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ON POINTWISE APPROXIMATION OF FOURIER SERIES BY TYPICAL MEANS

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1. **Notations and results.** Throughout the paper $f(x)$ will be a real-valued, 2π-periodic, L-integrable function on the real line such that $\int_{-\pi}^{\pi} f(x)dx = 0$. We denote the Fourier series of f by

$$S(f) \sim \sum_{k=1}^{\infty} A_k(f;\, x), \quad A_k(f;\, x) = a_k \cos kx + b_k \sin kx,$$

where a_k and b_k are its Fourier coefficients.

Let $X_{2\pi}$ be one of the function spaces $C_{2\pi}$, or $L^p_{2\pi}$, $1 \leqq p < \infty$, of the functions in question; the spaces are endowed with their usual norms. Let γ be a positive constant. If for an f in $X_{2\pi}$ the associated series

$$\sum_{k=1}^{\infty} k^{\gamma} A_k(f;\, x)$$

is the Fourier series of some function g in $X_{2\pi}$, we say that f has a *Riesz derivative of order* γ in $X_{2\pi}$ and we write

$$D^{\{\gamma\}} f := f^{\{\gamma\}} = g\,.$$

It is well known (see e. g., P. L. Butzer-K. Scherer [3, Ch. 4]) that the operator $D^{\{\gamma\}}$ is closed with domain

$$X_{2\pi}^{\{\gamma\}} = \{f \in X_{2\pi} : \sum_{k=1}^{\infty} k^{\gamma} A_k(f;\, x) \text{ is the Fourier series of a function } g \text{ in } X_{2\pi}\}$$

dense in $X_{2\pi}$. Since for an $f \in X_{2\pi}^{\{\gamma\}}$, $D^{\{\gamma\}} f = 0$ implies $f = 0$, $D^{\{\gamma\}}$ has an inverse $I^{\{\gamma\}}$. Its extension to the whole space $X_{2\pi}$ is the so-called *Riesz potential of order* γ. With $I^{\{\gamma\}} f := f_{\{\gamma\}}$ (or $:= f^{\{-\gamma\}}$)we have

$$S(f_{\{\gamma\}}) \sim \sum_{k=1}^{\infty} \frac{A_k(f;\, x)}{k^{\gamma}}\,.$$

The *typical means of order* γ

$$P_{n,\gamma}(f;\, x) = \sum_{k=1}^{n}\left(1 - \frac{k^\gamma}{(n+1)^\gamma}\right)A_k(f;\, x) \quad (\gamma > 0,\ n = 1, 2, \cdots)$$

of the Fourier sereis of a function f are closely related to its Riesz derivative $f^{\{\gamma\}}$. Indeed, the following theorem holds true.

THEOREM A. *Let* $\gamma > 0$. *If for the functions* f *and* g *in* $X_{2\pi}$

$$\lim_{n\to\infty} \|(n+1)^\gamma\{R_{n,\gamma} f - f\} - g\|_{X_{2\pi}} = 0,$$

then $f \in X_{2\pi}^{\{\gamma\}}$ *and* $f^{\{\gamma\}} = -g$, *and vice versa. If in particular* $g = 0$, *then* f *is the zero-function.*

The theorem is a modification of the *saturation theorem* for the typical means in $X_{2\pi}$. It has its roots in results due to A. Zygmund and B. Sz.-Nagy in the 1940's. The theorem itself was first formulated and proved by S. Aljančić for the space $C_{2\pi}$ and by G. Sunouchi-C. Watari for the spaces in question in the late 1950's. In the form of Theorem A it is due to P. L. Butzer-E. Görlich. For details we refer to P. L. Butzer-K. Scherer, loc. cit. Finally, we have to mention that G. Sunouchi [4] studied local versions of Theorem A.

It is the aim of this note to prove the following pointwise analogue of Theorem A.

THEOREM B. *Let* $f \in L_{2\pi}$ *be such that*

$$(1) \qquad \lim_{n\to\infty} R_{n,\gamma}(f;\, x) = f(x)$$

finitely for all x *in some interval* (a, b). *If there exists a finitely-valued, L-integrable function* $g(x)$ *in* (a, b) *such that*

$$(2) \qquad \lim_{n\to\infty} (n+1)^\gamma\{R_{n,\gamma}(f;\, x) - f(x)\} = g(x)$$

pointwise for all x *in* (a, b), *then* $f^{\{r-2\}}$ *belongs to* $L_{2\pi}$ *and for almost all* x *in* (a, b)

$$(3) \qquad f^{\{\gamma-2\}}(x) = Ax + B + \int_a^x dt \int_a^t g(u)du,$$

where A and B are some constants.

For $0<\gamma<1$, (3) *remains true even if* (1) *is violated in a denumerable set E of points, supposed that*

(4a) $$|R_{n,\gamma}(f;\ x)| = o(n^{1-\gamma}) \qquad (\textit{for all } x\in E).$$

For $\gamma=1$, (3) *remains true even if* (2) *is violated in a denumerable set E of points.*

For $\gamma>1$, (3) *remains true even if* (2) *is violated in a denumerable set of points E, supposed that*

(4b) $$|R_{n,\gamma}(f;\ x)-f(x)| = o(n^{1-\gamma}) \qquad (\textit{for all } x\in E).$$

For $\gamma=1$ (Fejér means) and $g(x)=0$, Theorem B is due to V. A. Andrienko [1].

The following corollaries are obvious consequences of Theorem B.

COROLLARY 1. *Let f and g be finitely-valued functions in* $L_{2\pi}$ *such that the conditions* (1) *and* (2) *of Theorem B are satisfied for all x except in a denumerable set E of points at which* (4) *holds, then* $f\in L_{2\pi}^{\{\gamma\}}$ *and for almost all x,* $f^{\{\gamma\}}(x) = -g(x)$.

