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ON THE MULTIPLIERS OF A^p(G)-ALGEBRAS

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1. Introduction and Preliminaries. Multiplier operators are frequently used in group algebras, especially in Fourier analysis. Various type of multipliers are investigated by Figà-Talamanca, Gaudry, Brainerd and Edwards, and Rieffel ect., see [8], [9], [10], [2] and [11] etc.

In this paper, our purpose is to characterize the multipliers of $A^p(G)$-algebras $1 \leqq p \leqq 2$ as a dual space $A_p(G)^*$ of $A_p(G)$ which we will define later, and hence $A_p(G)^*$ is isometrically isomorphic to the space of bounded regular measures if G is non-compact, locally compact abelian group. If G is an infinite compact abelian group, then $A_p(G)^*$ $1 \leqq p \leqq 2$ is isometrically isomorphic to the space of pseudomeasures, i.e. the dual space of the Fourier algebra $A(G)$. In section 3, we investigate also the multiplier spaces of $L^{p_1} \cap L^{p_2}(G)$ for $1<p_1, p_2<\infty$ and $L^1 \cap L^p(G)$ for $1<p<\infty$. The isomorphism theorem of $A^p(G)$-algebras is proved in section 4. Finally we consider the continuous linear mapping of $L^1(G)$ into $A^p(G)$ in which we characterize the space of operators from $L^1(G)$ into $A^p(G)$ with the function space $A^p(G)$ for $1<p \leqq 2$.

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Let G be a locally compact abelian group and $\hat{G}$ its character group. dx and $d\hat{x}$ denote the normalized Haar measures of G and $\hat{G}$ respectively. The space $A^p(G)$ denotes the subset of $L^1(G)$ consisting of those functions f whose Fourier transforms $\hat{f}$ belong to $L^p(\hat{G})$. We supply $A^p(G)$ with norm

$$\|f\|^p = \max (\|f\|_1, \|\hat{f}\|_p) \tag{1.1}$$

which is equivalent to the norm: $\|f\|_1 + \|\hat{f}\|_p$ for $f \in A^p(G)$. It is easy to see that $A^p(G)$, $1 \leqq p < \infty$ is a dense ideal in $L^1(G)$ and forms a semi-simple commutative Banach algebra with the norm $\| \ \|^p$ under convolution (see Larsen, Liu and Wang [9]). Since $\hat{f}$ is a bounded continuous function in $L^p(\hat{G})$, $\hat{f} \in L^r(\hat{G})$ for $p \leqq r$, we see that the $A^p(G)$ forms an ascending chain of dense ideals with respect to the

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index p, $1\leqq p<\infty$ in $L^1(G)$. We use the following general notations

$C_c(G)$: The continuous functions with compact support in G,

$C_0(G)$: The continuous functions vanishing at infinity on G,

$M(G)$: Bounded Radon measures on G.

Let $E=C_0(G)\times L^q(\hat{G})$. Then $E^*=M(G)\times L^p(\hat{G})$ for $1/p+1/q=1$, $1<p<\infty$. Denote

$H_q=$ The closure in E of $\{(f, -\hat{f}); f\in A^1(G)\}$,

$H_q^0=$ The subset in E^* such that if $(\sigma, \tau)\in E^*$, then $\sigma\tilde{f}-\tau\hat{f}=0$ for $(\tilde{f}, -\hat{f})\in H_q$,

where $\tilde{f}$ denotes the reflexive function of f: $x\to f(-x)$ and $\hat{f}$ the Fourier transform of f, and let

$K_q=$ The quotient space $E/H_q=C_0(G)\vee_{H_q}L^q(G)$.

The elements of K_q are denoted by $\{g, h\}$ and supplied with the norm

$$\|\{g, h\}\| = \inf\,\{\|g'\|+\|h'\|_q;\ \{g', h'\}=\{g, h\} \bmod H_q\}.$$

Then, if $1<p\leqq 2$, $1/p+1/q=1$, we have (see Liu and Rooij [7])

$$K_q^*=(E/H_q)^*\cong A^p(G) \text{ and } K_q^*=H_q^0. \tag{1.2}$$

As $p=1$, denote $H_\infty=$ The closure of $\{(\tilde{f}, -\hat{f}); f\in A^1(G)\}$ in $C_0(G)\times C_0(\hat{G})$ then

$$K_\infty^*=(C_0(G)\times C_0(\hat{G})/H_\infty)^*\cong A^1(G). \tag{1.3}$$

$A^p(G)$ may be considered to be a closed linear subspace of $E^*=M(G)\times L^p(\hat{G})$, the dual space of $E=C_0(G)\times L^q(\hat{G})$, the norm $\|f\|^p$ is identical with the norm $\|f\|_{E^*}$. Moreover, we can show the following

LEMMA 1.1. *For $f\in A^p(G)$, $1\leqq p\leqq 2$ and $g\in C_c$, we have*

$$\|f\|^p=\sup_{\|\{g,\hat{g}\}\|\leqq 1}\left|f*g(0)+\int_{\hat{G}}\hat{f}\cdot\hat{g}\,d\hat{x}\right| \tag{1.4}$$

where $\{g, h\}$ denotes the elements of $K_q=E/H_q$, $g\in C_0(G)$, $h\in L^q(\hat{G})$.

PROOF. For $f\in A^p(G)$ $1\leqq p\leqq 2$ and $g\in C_c(G)\subset L^2(G)$, we see that $\hat{f}\in L(\hat{G})\cap C_0(\hat{G})\subset L^2(\hat{G})$, $\hat{g}\in L^2(\hat{G})\cap C_0(\hat{G})$ and $f*g\in C_0(G)$, the Parseval's formula is applicable so that

$$\int_{\hat{G}} \hat{f}(\hat{x})\hat{g}(\hat{x})d\hat{x} = \int_G f(x)\tilde{g}(x)dx = f*g(0).$$

Consider the linear functional of the form

$$F(f) = \int_G f(x)\tilde{g}(x)dx + \int_{\hat{G}} \hat{f}(\hat{x})\hat{h}(\hat{x})d\hat{x} = t_f(\tilde{g}, \widehat{h})$$

for any $g, h \in C_c(G)$ and $(\tilde{g}, \hat{h}) \in C_0(G) \times L^q(\widehat{G})$. Since $(\tilde{g}, \hat{h}) = (\tilde{h}, \hat{g}) \bmod H_q$,

$$\|(\tilde{g}, \hat{h})\| = \|(\tilde{h}, \hat{g})\| = \inf (\|g\|_\infty + \|\hat{h}\|_q)$$

and

$$t_f(\tilde{g}, \hat{h}) = t_f(\tilde{h}, \hat{g}) = \frac{1}{2}(t_f(\tilde{g}, \hat{g}) + t_f(\tilde{h}, \hat{h})).$$

