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PROJECTIONS IN HILBERT SPACE II

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In [1] we showed, by using methods of group representations, that three projections suffice to generate all of $B(H)$. Here we want to show some variants of this result and also apply the method of group representations to the problem of two projections. This permits us to obtain without difficulty the recent results of Halmos [2] by an algebraic method. Throughout H will be a separable Hilbert space. Let P and Q be two projections. Then P and Q determine a unitary representation U of $G = Z_2 * Z_2$, the free product of two groups Z_2 of order two by

$$(1) \qquad U_\alpha = 1 - 2P,\ U_\beta = 1 - 2Q.$$

Here α and β are the generators of G, $\alpha^2 = \beta^2 = e$. With the generators $\gamma = \alpha\beta$ and α and the relations $\alpha\gamma\alpha = \gamma^{-1}$ and $\alpha^2 = e$, we see that G can also be considered as the infinite dihedral group. In particular G is the semidirect product of $\Gamma = \{\gamma^n \mid n = 0, \pm 1, \cdots\}$ and $\{e, \alpha \mid \alpha^2 = e\}$ may be replaced by: Z_2. Thus any irreducible representation of G is induced from a character of Γ and is at most two dimensional. As a consequence this shows that the W^*-algebra $\mathfrak{A}$ generated by P and Q is of type $I_{\leqq 2}$. The two dimensional irreducible representations of G have the canonical form

$$(2) \qquad U_\lambda(\gamma) = \begin{pmatrix} e^{i\lambda} & 0 \\ 0 & e^{-i\lambda} \end{pmatrix},\ U_\lambda(\alpha) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad 0 < \lambda < \pi.$$

By the unitary transformation $\frac{1}{\sqrt{2}}\begin{pmatrix} i & -i \\ 1 & 1 \end{pmatrix}$ this is brought to the equivalent form

$$(3) \qquad V_\lambda(\gamma) = \begin{pmatrix} \cos\lambda & -\sin\lambda \\ \sin\lambda & \cos\lambda \end{pmatrix},\ U_\lambda(\alpha) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Any unitary representation u of G can thus be written as a direct sum $U_1 \oplus U_2$ or $U_1 \oplus V_2$ of the parts, which are homogeneous with respect to the dimension of their irreducible components. Each U_2 and V_2 can be considered as a direct integral of their irreducible components

$$U_2 = \int U_\lambda d\mu(\lambda) \qquad V_2 = \int V_\lambda d\mu(\lambda) \tag{4}$$

where μ is a bounded Borel measure on $[0, \pi]$ with $\mu(\{0\}) = \mu(\{\pi\}) = 0$. Equivalently we can say that H can be written as $H = (K \oplus K) \oplus H_1$ such that U_1 acts on H_1 only and such that U_2 has the matrix representation on $K \oplus K$

$$U_2(\gamma) = \begin{pmatrix} W & 0 \\ 0 & W^* \end{pmatrix}, \quad U_2(\alpha) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{5}$$

where the spectrum of W is contained in the upper half unit circle, with 1 and -1 not in the point spectrum of W. Similarly

$$V_2(\gamma) = \begin{pmatrix} C & -S \\ S & C \end{pmatrix}, \quad V_2(\alpha) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{6}$$

where C and S are commuting selfadjoint operators, satisfying $C^2 + S^2 = 1$ and 0 is not in the point spectrum of S. This shows immediately that P and Q are in a generic position [2] iff the W^*-algebra they generate is homogeneous of type I_2. This is just the case if 1 and -1 are not eigenvalues of $U(\gamma)$. The method of the direct integral also makes clear, why the general situation can be extended so easily from the two by two matrix case. If P and Q are in generic position their unitary invariant is the same as that of the representation U_2. The invariant of U_2 is clearly $U_2(\gamma)$. From (6) we obtain easily

$$V_2(\beta) = \begin{pmatrix} -C & S \\ S & C \end{pmatrix}. \tag{7}$$

THEOREM 1. *Let P and Q be in a generic position. Then there exist a decomposition $H = K \oplus K$ of H and selfadjoint commuting operators C, S with $C^2 + S^2 = 1$ and* ker $S = 0$ *such that P and Q have the representation*

$$P = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \qquad Q = 1/2 \begin{pmatrix} 1+C & -S \\ -S & 1-C \end{pmatrix}$$

PROOF. Use (1), (6) and (7).

Other forms can easily be derived from this. (5) shows

THEOREM 2. *A unitary operator U can be written as a product of two symmetries iff $e^{i\lambda}$ belongs to the point (continuous) spectrum of U then $e^{-i\lambda}$ belongs to the point (continuous) spectrum of U and the corresponding multiplicities* agree.

