, 2$ let a_j and b_j be two points in the half plane Re $z_j>0$. If f is any quasiconformal mapping of Re $z_1>0$ onto Re $z_2>0$ with $f(a_1)=a_2$ and $f(b_1)=b_2$, then $\mu(e^{-2h_1}) \leqq K(f)\mu(e^{-2h_2})$, where $h_j=h(a_j, b_j)$ denotes the hyperbolic distance between a_j and b_j with respect to the half plane Re $z_j>0$, and $\mu(k) = m(2, k)$ as in (12) (See [13]). If we set $K=\mu(e^{-2h_1})/\mu(e^{-2h_2})$, then there exists a unique extremal mapping f_0 of this class satisfying $K(f_0)=K$.

We briefly describe f_0. First, by performing preliminary Möbius transformations of $\bar{R}^2$ onto itself we may assume that the given pairs of points are $k_1^{1/2}$, $k_1^{-1/2}$ and $k_2^{1/2}$, $k_2^{-1/2}$, and there is no loss in generality in assuming that $0<k_1<k_2<1$. It is easily checked (Cf. [13]) that $k_j=e^{-2h_j}$, $j=1, 2$.

Now for $j=1, 2$, let $z_j=x_j+iy_j$ and $w_j=u_j+iv_j$. Then (See [6; 14])

$$z_j = g_j(w_j) = k_j^{\frac{1}{2}} sn(w_j, k_j)$$

maps the rectangle R_j: $0<u_j<K_j$, $0<v_j<K_j'$ conformally onto the first quadrant $x_j>0$, $y_j>0$ of the z_j-plane, with

$$g_j(0)=0, \ \ g_j(K_j)=k_j^{\frac{1}{2}}, \ \ g_j(K_j+iK_j')=k_j^{-\frac{1}{2}}, \ \ g_j(iK_j')=\infty .$$

Let φ be the natural affine mapping of R_1 onto R_2. Then $f_0=g_2 \circ \varphi \circ g_1^{-1}$ is the required extremal mapping, after being continued by reflection in the real axis. We note that the stretchings of f_0 have the same form as in (9), hence that $K=K(f_0) = (K_1'K_2)/(K_1K_2')$ as in (10).

We now prove that the space mapping F_0 obtained by rotating f_0 about the imaginary axis is also quasiconformal, and we determine its dilatation.

THEOREM 2. *For $j=1, 2$ let a_j and b_j be two points in the half plane Re $z_j>0$. Let f_0 be the extremal mapping of least dilatation from the right half plane onto itself with $f_0(a_1)=a_2$ and $f_0(b_1)=b_2$. If F_0 is the mapping of $\bar{R}^3$ onto itself obtained by rotating f_0 about the imaginary axis in $\bar{R}^2$, then F_0 is a quasiconformal mapping of $\bar{R}^3$ onto itself with $K(F_0)=(k'_1K'_1)^2/(k_2'K_2')^2$, where $k_j=e^{-2h_j}$, $h_j=h(a_j, b_j)$ being the hyperbolic distance between the points a_j and b_j with respect to the half plane Re $z_j>0$.*

PROOF. If the mapping g_j is reflected in the segment $u_j=0$, $0<v_j<K_j'$, then g_j maps the rectangle $|u_j|<K_j$, $0<v_j<K_j'$ conformally onto the upper half

plane Im $z_j > 0$, while the extension of φ is still affine. It follows by (2), (9), and Lemma 1 that for Im $z_1 > 0$,

$$l_2 < \frac{y_2}{y_1} < L_2 . \tag{16}$$

Now let P_1 be any point in $\bar{R}^3 - \{0, \infty\} - C_1' \cup C_1''$, where C_j' and C_j'' will here represent the circles obtained by rotating the points $k_j^{1/2}$ and $k_j^{-1/2}$, respectively, about the imaginary axis. By symmetry we may assume that $P_1 = (x_1, y_1, 0)$, $x_1 \geqslant 0$, $y_1 \geqslant 0$. Then by [5, p.41; 6, #125. 01]