By (1.2) and (1.3),

$$\begin{aligned} \|f\|^p &= \sup_{\|(\tilde{g},\hat{h})\| \leqq 1} |t_f(\tilde{g}, \tilde{h})| = \sup_{\|(\tilde{h},\hat{g})\| \leqq 1} |t_f(\tilde{h}, \hat{g})| \\ &= \sup_{\|(\tilde{g},\hat{h})\| = \|(\tilde{h},\hat{g})\| \leqq 1} |t_f(\dot{g}, \hat{h})| \\ &= \sup_{\|(\tilde{g},\hat{h})\| = \|(\tilde{h},\hat{g})\| \leqq 1} \left|\frac{1}{2}(t_f(\tilde{g}, \hat{h}) + t_f(\tilde{h}, \hat{g}))\right| \\ &= \sup_{\|(\tilde{g},\hat{h})\| = \|(h,\hat{g})\| \leqq 1} \left|t_f\left(\left(\frac{g+h}{2}\right)^{\sim}, \left(\frac{g+h}{2}\right)^{\wedge}\right)\right| \\ &\leqq \sup_{\|(\tilde{g},\hat{g})\| \leqq 1} |t_f(\tilde{g}, \hat{g})|. \end{aligned}$$

Therefore

$$\sup_{\|(\tilde{g},\hat{g})\| \leqq 1} |t_f(\tilde{g}, \hat{g})| = \sup_{\|(\tilde{g},\hat{h})\| \leqq 1} |t_f(\tilde{g}, \hat{h})| = \|f\|^p. \qquad \text{Q. E. D.}$$

DEFINITION 1.2. *A multiplier T of $A^p(G)$ means a continuous linear operator on $A^p(G)$ which commutes with translation operator ρ_x for every $x \in G$ where ρ_x is defined by $\rho_x f(y) = f(y-x)$. In this paper we denote by $M(A^p)$ the set of all multipliers T of $A^p(G)$.*

The following proposition is immediate.

PROPOSITION 1.3. *A mapping T from $A^p(G)$ into itself is a multiplier of $A^p(G)$ if and only if T satisfies the following condition*

(1.5) $\qquad T(f*g) = Tf*g = f*Tg \qquad$ for any $f, g \in A^p(G)$.

(cf. Larsen [15]).

2. The multipliers of $A^p(G)$. By the preparation in previous section, we can define the space $A_p(G)$ for $1<p\leqq 2$ to be the set of all functions $u(x)$ such that

$$u=\sum_{i=1}^{\infty} f_i*g_i\,;\ f_i\in A^p(G),\ g_i\in C_q=\{g\in C_c;\ \{g,\hat{g}\}\in K_q\}$$

and $\sum_{i=1}^{\infty}\|f_i\|^p\|g_i\|<\infty$, where $\|g_i\|=\|\{g_i,\hat{g}_i\}\|$. (note that $\widehat{C_c(G)}$ is dense in $L^q(G)$ and $C_c(G)$ is dense in $C_0(G)$.)

Define $u\longrightarrow |||u|||_p$ by

$$|||u|||_p=\inf\left\{\sum_{i=1}^{\infty}\|f_i\|^p\|g_i\|;\ u=\sum_{i=1}^{\infty}f_i*g_i \text{ in } A_p(G)\right\} \tag{2.1}$$

the infimum being taken over all functions $f_i\in A^p(G)$, $g_i\in C_q$ for the representation of u. Since $\sum_{i=1}^{\infty}\|f_i\|^p\|g_i\|<\infty$,

$$\left\|\sum_{i=m}^{n}f_i*g_i\right\|_\infty\leqq\sum_{i=m}^{n}\|f_i\|^p g_i\|\longrightarrow 0 \text{ when } m,\ n\longrightarrow\infty,$$

we see that $u=\sum_{i=1}^{\infty}f_i*g_i$ is a uniformly continuous function on G, and the norm $|||u|||_p$ is stronger than the uniform norm. It can be shown that the space $A_p(G)$ is a dense subspace of $C_0(G)$[1)] and so we may consider the dual space $A_p(G)^*$ which contains the space of bounded Radon measures. The following proposition is not hard to prove (cf. Gaudry [2]).

PROPOSITION 2.1. *The space $A_p(G)$ is a dense linear subspace of $C_0(G)$ and is a Banach space with respect to the norm $|||\ |||_p$, and thus the topology so defined is stronger than the uniform topology and also stronger than the topology induced from $A^p(G)$.*

THEOREM 2.2. *The multiplier space $M(A^p)$ for $1<p\leqq 2$ is isometrically isomorphic to the topological dual $A_p(G)^*$ of $A_p(G)$.*

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1) Since C_c is L^1-dense in A^p, it suffices to show that $\{f*g: f,\ g\in C_c\}$ is uniformly dense in C^0. The algebra of continuous functions on G generated by $\{f*g;\ f,\ g\in C_c\}$ is a self-adjoint subalgebra of C_0 and separates points of G, thus it is uniformly dense in $C_0(G)$ by Stone-Weierstrass theorem.

PROOF. Suppose that $T \in M(A^p)$ and define the linear functional μ on $A_p(G)$ by

$$\mu(u) = \sum_{i=1}^{\infty} \left(\int_G Tf_i(x) \tilde{g}_i(x)\, dx + \int_{\hat{G}} \widehat{Tf_i}(\hat{x})\, \hat{g}_i(\hat{x})\, d\hat{x} \right)$$
$$= \sum_{i=1}^{\infty} \left(Tf_i * g_i(0) + \int_{\hat{G}} \widehat{Tf_i}(\hat{x})\, \hat{g}_i(\hat{x})\, d\hat{x} \right)$$

for $u = \sum_{i=1}^{\infty} f_i * g_i$ in $A_p(G)$ with $f_i \in A^p(G)$, $g_i \in C_c(G)$ and $\sum_{i=1}^{\infty} \|f_i\|^p \|g_i\| < \infty$. This μ is well-defined. To show this, it suffices to show that if $u = \sum_{i=1}^{\infty} f_i * g_i = 0$, and $\sum_{i=1}^{\infty} \|f_i\|^p \|g_i\| < \infty$ then $\mu(u) = 0$.

Let $\{e_\alpha\}$ be an approximate identity of $A^p(G)$ (cf. Lai [3]) and let $h_\alpha = Te_\alpha$, then

$$h_\alpha * f \longrightarrow Tf \qquad \text{in } A^p(G)\text{-norm for } f \in A^p(G),$$

and we have, for $f \in A^p(G)$ and $g \in C_c(G)$,

$$\|h_\alpha * f * g\|_\infty \leqq \int_G |e_\alpha * Tf(x)|\ |g(-x)|\ dx$$
$$\leqq \|e_\alpha\|_1 \|Tf\|_1 \|g\|_\infty \qquad (\|e_\alpha\|_1 \leqq C).$$

By the assumption, the series $u = \sum_{i=1}^{\infty} f_i * g_i$ is uniformly convergent on G and $u = \sum_{i=1}^{\infty} f_i * g_i = 0$, we have

$$h_\alpha * u = \sum_{i=1}^{\infty} h_\alpha * f_i * g_i = 0\,.$$

But it is easy to see that

$$\lim_\alpha \sum_{i=1}^{\infty} h_\alpha * f_i * g_i(0) = \sum_{i=1}^{\infty} Tf_i * g_i(0) = 0\,.$$

On the other hand, if $f \in A^p(G)$, $g \in C_c(G)$, we have $\hat{f} \in C_0 \cap L^2(\hat{G})$ and $\hat{g} \in C_0 \cap L^2(\hat{G})$, so by Parseval's formula,

$$\int_{\hat{G}} \hat{f}(\hat{x})\, \hat{g}(\hat{x}) d\hat{x} = f * g(0),$$

hence we obtain, under the same assumption,

$$\sum_{i=1}^{\infty}\int_{\hat{G}} \widehat{Tf_i}(\hat{x})\,\hat{g}_i(\hat{x})\,d\hat{x} = \sum_{i=1}^{\infty} Tf_i * g_i(0) = 0\,.$$

Thus $\mu(u) = 0$ and μ is well-defined.