If one of the projections, say P, is finite dimensional, $\dim P = n < \infty$, the space $K \oplus K$ in our construction has at most the dimension $2n$. Thus there exists a central projection Z of $\mathfrak{A}$ with $P \leqslant Z$ and $\dim Z \leqslant 2n$. This result has a curious consequence. Let $A_1, \ldots, A_m$ be operators on H of finite rank and let Q be an arbitrary projection on H, then the W^*-algebra generated by $A_1, \ldots, A_m$ and Q is not $B(H)$ if H is infinite dimensional. To see this let P be the join of all ranges of $A_1, A_1^*, \ldots, A_m, A_m^*$. Then $\dim P = l < \infty$ and there exists a projection Z with $P \leqslant Z$, $\dim Z \leqslant 2l$ and $ZQ = QZ$. Clearly also $ZA_i = A_i Z = A_i$, $i = 1, \ldots, m$. That this result is optimal follows from the next theorem.

THEOREM 3. *Let H be a separable infinite dimensional Hilbert space, then $B(H)$ is generated by three projections P_1, P_2 and P_3, which we may choose such that $P_1 \cdot P_2 = 0$ and $\dim P_2 = n$ with $n = 1, 2, \cdots\cdots$ or ∞.*

For this we need the following lemma.

LEMMA. *$B(H)$ is generated by a positive invertible operator A and a projection P of arbitrary finite dimension.*

PROOF. Let $B(H) = l^2(Z)$, Z the integers, and let U be the bilateral shift on H. $U\,\varepsilon_n = \varepsilon_{n+1}$, where $\{\varepsilon_n\}_{n \in Z}$ is an orthonormal basis of H. U generates a maximal abelian subalgebra $\mathfrak{B}$ of $B(H)$, and we can always find a positive invertible generator A of $\mathfrak{B}$. Let P be the projection on $[\varepsilon_0, \cdots\cdots, \varepsilon_{n-1}]$ and assume the operator C commutes with P and A. Then $C \in \mathfrak{B}$ and we can write $C = \Sigma\, c_n U^n$ with $\Sigma |c_n|^2 < \infty$. It is now easy to see that such a C leaves $[\varepsilon_0, \cdots\cdots, \varepsilon_{n-1}]$ invariant only if $C = \lambda 1$.

PROOF OF THE THEOREM. Write $H = K \oplus K$ and define

$$P_1 = \begin{pmatrix} 1-p & 0 \\ 0 & 0 \end{pmatrix}, \quad P_2 = \begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix} \text{ and } P_3 = \begin{pmatrix} a & t \\ t & 1-a \end{pmatrix}$$

with $t = [a(1-a)]^{1/2}$, where a and p are chosen as generators of $B(K)$ as in the lemma such that $0 < a < 1$ and $\dim p = n$. Then the P_i are projections with the required properties. Let C be an operator which commutes with all P_i, $i = 1, 2, 3$. Then $C(P_1 + P_2) = (P_1 + P_2)C$ shows that C is diagonal $C = \begin{pmatrix} c_1 & 0 \\ 0 & c_2 \end{pmatrix}$. Then $CP_2 = P_2 C$ and $CP_3 = P_3 C$ give $c_1 p = p c_1$, $a c_1 = c_1 a$ and $t c_1 = c_2 t$. By assumption $c_1 = \lambda 1$ and $c_1 = c_2$.

We should remark that three projections P_1, P_2, P_3 with $P_1 \cdot P_2 = 0$ and dim $P_3 = n < \infty$ do not generate $B(H)$ if H is infinite dimensional. Since P_1, P_2 and P_3 form an irreducible system, the C^*-algebra generated by these projections contains all compact operators. Thus any compact operator can be approximated uniformly by polynomials in these three projections. To return to our initial consideration, one may ask whether a pair $\{P, E\}$, consisting of a projection P and an idempotent E can be discussed in a similar geometric fashion as in [2] or in a similar algebraic fashion as above. However the next result shows that this is impossible in general.

THEOREM 4. *Let H be a separable infinite dimensional Hilbert space, then $B(H)$ is generated by an idempotent E and a projection P. P may be chosen to be finite dimensional of arbitrary dimension n.*

PROOF. Let $H = K \oplus K$ and let $P = \begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix}$ and $E = \begin{pmatrix} 1 & a \\ 0 & 0 \end{pmatrix}$, where $0 < a < 1$ and p are chosen as in the lemma. Let $C = C^*$ commute with P and E. Since $CEE^* = EE^*C$ we see that C is diagonal $C = \begin{pmatrix} c_1 & 0 \\ 0 & c_2 \end{pmatrix}$ with $c_1 a = a c_1$. Then $CP = PC$ gives $c_1 p = p c_1$ or $c_1 = \lambda 1$ by assumption. $CE = EC$ shows as before that $c_1 = c_2 = \lambda 1$.

REFERENCES

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