we get

$$\frac{x_2}{x_1} \div L_2 = \frac{K_1}{K_2} \frac{s_2 D_2}{s_1 D_1} \left[\frac{1 - s_1^2 D_1^2}{1 - s_2^2 D_2^2} \right]^{\frac{1}{2}} \left[\frac{D_1^2 - k_1^2 s_1^2}{D_2^2 - k_2^2 s_2^2} \right]^{\frac{1}{2}}, \tag{17}$$

where

$$\begin{aligned} s_j &= sn(u_j, k_j), \quad c_j = cn(u_j, k_j), \quad d_j = dn(u_j, k_j), \\ S_j &= sn(v_j, k_j'), \quad C_j = cn(v_j, k_j'), \quad D_j = dn(v_j, k_j'). \end{aligned} \tag{18}$$

It is easily checked, using (17) and [5, p.9; 6, #121. 00] that the stretchings of F_0 at P_1 are

$$\begin{cases} L_2, l_2, \dfrac{x_2}{x_1} & \text{for } x_1 > 0, y_1 > 0 \\ L_2, l_2, L_2 & \text{for } x_1 = 0, y_1 > 0, \\ L_2, l_2, \dfrac{x_2}{x_1} = \left[\dfrac{c_1 d_1}{s_1} \div \dfrac{c_2 d_2}{s_2} \right] \dfrac{K_1}{K_2} L_2 & \text{for } y_1 = 0;\ 0 < x_1 < k_1^{\frac{1}{2}} \text{ or } x_1 > k_1^{-\frac{1}{2}}, \\ L_2, l_2, \dfrac{x_2}{x_1} = \left[\dfrac{S_1 C_1}{D_1} \div \dfrac{S_2 C_2}{D_2} \right] \dfrac{k_1'^2 K_1}{k_2'^2 K_2} L_2 & \text{for } y_1 = 0;\ k_1^{\frac{1}{2}} < x_1 < k_1^{-\frac{1}{2}}. \end{cases} \tag{19}$$

Now by [5, p.38; 6, #125. 01]

$$\frac{x_2}{x_1} \div \frac{y_2}{y_1} = \left[\frac{c_1 d_1}{s_1} \div \frac{c_2 d_2}{s_2} \right] \left[\frac{S_1 C_1}{D_1} \div \frac{S_2 C_2}{D_2} \right], \tag{20}$$

while the inequalities of [4] give the sharp bounds

(21) $$\frac{K_2}{K_1} \leqq \frac{c_1 d_1}{s_1} \div \frac{c_2 d_2}{s_2} \leqq \frac{k_1'^2 K_1}{k_2'^2 K_2}, \quad \frac{1+k_2}{1+k_1} \leqq \frac{S_1 C_1}{D_1} \div \frac{S_2 C_2}{D_2} \leqq \frac{K_1'}{K_2'}.$$

From (16), (20), and (21) we thus obtain

(22) $$\frac{(1+k_2)K_2}{(1+k_1)K_1} l_2 < \frac{x_2}{x_1} < \frac{k_1'^2 K_1 K_1'}{k_2'^2 K_2 K_2'} L_2 \text{ for } x_1 > 0,\ y_1 > 0.$$

Since obviously $(1+k_2)K_2 > (1+k_1)K_1$ in (22) and since the inequalities of [4] show that the coefficient of L_2 in (22) is not less than 1, it follows from (7) that the space mapping F_0 has dilatation $K(F_0)$ satisfying

(23) $$K(F_0) \leqq \frac{k_1'^2 K_1 K_1'}{k_2'^2 K_2 K_2'} K(f_0) = \frac{k_1'^2 K k_1'^2}{k_2' K_2'^2}.$$