The mapping $T \to \mu$ is evidently injective, we will show that it is an isometry. By Lemma 1. 1,

$$|\mu(u)| \leqq \sum_{i=1}^{\infty} \|Tf_i\|^p \|g_i\| \leqq \|T\| \sum_{i=1}^{\infty} \|f_i\|^p \|g_i\|,$$

it follows that

$$|\mu(u)| \leqq \|T\| \; |||u|||_p\,.$$

Therefore

$$\|\mu\| \leqq \|T\|\,.$$

On the other hand, it follows from (1. 2) and Lemma 1. 1 that

$$\begin{aligned}\|T\| &= \sup_{\|f\|^p \leqq 1} \|Tf\|^p \\ &= \sup_{\|f\|^p \leqq 1,\ \|g\| \leqq 1} |\mu(f*g)| \\ &\leqq \sup_{|||f*g|||_p \leqq 1} |\mu(f*g)| \leqq \|\mu\|\,.\end{aligned}$$

Hence $\|T\| = \|\mu\|$.

Finally, we want to show that $T \to \mu$ is surjective.

Suppose that $\mu \in A_p(G)^*$ and for an arbitrary fixed $f \in A^p(G)$, define the linear functional

$$g \to \mu(f*g) = t(g) \qquad \text{for } g \in C_c(G)\,.$$

Thus $|t(g)| \leqq \|\mu\| \; \|f\|^p \; \|g\|$, $t(g)$ may be extended to an element of $K_q{}^*$, and hence t defines a unique (element) function, say Tf, in the dual space $A^p(G)$ of $K_q = E/H_q$, it follows from (1, 2) that

$$Tf*g(0) + \int_{\hat{G}} \widehat{Tf}(\hat{x})\, g(\hat{x})\, d\hat{x} = \mu(f*g) = t(g)\,.$$

Since μ is a bounded linear functional on $A_p(G)$,

$$\begin{aligned}|t(g)| &= |\mu(f*g)| \\ &\leqq \|\mu\| \; |||f*g|||_p \leqq \|\mu\| \|f\|^p \; \|g\|\end{aligned}$$

and since t defines Tf, by (1. 2) $\|t\| = \|Tf\|^p$, we see that

$$\|Tf\|^p \leqq \|\mu\| \ \|f\|^p,$$

this implies that $\|T\| \leqq \|\mu\|$. Hence T is a bounded linear operator on $A^p(G)$. Actually it is a multiplier of $A^p(G)$; for if $y \in G$ and $f \in A^p(G)$, $g \in C_c(G)$, we have

$$\begin{aligned} T(\rho_y f)*g(0) + \int_{\hat{G}} \widehat{T\rho_y f} \cdot \hat{g} dx &= \mu(\rho_y f*g) \\ &= \mu(f*\rho_y g) \\ &= Tf*\rho_y g(0) + \int_{\hat{G}} \widehat{Tf}(\hat{x})\, \widehat{\rho_y g}(\hat{x})\, d\hat{x} \\ &= \rho_y Tf*g(0) + \int_{\hat{G}} \widehat{\rho_y Tf}(\hat{x})\, \hat{g}(\hat{x})\, d\hat{x} \end{aligned}$$

whence $\rho_y(Tf) = T(\rho_y f)$ for any $f \in A^p(G)$, i. e. T commutes with translation, by definition 1. 2, $T \in M(A^p)$. Q. E. D.

REMARK 2. 3. For $A^1(G)$, we define the space $A_1(G)$ consisting of all the functions u of the form

$$u = \sum_{i=1}^{\infty} f_i*g_i \text{ with } f_i \in A^1(G),\ g_i \in \{g \in C_c(G);\ \{g, \hat{g}\} \in K_\infty\}$$

such that

$$\sum_{i=1}^{\infty} \|f_i\|^1 \ \|g_i\| < \infty,$$

where $\|g_i\| = \inf \{\|g_i'\|_\infty + \|\hat{g}_i'\|_\infty;\ \{g_i', \hat{g}_i'\} = \{g_i, \hat{g}_i\} \in K_\infty\}$. Here K_∞ is defined in section 1. The norm of $A_1(G)$ is defined by the same way like as $A_p(G)$, for $1 < p \leqq 2$. Then by (1. 3) we have the following

COROLLARY 2. 4. *The space $M(A^1)$ is isometrically isomorphic to the dual space $A_1(G)^*$ of $A_1(G)$.*

DISCUSSION 2. 5. The above characterizations for multipliers of $A^p(G)$, $1 \leqq p \leqq 2$ are representing different function spaces which depend on the group of compact or non-compact.

1^0. The case of non-compact group G.

Let $\mu \in A_p(G)^*$ be arbitrarily and take $f \in A^p(G)$. Define

$$g \to \mu(f*g) = 2T_\mu f*g(0) = t_f(g) \qquad \text{for all } g \in C_q \subset C_0.$$

Then

$$|t_f(g)| = 2|T_\mu f*g(0)| \leqq 2\|T_\mu f\|_1 \|g\|_\infty$$
$$\leqq 2\|T_\mu\| \|f\|^p \|g\|_\infty,$$

and T_μ is a multiplier of $A^p(G)$ into $L^1(G)$. Hence when G is non-compact, it can be shown by the same argument of Figà-Talamanca and Gaudry [14; Theorem 3.1] that there exists $\nu \in M(G)$ such that

$$\nu*f = T_\mu f \qquad \text{for all } f \in A^p(G)$$

and $\|\nu\| = \|T_\mu\| = \|\mu\|$. Therefore we have

COROLLARY 2.6. *Let G be a non-compact, locally compact abelian group. Then for* $1 \leqq p \leqq 2$,

$$A_p(G)^* \cong M(G).$$

Let $A(G)$ be the space of functions which are the Fourier transforms of functions in $L^1(\hat{G})$. $A(G)$ forms a Banach algebra, called the Fourier algebra, under pointwise product with the same norm of $L^1(\hat{G})$. It is precisely the convolution of two functions in $L^2(G)$. We denote by $P(G)$ the space of bounded linear functionals of $A(G)$, each element of $P(G)$ is called a pseudo-measure on G.

2°. The case of infinite compact group.

If G is an infinite compact abelian group, then $A^p(G) \subset L^2(G)$ $1 \leqq p \leqq 2$. Let $\mu \in A_p(G)^*$ be arbitrarily, there corresponds a multiplier $T_\mu \in M(A^p)$ such that

$$\mu(f*g) = 2T_\mu f*g(0) \qquad \text{for } f \in A^p(G) \text{ and } g \in C_q \subset L^2.$$

Define

$$\nu(f*g) = \frac{1}{2}\mu(f*g) = T_\mu f*g(0).$$

Then

$$|v(f*g)| = \left|\int_G Tf(x)\, g(-x)\, dx\right|$$
$$= \left|\int_{\hat{G}} \widehat{T_\mu f}(\hat{x})\, \hat{g}(\hat{x})\, d\hat{x}\right| \qquad \text{(Parseval's identity)}$$
$$= \left|\int_{\hat{G}} \varphi(\hat{x}) \hat{f}(\hat{x})\, \hat{g}(\hat{x})\, d\hat{x}\right|,$$

φ is a bounded continuous function on $\hat{G}$ (cf. Wang [13],

$$\leqq \|T_\mu\| \ \|f*g\|_{A(G)}$$

since $\|\varphi\|_\infty \leqq \|T_\mu\|$ (cf. Wang [13]). Since $A^p(G)*C_q'(G)$ is dense in $A(G)$, ν may be defined on all of $A(G)$ such that

$$\|\nu\|_{A(G)} \leqq \|T_\mu\| = \|\mu\|_{A_p(G)}\,.$$

Hence ν is a pseudo-measure in $P(G)$.

For $f \in A^p(G)$, $g \in C_q(G)$, we have

$$f*g = \frac{1}{2}\,[(f*(g-h)) + (f*(g+h))]$$

where h varies in $A^1(G)$. Since

$$\begin{aligned}\|f*(g-h)\|_{A(G)} &= \|\hat{f}(g-h)\hat{\ }\|_1\\ &\leqq \|\hat{f}\|_p\|(g-h)\hat{\ }\|_q\end{aligned}$$

and

$$\begin{aligned}\|f*(g+h)\|_{A(G)} &\leqq \|\hat{f}\|_p\ \|(g+h)\|_q\\ &\leqq \|\hat{f}\|_p\|g+h\|_\infty,\end{aligned}$$

if the Haar measure of G is taken to be 1, where $1/p+1/q=1$, we have

$$\|f*g\|_{A(G)} \leqq \frac{1}{2}\|\hat{f}\|_p(\|g+h\|_\infty + \|(g-h)\hat{\ }\|_q)\,.$$

Similarly, for h replaced by $\tilde{h}$, we have

$$\|f*g\|_{A(G)} \leqq \frac{1}{2}\|\hat{f}\|_p(\|g+\tilde{h}\|_\infty + \|(g-\tilde{h})\hat{\ }\|_q)\,.$$

Then

$$\begin{aligned}\|f*g\|_{A(G)} \leqq \frac{1}{4}\|\hat{f}\|_p[(\|g+\tilde{h}\|_\infty &+ \|(g-h)\hat{\ }\|_q)\\ &+ (\|g+h\|_\infty + \|(g-\tilde{h})\hat{\ }\|_q)]\,.\end{aligned}$$

By taking the infimum over $h \in A^1(G)$ so that

$$\{g, \hat{g}\} = \{g+\tilde{h}, \hat{g}-\hat{h}\} \qquad \text{mod } H_q$$

where $q=\infty$ if $p=1$ and $q=p/(p-1)$ if $1<p\leqq 2$, we obtain

$$\|f*g\|_{A(G)} \leqq \frac{1}{2} \|f\|^p \|g\| .$$

Now if $u = \sum_{i=1}^{\infty} f_i * g_i$ with $\sum_{i=1}^{\infty} \|f_i\|^p \|g_i\| < \infty$, then for any n,

$$\left\| \sum_{i=1}^{n} f_i * g_i \right\|_{A(G)} \leqq \sum_{i=1}^{n} \|f_i * g_i\|_{A(G)}$$

$$\leqq \frac{1}{2} \sum_{i=1}^{\infty} \|f_i\|^p \|g_i\| < \infty,$$

and so

$$\|u\|_{A(G)} \leqq \frac{1}{2} |||u|||_p .$$

Hence for any $\nu \in P(G)$,

$$|\nu(u)| \leqq \|\nu\|_{A(G)} \|u\|_{A(G)} \leqq \frac{1}{2} \|\nu\|_{A(G)} |||u|||_p .$$

and

$$\|\nu\|_{A_p(G)} \leqq \frac{1}{2} \|\nu\|_{A(G)} .$$

Thus for any $\nu \in P(G)$ there corresponds a unique $\mu \in A_p(G)^*$ such that

$$2\nu(f*g) = \mu(f*g) = 2T_\nu f*g(0)$$

and

$$\|\nu\|_{A(G)} = \|\mu\|_{A_p}(G) .$$

Therefore we have the following

COROLLARY 2.7. *Let G be an infinite compact abelian group. Then for $1 \leqq p \leqq 2$, there is an isometric isomorphism mapping $A_p(G)^*$ onto the space $P(G)$ of the pseudo-measures.*

3. The multipliers of $D^{p_1,p_2}(G) = L^{p_1}(G) \cap L^{p_2}(G)$, $1 < p_1, p_2 < \infty$. We supply the norm of $D^{p_1,p_2}(G) = L^{p_1}(G) \cap L^{p_2}(G)$ by

(3.1) $$\|f\|_\wedge = \max (\|f\|_{p_1}, \|f\|_{p_2}) .$$

Then $D^{p_1,p_2}(G)$ is a Banach space (not necessary an algebra if G is not compact) with respect to the norm $\|f\|_\wedge$.

Let

$$S_{q_1,q_2} = \{g(x) \mid g(x) = g_1(x) + g_2(x) \text{ and } (g_1, g_2) \in L^{q_1}(G) \times L^{q_2}(G)\} .$$

We supply it with the norm

$$(3.2) \qquad \|g\|_\vee = \inf \{\|g_1'\|_{q_1} + \|g_2'\|_{q_2} \text{ for } g = g_1' + g_2' \text{ with } (g_1', g_2') \in L^{q_1} \times L^{q_2}\},$$

then S_{q_1,q_2} is a Banach space. It is known that D^{p_1,p_2} and S_{q_1,q_2} are reflexive and (see Liu and Wang [6: Theorem 4])

$$(3.3) \qquad S^*_{p_1,p_2} \cong D^{p_1,p_2}(G) \qquad \left(1 < p_1, p_2 < \infty,\ \frac{1}{p_i} + \frac{1}{q_i} = 1 \quad i = 1, 2\right).$$

Since $L^1(G)$ is a Banach algebra under convolution, D^{p_1,p_2} becomes a left $L^1(G)$-module when elements of $L^1(G)$ act on D^{p_1,p_2} by convolution on the left.