To show that (23) holds with equality it is sufficient to show that the coefficient of L_2 in (22) cannot be replaced by a smaller number. By (19) and [4], $(x_2/x_1) \div L_2$ approaches $(k_1'^2 K_1 K_1')/(k_2'^2 K_2 K_2')$ as a limit as z_1 tends to $k_1^{1/2}$ along the segment $y_1=0$, $k_1^{1/2} < x_1 < k_1^{-1/2}$. By the continuity of $(x_2/x_1) \div L_2$ on this segment as a function of z_1 (Cf. (17)), we conclude that the second inequality in (22) is sharp. Therefore F_0 is a differentiable $(k_1'K_1')^2/(k_2'K_2')^2$-quasiconformal mapping of $\bar{R}^3 - \{0, \infty\} - C_1' \cup C_1''$ onto $\bar{R}^3 - \{0, \infty\} - C_2' \cup C_2''$ and hence, by removing the singularities, a (generalized) quasiconformal mapping of $\bar{R}^3$ onto itself with the same dilatation.

COROLLARY 2. *For $j=1,2$ let γ_j denote the ray $x_j=0, y_j \geqslant 0$, and let $z_2 = f_0(z_1)$ be the extremal quasiconformal mapping of Theorem 2 carrying the first quadrant of the z_1-plane onto the first quadrant of the z_2-plane, with $f_0(0)$, $f_0(k_1^{1/2}) = k_2^{1/2}$, $f_0(k_1^{-1/2}) = k_2^{-1/2}$, and $f_0(\infty) = \infty$. Then*

$$\frac{(1+k_2)K_2}{(1+k_1)K_1} \leqq \frac{\tan \frac{\pi}{2} \omega(\gamma_1, z_1)}{\tan \frac{\pi}{2} \omega(\gamma_2, z_2)} \leqq \frac{k_1'^2 K_1 K_1'}{k_2'^2 K_2 K_2'},$$

where $\omega(\gamma_j, z_j)$ denotes the harmonic measure of γ_j at z_j with respect to the first quadrant of the z_j-plane. These bounds are sharp.

PROOF. Since $\omega(\gamma_j, z_j) = 2\theta_j/\pi$, where $\theta_j = \arg z_j$, and since $(\tan \theta_1)/(\tan \theta_2) = (x_2/x_1)/(y_2/y_1)$, this result follows directly from (20) and (21) above. The bounds are sharp because the inequalities in (21) are sharp.

COROLLARY 3. *Let g be the extremal quasiconformal mapping of a* *quadrilateral* G_1 *onto a quadrilateral* G_2 *with* $g(\gamma_1)=\gamma_2$, *where* γ_1 *and* γ_2 *are sides of* G_1 *and* G_2, *respectively. For* $j=1,2$ *let* $\omega(\gamma_j, z_j)$ *denote the harmonic measure of* γ_j *with repect to* G_j *at* z_j. *If* $a_2=\text{mod } G_2<\text{mod } G_1=a_1$, *then*

$$1<\frac{\omega(\gamma_1, z_1)}{\omega(\gamma_2, z_2)}<A,$$

where $A=O(a_2e^{\pi/a_2})$ *as* a_2 *tends to* 0. *This is the best possible result as to order.*

PROOF. We may assume that G_j is the rectangle $0<x_j<1$, $0<y_j<a_j$, where $a_j=K_j'/K_j$, $0<k_1<k_2<1$, and that γ_j is the vertical segment $x_j=0$, $0\leqq y_j\leqq a_j$. Then (Cf. beginning of § 3) G_j may be mapped conformally onto the first quadrant of the w_j-plane by means of

$$w_j = k_j^{\frac{1}{2}} sn(K_jz_j, k_j),$$

with γ_j being carried onto the ray $u_j=0$, $v_j\geqslant 0$.