Define the space $D_{p_1,p_2}(G)$ to be the set of all functions $u(x)$ of the form

$$u = \sum_{i=1}^{\infty} f_i * g_i\, ;\ f_i \in D^{p_1,p_2}(G),\ g_i \in C_c(G) \subset S_{q_1,q_2}$$

with $\sum_{i=1}^{\infty} \|f_i\|_\wedge \|g_i\|_\vee < \infty$, ($C_c$ is dense in S_{q_1,q_2}) and define $u \to |||u|||_{p_1,p_2}$ by

$$(3.4) \qquad |||u|||_{p_1,p_2} = \inf\left\{\sum_{i=1}^{\infty} \|f_i\|_\wedge \|g_i\|_\vee;\ u = \sum_{i=1}^{\infty} f_i * g_i \text{ in } D_{p_1,p_2}\right\}$$

the infimum being taken over all the representations for u in D_{p_1,p_2}. Evidently, $|||u|||_{p_1,p_2}$ is a norm of $D_{p_1,p_2}(G)$.

It is easy to see that $D_{p_1,p_2}(G)$ is a dense linear subspace of $C_0(G)$ and the same like as proposition 2.1, we have the following

PROPOSITION 3.1. *The space $D_{p_1,p_2}(G)$ is a dense linear subspace of $C_0(G)$ and is a Banach space with respect to the new norm $|||\ |||_{p_1,p_2}$ and the topology so defined is not weaker than the uniform norm topology.*

We say that a multiplier T of $D^{p_1,p_2}(G)$ means a bounded linear operator on $D^{p_1,p_2}(G)$ which commutes with translation operators and denote the multiplier space of $D^{p_1,p_2}(G)$ by $M(D^{p_1,p_2})$.

THEOREM 3.2. *Let G be a locally compact abelian group. The multiplier space $M(D^{p_1,p_2})$ is isometrically isomorphic to $D_{p_1,p_2}(G)^*$, the conjugate space of $D_{p_1,p_2}(G)$.*

PROOF. For any $T \in M(D^{p_1,p_2})$, define

$$\mu(u) = \sum_{i=1}^{\infty} Tf_i*g_i(0) \tag{3.5}$$

for $u = \sum_{i=1}^{\infty} f_i*g_i$ in D_{p_1,p_2}. μ is well-defined, i. e. $\mu(u)$ is independent of the particular representation of u chosen. To show this it suffices to show that if $u = \sum_{i=1}^{\infty} f_i*g_i = 0$ in $D_{p_1,p_2}(G)$ and $\sum_{i=1}^{\infty} \|f_i\|_\wedge \|g_i\|_\vee < \infty$, then $\sum_{i=1}^{\infty} Tf*g_i(0) = 0$.

Let $\{e_\alpha\}$ be an approximate identity for $L^1(G)$ with $\|e_\alpha\|_1 = 1$. Since $L^1*L^p(G) = L^p(G)\,(1<p<\infty)$, $e_\alpha*f \in D^{p_1,p_2}$ for all $f \in D^{p_1,p^*}$ and

$$\|e_\alpha*f - f\|_\wedge \longrightarrow 0$$

for the limit taking over the index α. Then

$$|T(e_\alpha*f_i)*g_i(0) - Tf_i*g_i(0)| > \|T\| \; \|e_\alpha*f_i - f_i\|_\wedge \; \|g_i\|_\vee \longrightarrow 0,$$

we have

$$\lim_\alpha T(e_\alpha*f_i)*g_i(0) = Tf_i*g_i(0).$$

Since $u = \sum_{i=1}^{\infty} f_i*g_i = 0$ and the convergence of the series $\sum_{i=1}^{\infty} f_i*g_i$ is uniform, we see that

$$\begin{aligned}
\sum_{i=1}^{\infty} T(e_\alpha*f_i)*g_i(\cdot) &= \sum_{i=1}^{\infty} \int \rho_y T(e_\alpha*f_i)(\cdot)\, g_i(y)\, dy \\
&= \sum_{i=1}^{\infty} \int T\rho_y(e_\alpha*f_i)(\cdot)\, g_i(y)\, dy \\
&= \sum_{i=1}^{\infty} T(e_\alpha*f_i*g_i)(\cdot) \\
&= T\left(e_\alpha * \sum_{i=1}^{\infty} (e_\alpha*g_i)\right)(\cdot) = 0.
\end{aligned}$$

and then for any large integer N,

$$\left|\sum_{i=1}^{\infty} Tf_i*g_i(0)\right| = \left|\sum_{i=1}^{\infty} Tf_i*g_i(0) - \sum_{i=1}^{\infty} T(e_\alpha*f_i)*g_i(0)\right|$$
$$\leqq \left|\sum_{i=1}^{N} Tf_i*g_i(0) - \sum_{i=1}^{N} T(e_\alpha*f_i)*g_i(0)\right|$$
$$+ 2\|T\| \sum_{i=N+1}^{\infty} \|f_i\|_\wedge \|g_i\|_\vee$$

the right hand side of this last inequality can be made arbitrarily small by taking a sufficiently large positive integer N, and then passing to the limit with respect to α. Therefore we conclude that

$$\sum_{i=1}^{\infty} Tf_i*g_i(0) = 0\,.$$

It is obvious that the mapping $T \to \mu$ is injective(one to one). We show that it is an isometry. Indeed,

$$|\mu(u)| \leqq \sum_{i=1}^{\infty} |Tf_i*g_i(0)|$$
$$\leqq \sum_{i=1}^{\infty} \|Tf_i\|_\wedge \|g_i\|_\vee$$
$$\leqq \|T\| \sum_{i=1}^{\infty} \|f_i\|_\wedge \|g_i\|_\vee$$

implies that

$$|\mu(u)| \leqq \|T\| \; |||u|||_{p_1,p_2}\,.$$

Hence $\|\mu\| \leqq T\|$.

On the other hand,

$$\|T\| = \sup_{\substack{\|f\|_\wedge \leqq 1 \\ \|g\|_\vee \leqq 1}} |Tf*g(0)| \leqq \sup_{|||f*g|||_{p_1,p_2} \leqq 1} |\mu(f*g)| \leqq \|\mu\|\,.$$

(see (3. 3)). Therefore

$$\|T\| = \|\mu\|.$$

Finally we show that the mapping $T \to \mu$ is surjective (onto).

Suppose that $\mu \in D_{p_1,p_2}(G)^*$ and $f \in D^{p_1,p_2}(G)$, define

$$g \longrightarrow \mu(f*g) = t(g) \qquad \text{on } C_c(G) \subset S_{q_1,q_2}\,.$$

By Hahn Banach theorem, the bounded linear functional t can be extended to S_{q_1,q_2}

and

$$|\mu(f*g)| \leqq \|\mu\| \, \|f\|_{\wedge}\|g\|_{\vee} \text{ for } f\in D^{p_1,p_2},\ g\in S_{q_1,q_2}.$$

It follows from (3. 3) that there is a unique $Tf\in D^{p_1,p_2}$ such that

$$Tf*g(0)=\mu(f*g)=t(g) \qquad \text{for } g\in C_c(G)\subset S_{q_1,q_2},$$

and $\|Tf\|_{\wedge}\leqq\|\mu\|\,\|f\|_{\wedge}$. Hence T is a continuous linear operator on D^{p_1,p_2}. It remains to show this bounded operator T is actually a multiplier on $D^{p_1,p_2}(G)$. Indeed, for any $f\in D^{p_1,p_2}$, $g\in S_{q_1,q_2}$, and $a\in G$, we see that $\rho_a f\in D^{p_1,p_2}$ and $\rho_a g \in S_{q_1,q_2}$. Then

$$\begin{aligned} T(\rho_a f)*g(0) &= \mu(\rho_a f*g)=\mu(f*\rho_a g)\\ &= Tf*\rho_a g(0)=\rho_a Tf*g(0) \end{aligned}$$

holds for arbitrary function g in S_{q_1,q_2}, we have

$$T\rho_a f=\rho_a Tf\in D^{p_1,p_2}\cong S^*_{p_1,q_2}$$

for every $f\in D^{p_1,p_2}(G)$. Hence $T\rho_a=\rho_a T$. This shows that $T\in M(D^{p_1,p_2})$.

Q. E. D.

REMARK 3. 3. When G is compact abelian group, then $D^{p_1,p_2}(G)=L^r(G)$, $r=\max(p_1,p_2)$, is a commutative Banach algebra under convolution. In this case, the multiplier problem reduces to the case of general Lebesgue spaces $L^r(G)\,(1<r<\infty)$ (see [8], [9] and also [13]). The characterization of $D_{p_1,p_2}(G)^*$ is depending on the index r, $1<r<\infty$. Since $M(L^r)=M(L^{r'})$ for $1/r+1/r'=1$, the multiplier space of $L^r(G)$ for $1<r<2$ and for $2<r<\infty$ are the same. Thus we divide it in the following two cases

(i) $r=2$. We refer to Corollary 2. 7 that $D_{p_1,p_2}(G)^*\cong P(G)$.

(ii) $2<r<\infty$. In this case $L^r\subset L^2\subset L^1$ and for $g\in C(G)\subset S_{q_1,q_2}$, $g=g_1+g_2$ with norm

$$\|g\|_{\vee}=\inf\,(\|g_1\|_{q_1}+\|g_2\|_{q_2})=\|g\|_{r'}$$

since $\|g\|_{q_1}$, $\|g\|_{q_2}\geqq\|g\|_{r'}$, the infimum norm can be chosen so that g_1 or $g_2=0$. We will show that

$$D_{p_1,p_2}(G)^* \longrightarrow P(G)$$

is continuous.

For $\mu\in D_{p_1,p_2}(G)^*$, there is a multiplier $T_\mu\in M(D^{p_1,p_2})$ such that

$$\mu(f*g)=T_\mu f*g(0) \qquad \text{for any } f\in D^{p_1,p_2}(G) \text{ and } g\in C(G)\subset S_{q_1,q_2}.$$

Define ν by

$$\nu(f*g)=\mu(f*g) = T_\mu f*g(0) \qquad f \in D^{p_1,p_2} = L^r \subset L^2 \text{ and } g \in C(G) \subset L^2 .$$

Then

$$\begin{aligned}\nu(f*g)| &= \left|\int_G T_\mu f(x)g(-x)\, dx\right| \\ &\leqq \|T_\mu\| \, \|f*g\|_{A(G)}\end{aligned}$$

Since $D_{p_1,p_2}(G)$ is dense in $A(G)$, ν defines on all of $A(G)$ such that

$$\|\nu\|_{A(G)} \leqq \|T_\mu\| = \|\mu\|_{D_{p_1,p_2}}$$

and hence ν is a pseudomeasure.

Note that for any $\nu \in P(G)$, and $f \in L^r(G)$, $g \in C(G) \subset L^{r'}(G)$,

$$|\mu(f*g)| = |\nu(f*g)| \leqq \|\nu\|_{A(G)} \|f\|_r \|g\|_2$$

but the right hand side does not necessarily dominated by $\|\nu\|_{A(G)} \|f\|_r \|g\|_{r'}$ since $\|g\|_2 > \|g\|_{r'}$ in general. Hence we can not obtain $\|\mu\|_{D_{p_1,p_2}} \leqq C\|\nu\|_{A(G)}$. Consequently, we obtain (cf. Larsen [15: Theorem 4. 3. 2])

COROLLARY 3. 4. *Let G be a compact abelian group. Then for $1 < p_1 \neq 2 \neq p_2 < \infty$, there is a continuous algebra isomorphism from $D_{p_1,p_2}(G)^*$ into $P(G)$, the space of pseudomeasures.*

Using the argument, mutatis mutandis, like as Theorem 3. 2, we can characterize the multipliers of $L^1(G) \cap L^p(G)\,(1 < p < \infty)$. We give the norm of $D^p(G) = L^1 \cap L^p(G)$ by

$$\|\!|f|\!\| = \max\,(\|f\|_1, f\|_p)\,. \tag{3. 6}$$

Then $D^p(G)$ is a Banach algebra under convolution and is a dense ideal of $L^1(G)$. In particular if $p=2$, $D^2(G) = L^1 \cap L^2(G) = A^2(G)$. Let

$$S_q = \{g(x)\,|\,g(x) = g_1(x) + g_2(x) \text{ with } (g_1, g_2) \in C_0 \times L^q(G)\}$$

and the norm is defined by

$$\|g\| = \inf\,\{\|g_1'\|_\infty + \|g_2'\|_q \text{ for } g = g_1' + g_2',\ (g_1', g_2') \in C_0 \times L^q\}\,. \tag{3. 7}$$

It is known that (see Liu and Wang [6: Theorem 5])

$$S_q^* \cong D^p(G) \qquad \left(1 < p < \infty, \frac{1}{p} + \frac{1}{q} = 1\right). \tag{3.8}$$

Define the space $D_p(G)$ to be the set of all functions $u(x)$ of the form:

$$u = \sum_{i=1}^{\infty} f_i * g_i\,;\ f_i \in D^p(G),\ g_i \in C_c(G) \subset S_q \text{ with } \sum_{i=1}^{\infty} |||f_i|||\, \|g_i\| < \infty\,.$$

The space $D_p(G)$ will be endowed the norm

$$|||u|||_p = \inf\left\{\sum_{i=1}^{\infty} |||f_i|||\ \|g_i\|;\ u = \sum_{i=1}^{\infty} f_i * g_i \text{ in } D_p(G)\right\}, \tag{3.9}$$

the infimum being taken over all $f_i \in D^p(G)$ and $g_i \in C_c(G) \subset S_q$ for the representation of u in $D_p(G)$. By the same argument of Theorem 3.2, we have the following

THEOREM 3.5. *The multiplier space $M(D^p)$ is isometrically isomorphic to $D_p(G)^*$, the dual space of $D_p(G)$.*