Next, identify the z_1- and z_2-planes so that the corresponding axes coincide. Then clearly $G_2<G_1$, $\gamma_2<\gamma_1$, and $z_1\in G_1$ lies vertically above its image $z_2=g(z_1)\in G_2$. As in the proof of Lemma 1, $G_2'<G_1$ and $\gamma_2'<\gamma_1$, where G_2' and γ_2' denote the translates of G_2 and γ_2, respectively, vertically by an amount $|z_1-z_2|$. From the conformal invariance of harmonic measure and the maximum principle for harmonic functions we then easily obtain

$$\frac{2}{\pi}\theta_1=\omega(\gamma_1, z_1)>\omega(\gamma_2', z_1)=\omega(\gamma_2, z_2)=\frac{2}{\pi}\theta_2 .$$

Since, for $\theta>0$, $\theta/(\tan\theta)$ is a strictly decreasing function of θ, we have

$$1<\frac{\theta_1}{\theta_2}<\frac{\tan\theta_1}{\tan\theta_2}. \tag{24}$$

Then (24) and Corollary 2 yield

$$1<\frac{\theta_1}{\theta_2}<\frac{k_1'^2K_1K_1'}{k_2'^2K_2K_2'},$$

the right side being $O(a_2e^{\pi/a_2})$ as a_2 tends to 0 with a_1 fixed, according to [5, p.21; 6, #112. 04]. We see that the order is correct because the upper bound in Corollary 2 is sharp.

4. Rotation of the extremal mapping for the case of one interior point and two boundary points. For $j=1,2$ let α_j, β_j, δ_j, be a triple of points in $\bar{R}^2$ with

$$\operatorname{Im} \alpha_j = \operatorname{Im} \beta_j = 0, \ \operatorname{Im} \delta_j > 0 .$$

If f is any quasiconformal mapping of $\operatorname{Im} z_1 > 0$ onto $\operatorname{Im} z_2 > 0$ with $f(\alpha_1) = \alpha_2$, $f(\beta_1) = \beta_2$, $f(\delta_1) = \delta_2$, then $\mu(\sin \pi\omega_1/2) \leqq K(f)\mu(\sin \pi\omega_2/2)$, where ω_j denotes the harmonic measure of the segment $\alpha_j\beta_j$ at δ_j with respect to the upper half plane and $\mu(k) = m(2, k)$ as in (12) (See [13]). If we take $K = \mu(\sin \pi\omega_1 2/)/\mu(\sin \pi\omega_2/2)$, then there exists a unique extremal mapping f_0 of this class satisfying $K(f_0) = K$.

The mapping f_0 is easily described. First, by performing preliminary Möbius transformations of $\bar{R}^2$ onto itself we may assume that the given pair of triples are $-1, 1, ia_1$ and $-1, 1, ia_2$, and that $0 < a_2 < a_1 < \infty$. Let

$$k_j = (1 + a_j^2)^{-\frac{1}{2}}, \ j = 1, 2,$$

so that $0 < k_1 < k_2 < 1$ and $a_j = k_j'/k_j$. It is easily verified that $k_j = \sin \pi\omega_j/2$. Then for $j = 1, 2$, let $z_j = x_j + iy_j$ and $w_j = u_j + iv_j$. Next,

$$z_j = g_j(w_j) = cn(w_j, k_j),$$

where cn denotes Jacobi's elliptic cosine function [6; 14], maps the rectangle R_j: $0 < u_j < K_j$, $-K_j' < v_j < 0$ conformally onto the first quadrant of the z_j-plane with

$$g_j(-iK_j') = \infty, \ g_j(K_j - iK_j') = ia_j, \ g_j(K_j) = 0, g_j(0) = 1 .$$

Finally, let φ be the natural affine mapping of R_1 onto R_2. Then $f_0 = g_2 \circ \varphi \circ g_1^{-1}$ is the required extremal mapping, after being extended by reflection in the imaginary axis. This mapping is differentiable at all points of $\operatorname{Im} z_1 > 0$ except ia_1.