REMARK 3.6. *If G is a non-compact locally compact abelian group, then by the argument, mutatis mutandis, like as Corollary* 2.6 *and Figà-Talamanca and Gaudry* [14: Theorem 3.2], *we can derive that*

$$D_p(G)^* \cong M(G)\,.$$

4. Isomorphisms of $A^p(G)$-algebras.

From [14: Theorem 3.1], it is obvious that for any multiplier $T \in M(A^p)$, there is a unique bounded measure $\mu \in M(G)$ such that

$$Tf = \mu * f \qquad \text{for every } f \in A^p(G)$$

provided that G is non-compact locally compact abelian group. Using this representation, we have the following

THEOREM 4.1. *Let G_1 and G_2 be locally compact abelian groups and ψ be an algebraic isomorphism of $A^p(G_1)$ onto $A^p(G_2)$ $1 \leqq p < \infty$. Suppose that one of $\hat{G}_1$ and $\hat{G}_2$ is connected, then ψ induces a homeomorphic* isomor*phism τ carrying G_2 onto G_1. Furthermore, $\psi f(x) = C\hat{x}(x)\, f(\tau x)$ for $f \in A_P(G_1)$ where $\hat{x}(x)$ is a fixed character on G_2 and C a constant depending only on the choice of Haar measure in G_2.*

We note that the maximal ideal space of $A^p(G)$ can be identified with the character group $\hat{G}$ (see Larsen, Liu and Wang [5: Theorem 4]). Since the isomorphism ψ of $A^p(G_1)$ onto $A^p(G_2)$ maps the maximal ideals of $A^p(G_1)$ onto the maximal ideals of $A^p(G_2)$, ψ induces to a homeomorphism carrying $\hat{G}_1$ onto $\hat{G}_2$. Therefore if one of $\hat{G}_1$ and $\hat{G}_2$ is connected then both of $\hat{G}_1$ and $\hat{G}_2$ are connected. Hence G_1 and G_2 are non-compact and then the result of [14] is applicable.

First we show the following lemma which will be useful in the proof of theorem.

LEMMA 4.2. *Let* $\mu \in M(G)$. *If* $\mu * f = 0$ *for all* $f \in A^p(G)$, *then* $\mu = 0$.

PROOF. Suppose that K is any compact set in $\hat{G}$, then there exists $k \in L^1(G)$ such that $\hat{k} = 1$ on K and $\hat{k}$ has compact support in $\hat{G}$, we see that $k \in A^p(G)$. Therefore for $\mu \in M(G)$,

$$\mu * k = 0 \qquad \text{implies} \qquad \hat{\mu} \cdot \hat{k} = 0 .$$

That is $\hat{\mu}(\hat{x}) = 0$ for all $\hat{x} \in K$. Since K is an arbitrary compact set in $\hat{G}$, this implies $\hat{\mu}(\hat{x}) = 0$ for all $\hat{x} \in \widehat{G}$. Hence $\mu = 0$, by uniqueness theorem. Q. E. D.

PROOF OF THEOREM 4.1. Take $u \in M(G_2)$. For any $f \in A^p(G)$, we define an operator T on $A^p(G_1)$ by

$$Tf = \psi^{-1}(u * \psi f) . \tag{4.1}$$

It is well-defined since $A^p(G)$ is an ideal of $M(G)$. Since the algebras $A^p(G_1)$ and $A^p(G_2)$ are semi-simple and commutative, ψ is bicontinuous (cf. Rudin [16: 4.1]) Hence T is a bounded operator on $A^p(G_1)$ and

$$\begin{aligned} T(f * g) &= \psi^{-1}(u * \psi(f * g)) \\ &= \psi^{-1}(u * \psi f) * g \\ &= Tf * g, \end{aligned}$$

T is a multiplier of $A^p(G_1)$. By assumption, one of $\hat{G}_1$ and $\hat{G}_2$ is connected so both $\hat{G}_1$ and $\hat{G}_2$ are connected. Therefore G_1 and G_2 are non-compact, and there exists uniquely a μ in $M(G_1)$ such that

$$\mu * f = Tf = \psi^{-1}(u * \psi f) . \tag{4.2}$$

This μ is uniquely determined by u, we can define a mapping Φ of $M(G_2)$ into $M(G_1)$ by

$$(4.3) \qquad \Phi u*f = \Psi^{-1}(u*\Psi f).$$

We shall show that Φ is an isomorphism of $M(G_2)$ onto $M(G_1)$.

Let $u, v \in M(G_2)$ and f be any element in $A^p(G_1)$. It follows from (4. 3) that

$$\begin{aligned}\Phi(u*v)*f &= \Psi^{-1}((u*v)*\Psi f)\\ &= \Psi^{-1}(u*\Psi(\Psi^{-1}(v*\Psi f)))\\ &= \Phi u*\Psi^{-1}(v*\Psi f)\\ &= (\Phi u*\Phi v)*f.\end{aligned}$$

Since f is arbitrary in $A^p(G_1)$, by Lemma 4. 2,

$$(4.4) \qquad \Phi(u*v) = \Phi u*\Phi v$$

while the linearity of Φ is obvious, Φ is a homomorphism.

For any $\mu \in M(G_1)$, define an operator S on $A^p(G_2)$ by

$$(4.5) \qquad Sg = \psi(\mu*\Psi^{-1}g)$$

for any g in $A^p(G_2)$. Then the same arguments as we have done before show that S is a multiplier of $A^p(G_2)$. Hence there exists $u \in M(G_2)$ such that

$$u*g = Sg = \Psi(\mu*\psi^{-1}g)$$

or

$$\psi^{-1}(u*g) = \mu*\psi^{-1}g.$$

Since ψ is an onto isomorphism, we have

$$(4.6) \qquad \psi^{-1}(u*\psi f) = \mu*f$$

for any $f \in A^p(G_1)$. And so by (4. 3), $\Phi u = \mu$. This shows that Φ is an onto map. If $\Phi u = 0$, then $\psi^{-1}(u*\psi f) = 0$. Hence $u*\psi f = 0$, which implies $u = 0$ (see Lemma 4. 2) proving the one-to-one property of Φ. Therefore Φ is an isomorphism of $M(G_2)$ onto $M(G_1)$. Since both algebras $M(G_1)$ and $M(G_2)$ are semi-simple and commutative, Φ is bicontinuous. Now for function $g \in A^p(G_2)$,

$$\Phi g*f = \psi^{-1}(g*\psi f) = \psi^{-1}(g)*f$$

for any $f \in A^p(G_1)$. Hence $\Phi g = \psi^{-1}g$ (Lemma 4. 2) proving that $\Phi|_{A^p(G_2)} = \psi^{-1}$. Since the algebra $A^p(G_2)$ is dense in $L^1(G_2)$, $\Phi|_{L^1(G_2)}$ becomes an isomorphism of

$L^1(G_2)$ onto $L^1(G_1)$ (see Rudin [16], Theorem 4. 6. 4). Then the result of Beurling and Helson is applicable (cf. Rudin [16: 4. 7. 2]) and hence the theorem is complete.
Q. E. D.

REMARK. It is remarkable that the proof of Theorem 4. 1 can be taken over for a general theorem on any dense subalgebras of $L^1(G)$ as following

THEOREM 4. 3. *Let G be a locally compact abelian group and $S(G)$ be a Banach subalgebra of $L^1(G)$ with respect to some norm and it is a dense ideal of $L^1(G)$ and the maximal ideal space is identified with $\hat{G}$. Suppose that the multipliers of $S(G)$ can be characterized by the bounded measures. Then the algebraic isomorphism Φ of $S(G_1)$ onto $S(G_2)$ can be reduced to a topological isomorphism τ carrying G_2 onto G_1 provided one of $\hat{G}_1$ and $\hat{G}_2$ is connected. Furthermore,*

$$\Phi f(x) = C\hat{x}(x)f(\tau x) \qquad \text{for } f \in S(G_1)$$

where $\hat{x}$ is a fixed character on G_2 and C a constant depending only on the choice of the Haar measure in G_2.

By [14: Theorem 3, 2] and the above theorem, it is immediately that

COROLLARY 4. 4. *Let G_1 and G_2 be locally compact abelian groups and Φ be an algebraic isomorphism of $D^p(G_1)$ onto $D^p(G_2)$ $(1<p<\infty)$, then G_1 and G_2 are topological isomorphic provided that one of $\hat{G}_1$ and $\hat{G}_2$ is connected.*

5. Additional remark for the continuous linear mappings from $L^1(G)$ to $A^p(G)$. Let A be a normed algebra and B be an A-module normed linear space. Consider the normed linear space $M(A, B)$ of all continuous linear mappings T: $A \to B$ that have the property

$$T(a*x) = a*Tx \qquad \text{for all } a, x \in A.$$

Evidently $M(A, A)$ is the space of all multipliers of A, and since for any $b \in B$,

$$a \longrightarrow a*b \in B \qquad \text{for all } a \in A,$$

the space $M(A, B)$ contains all of B.

Concerning the class $M(A, B)$, there are many characterizations which are known. For examples, (5.1) $M(L^1(G),\ L^1(G)) \cong M(G)$;

(5.2) $M(L^1(G),\ L^p(G)) \cong L^p(G) \quad 1<p<\infty$;

(5.3) $M(A^p(G),\ L^1(G)) \cong M(A^p(G),\ A^p(G)) \quad 1 \leqq p < \infty$

$\cong M(G)$ if G is non-compact abelian.

Liu and Rooij [7] proved the following

LEMMA 5.1. *Let A be a normed algebra with bounded approximate identity $\{e_\alpha\}$ with $\|e_\alpha\|_A \leqq 1$ and B a normed right A-module such that $x*e_\alpha \to x$ for all $x \in B$, where limit being taken over α. Then there is a natural isometry*

(5.4) $$M(A,\ B') \cong B'$$

where B' denotes the dual space of B.

Since the Lebesgue space $L^p(G)$ is reflexive $1<p<\infty$, (5.2) follows directly from this lemma.

Using this lemma, Liu and Rooij [7: Proposition 2.9] show that

(5.5) $$M(L^1(G),\ A^1(G)) \cong A^1(G).$$

We ask that whether the space $M(L^1(G),\ A^p(G))$ of operators for $p>1$ can be characterized as a function space. There is a slight extension of (5.5) to the case of $1<p\leqq 2$. That states as following

PROPOSITION 5.2. *Let G be a locally compact abelian group. The algebra $A^p(G)$ is an $L^1(G)$-module under convolution and*

(5.6) $$M(L^1(G),\ A^p(G)) \cong A^p(G) \qquad \text{for } 1<p\leqq 2.$$

The proof of this theorem can be proved likewise, mutatis mutandis, as that for Proposition 2.9 in Liu and Rooij [7]. The only task is to show that the space $L^q(\hat{G})$ (so does $C_0(G)\vee_{Hq}L^q(\hat{G})$) is also $L^1(G)$-module where $1/p+1/q=1$, $1<p\leqq 2$. Now we sketch simply the proof as follows.

PROOF OF PROPOSITION 5.2. Since $A^p(G)$ is an ideal of $L^1(G)$ and

$$\|f*h\|^p \leqq \|f\|_1\ \|h\|^p \qquad \text{for } f\in L^1(G) \text{ and } h \in A^p(G),$$

$A^p(G)$ is $L^1(G)$-module. We will use Lemma 5.1 to show the identity (5.6).

Let $\{e_\alpha\}$ be an approximate identity for $L^1(G)$ such that $\|e_\alpha\|_1 \leqq 1$ for all α. It is clear that $C_0(G)$ is a normed module over $L^1(G)$. For $L^q(\hat{G})$, we define

$$f \triangle g = \hat{f} \cdot g \qquad \text{for } f \in L^1(G) \text{ and } g \in L^q(\hat{G}).$$

Then we have

$$\|f \triangle g\|_q = \|\hat{f} \cdot g\|_q \leqq \|f\|_1 \ \|g\|_q. \tag{5.7}$$

On the other hand,

$$\lim_\alpha (e_\alpha \triangle g) = \lim_\alpha \hat{e}_\alpha \cdot g = g \qquad \text{for all } g \in L^q(\hat{G}). \tag{5.8}$$

Indeed, for $1 < p \leqq 2$, the Fourier transforms $\widehat{L^p}(G)$ is dense in $L^q(\hat{G})$ and

$$\|\hat{\tilde{e}}_\alpha \hat{h} - \hat{h}\|_q \leqq \|\tilde{e}_\alpha * h - h\|_p \longrightarrow 0 \qquad \text{for all } h \in L^p(G)$$

i. e. $\lim_\alpha \hat{\tilde{e}}_\alpha \hat{h} = \lim_\alpha \widehat{\tilde{e}_\alpha * h} = \hat{h}$ for all $h \in L^p(G)$,

implies that (5. 8) holds. Hence $L^q(\hat{G})$ is an $L^1(G)$-module by (5. 7) and (5. 8).

Next we show that H_q, the closure of $\{(\tilde{h}, -\hat{h}) \mid h \in A^1(G)\}$ in $C_0(G) \times L^q(\hat{G})$, is an $L^1(G)$-module, it is immediately that

$$\begin{aligned} f*(\tilde{h}, -\hat{h}) &= ((\tilde{f}*h)^{\sim}, -f\Delta\hat{h}) \\ &= ((\tilde{f}*h)^{\sim}, -(\tilde{f}*h)^{\wedge}) \in H_q. \end{aligned}$$

Consequently, $C_0(G) \vee_{Hq} L^q(\hat{G})$ is $L^1(G)$-module and

$$\lim_\alpha e_\alpha * u = u \qquad \text{for all } u \in C_0(G) \vee_{Hq} L^q(\hat{G}).$$

Therefore

$$M(L^1(G), (C_0(G) \vee_{Hq} L^q(G))^*) \cong (C_0(G) \vee_{Hq} L^q(\hat{G}))^* \cong A^p(G)$$

or

$$M(L^1(G), A^p(G)) \cong A^p(G) \qquad \text{for } 1 < p \leqq 2. \qquad \text{Q. E. D.}$$

REMARK. For $p > 2$. the characterization of $M(L^1(G), A^p(G))$ is an open question.

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