To calculate the dilatation of f_0, let P_1 be any point in the first quadrant of the z_1-plane and let $P_2 = f_0(P_1)$, $Q_1 = g_1^{-1}(P_1)$, and $Q_2 = g_2^{-1}(P_2)$. The maximum and minimum stretchings $L_2(P_1)$ and $l_2(P_1)$ of f_0 at P_1 can be written as in (9), whence, as in previous work, $K = K(f_0) = (K_1'K_2)/(K_1K_2')$.

We now prove that the space mapping F_0 obtained by rotating f_0 about the real axis is also quasiconformal.

THEOREM 3. *For $j = 1, 2$ let α_j, β_j, δ_j be a triple of points in $\bar{R}^2$ with*

$$\operatorname{Im} \alpha_j = \operatorname{Im} \beta_j = 0, \ \operatorname{Im} \delta_j > 0 .$$

Let f_0 be the extremal quasiconformal mapping of least dilatation from the half plane Im $z_1>0$ *onto the half plane* Im $z_2>0$ *with* $f_0(\alpha_1)=\alpha_2$, $f_0(\beta_1)=\beta_2$, $f_0(\delta_1)=\delta_2$, *and let* $K(f_0)=K$. *If* F_0 *is the mapping of* $\bar{R}^3$ *onto itself obtained by rotating* f_0 *about the real axis in* $\bar{R}^2$, *then* F_0 *is a quasiconformal mapping of* $\bar{R}^3$ *onto itself with* $K\leqq K(F_0)\leqq K^3$.

PROOF. As already remarked, we may assume that the given triples are $-1,1, ia_1$ and $-1,1, ia_2$, with $0<a_2<a_1<\infty$. As in earlier problems, this normalization does not affect the dilatation of the space mapping F_0.

Now let P_1 be any point in $\bar{R}^3-\{-1,1\}-C_1$, where C_j will here represent the circle obtained by rotating the point ia_j about the real axis. By symmetry we may assume that $P_1=(x_1, y_1, 0)$, $x_1\geqslant 0$, $y_1\geqslant 0$. The three stretchings of F_0 at P_1 are easily seen to be

$$\begin{cases} L_2,\ l_2,\ \dfrac{y_2}{y_1} & \text{for } y_1>0,\ \ x_1\geqslant 0, \\ L_2,\ l_2,\ l_2 & \text{for } y_1=0,\ \ 0\leqq x_1<1, \\ L_2,\ l_2,\ L_2 & \text{for } y_1=0,\ \ x_1>1. \end{cases}$$

If we show that, for $y_1>0$ and $x_1\geqslant 0$,

$$\frac{1}{K}l_2<\frac{y_2}{y_1}<KL_2, \tag{25}$$

it will follow from (7) in § 1. 3 that $K\leqq K(F_0)\leqq K^3$.

To establish (25) we note first that $g_j(w_j)^2$ maps the rectangle R_j conformally onto the half plane Im $z_j>0$. Hence by Lemma 1 and (9),

$$\frac{r_2}{r_1}l_2<\frac{x_2y_2}{x_1y_1}<\frac{r_2}{r_1}L_2, \tag{26}$$

where $r_j=(x_j^2+y_j^2)^{1/2}$, $j=1, 2$. Now by [5, p.38; 6, #125. 01]

$$\frac{x_j}{r_j}=\frac{\text{Re } cn(w_j, k_j)}{|cn(w_j, k_j)|}=\left[1+\frac{s_j^2d_j^2}{c_j^2}\,\frac{S_j^2D_j^2}{C_j^2}\right]^{-\frac{1}{2}}, \tag{27}$$

where $s_j, c_j, d_j, S_j, C_j, D_j$ have the meaning assigned in (18).

Next it follows from [4] and [5, (29), p.13; 6, #122. 03] that

$$\frac{K_1}{K_2}\leqq\frac{s_2d_2}{c_2}\div\frac{s_1d_1}{c_1}\leqq\frac{K_2}{K_1},\ \ \frac{K_2'}{K_1'}\leqq\frac{S_2D_2}{C_2}\div\frac{S_1D_1}{C_1}\leqq\frac{K_1'}{K_2'}, \tag{28}$$

and hence by (27) that

$$\frac{1}{K} \leqq \frac{x_2/x_1}{r_2/r_1} \leqq K.$$

Because of (26) this gives (25). Thus F_0 is a differentiable K^3-quasiconformal mapping of $\bar{R}^3-\{-1,1\}-C_1$ onto $\bar{R}^3-\{-1,1\}-C_2$, and hence a (generalized) quasicon-formal mapping of $\bar{R}^3$ onto itself satisfying $K \leqq K(F_0) \leqq K^3$ as claimed.

REFERENCES

[1] L. V. AHLFORS, Confomal Mapping, Notes from lectures at Oklahoma A. &M. College, Summer 1951.

[2] G. D. ANDERSON, The coefficients of quasiconformality of ellipsoids, Ann. Acad. Sci. Fenn. Ser. AI, 411(1967), 1-14.

[3] G. D. ANDERSON, Symmetrization and extremal rings in space, Ann. Acad. Sci. Fenn. Ser. AI, 438(1969), 1-24.

[4] G. D. ANDERSON, and M. K. Vamanamurthy, Affine mappings and elliptic functions, Publ. de l'Inst. Math. (Yogoslavia), tome 11(25), (1971), 49–51.

[5] F. BOWMAN, Introduction to Elliptic Functions with Applications, Dover, New York, 1961.

[6] P. F. BYRD AND M. D. FRIEDMAN, Handbook of Elliptic Integrals for Engineers and Physicists, Springer, Berlin, 1954.

[7] F. W. GEHRING, Symmetrization of rings in space, Trans. Amer. Math. Soc., 101 (1961), 499-519.

[8] F. W. GEHRING, Rings and quasiconformal mappings in space. Trans. Amer. Math. Soc., 103(1962), 353-393.

[9] F. W. GEHRING, AND J. VÄISÄIÄ, The coefficients of quasiconformality of domains in space, Acta Math., 114(1965), 1-70.

[10] J, HERSCH, Contribution à la théorie des fonctions pseudo-analytiques, Comment. Math. Helv. 29, 1955, 301-337.

[11] J. HERSCH AND A. PFLUGER, Généralisation du lemme de Schwarz et du principe de la mesure harmonique pour les fonctions pseudo-analytiques, C. R. Acad. Sci. Paris, 234 (1952), 43-45.

[12] K. IKOMA, On the distortion and correspondence under quasiconformal mappings in space, Nagoya Math. J., 24(1965), 175-203.

[13] J. A. KELINGOS, Characterizations of quasiconformal mappings in terms of harmonic and hyperbolic measure, Ann. Acad. Sci. Fenn. Ser. AI, 368(1965), 1-16.

[14] H. KOBER, Dictionary of Conformal Representations, Dover, New york, 1957.

[15] O. LEHTO, AND K. I. VIRTANEN, Quasikonforme Abbildungen, Springer, Berlin, 1965.

[16] Z. NEHARI, Conformal Mapping, McGraw-Hill, New York, 1952.

[17] R. NEVANLINNA, Eindeutige Analytische Funktionen, Springer, Berlin, 1953.

[18] B. V. ŠABAT, On the theory of quasiconformal mappings in space, Soviet Math., 1(1960), 730-733.

[19] A. V. SYČEV, Quasiconformal mappings in space, Dokl. Akad. Nauk. SSSR. 166(1966), 298-300.

[20] M. VIRSU, On the linear dilatation of quasiconformal mappings in space, Duke Math. J., 38(1971), 569–574.

DEPARTMENT OF MATHEMATICS,
MICHIGAN STATE UNIVERSITY,
EAST LANSING, MICHIGAN, U. S. A.
AND
DEPARTMENT OF MATHEMATICS,
MADURAI UNIVERSITY,
MADURAI, INDIA