COROLLARY 2. *Let f and g belong to* $C_{2\pi}$ *such that*

$$\lim_{n\to\infty} (n+1)^{\gamma}\{R_{n,\gamma}(f;\ x)-f(x)\} = g(x)$$

pointwise everywhere, then $f\in C_{2\pi}^{\{\gamma\}}$ *and* $f^{\{\gamma\}}=-g$, *and vice versa.*

The second corollary substantially weakens the statement of Theorem A for the space $X_{2\pi}=C_{2\pi}$.

REMARK. The function f_γ given by

(5) $$S(f_\gamma)\sim\sum_{k=1}^{\infty}\frac{\cos kx}{k^{\gamma}} \qquad (\gamma>0)$$

shows that if there is only one exceptional point x_0 (mod 2π) in R for which (4) is violated then the statement of Corollary 1 is wrong. Indeed, the series (5) converges for all $x\neq 0$ (mod 2π) and the associated function f_γ belongs to

$L_{2\pi}$. Moreoevr, for all $x \neq 0 \pmod{2\pi}$

$$\lim_{n\to\infty} (n+1)^{\gamma}\{R_{n,\gamma}(f_\gamma;\, x) - f_\gamma(x)\} = \frac{1}{2},$$

while for $x_0 = 0 \pmod{2\pi}$

$$|R_{n,\gamma}(f_\gamma;\, x_0)| = \begin{cases} \Omega(n^{1-\gamma})\ (n\to\infty), & 0<\gamma<1, \\ \Omega(\log n)\ (n\to\infty), & \gamma = 1, \end{cases}$$

and

$$|R_{n,\gamma}(f_\gamma;\, x_0) - f(x_0)| = \Omega(n^{1-\gamma})\ (n\to\infty),\ \gamma>1.$$

We conclude this section with the formulation of a third theorem although it seems to be known, see G. Sunouchi [4]. The proof follows directly from relation (11) and the Fejér-Lebesgue theorem.

THEOREM C. *Let f be a function in $L_{2\pi}^{[\gamma]}$. For almost all x*

$$\lim_{n\to\infty} (n+1)^{\gamma}\{R_{n,\gamma}(f;\, x) - f(x)\} = -f^{[\gamma]}(x).$$

2. **Proof of Theorem B**. The proof is based on two lemmas. Lemma 1 is a uniqueness theorem for (C, 1)-summable trigonometric series. It can be obtained out of Verblunsky's uniqueness thoerems for Abel summable trigonometric series (cf. A. Zygmund [6, p. 352ff]). Moreover, Lemma 1 is a very special form of results due to F. Wolf [5] about (C, λ)-summable series. In Lemma 2 we rewrite condition (2) so that Theorem B can be concluded directly out of Lemma 1.

LEMMA 1. *Let $\sum_{k=1}^{\infty} A_k(x)$ be a trigonometric series. If in some interval (a, b) the limit*

$$\lim_{n\to\infty} \sigma_n(x) := \lim_{n\to\infty} \sum_{k=1}^{n} \left(1 - \frac{k}{n+1}\right) A_k(x) = g(x) \tag{6}$$

exists finitely, except in a denumerable set E, with $g(x)$ L-integrable, and if for all $x \in E$

$$\sigma_n(x) = o(n) \qquad (n\to\infty), \tag{7}$$

then $\sum_1^\infty A_k(x)/k^2$ *is the Fourier series of some function* F *in* $L_{2\pi}$ *and for almost all* x *in* (a,b)

$$F(x) = Ax + B + \int_a^x dt \int_a^t g(u)du. \tag{8}$$

If, moreover, the interval (a,b) *contains* $[0,2\pi)$ *then* $\sum_1^\infty A_k(x)$ *is the Fourier series of* g.

LEMMA 2. *Let* f *in* $L_{2\pi}$ *be such that for some* x, $R_{n,\gamma}(f;\ x)\to c(x)$ *finitely as* $n\to\infty$. *The limit*

$$\lim_{n\to\infty} (n+1)^\gamma\{R_{n,\gamma}(f;\ x)-c(x)\}$$

exists finitely if, and only if, the limit

$$\lim_{n\to\infty} -\sigma_n^{\{\gamma\}}(f;\ x) := \lim_{n\to\infty}\sum_{k=1}^n\left(1-\frac{k}{n+1}\right)(-k^\gamma)A_k(f;\ x) \tag{9}$$

exists, and both limits are equal.

PROOF. By the use of the identity

$$R_{n-1,\gamma}(f;\ x)-R_{n,\gamma}(f;\ x) = \left\{\frac{1}{n^\gamma}-\frac{1}{(n+1)^\gamma}\right\}\sum_{k=1}^n(-k^\gamma)A_k(f;x)\ (n=1,2,\cdots),$$

see P. L. Butzer-S. Pawelke [2], we obtain the relation

$$R_{n,\gamma}(f;\ x)-c(x) = \sum_{k=n+1}^\infty\left\{\frac{1}{k^\gamma}-\frac{1}{(k+1)^\gamma}\right\}\sum_{j=1}^k(-j^\gamma)A_j(f;x).$$

Introducing the abbreviations $s_n=\sum_{k=1}^n(-k^\gamma)A_k(f;\ x)$ and $t_n=R_{n,\gamma}(f;\ x)-c(x)$, the latter equation simply reads

$$t_n = \sum_{k=n+1}^\infty\left\{\frac{1}{k^\gamma}-\frac{1}{(k+1)^\gamma}\right\}s_k. \tag{10}$$

It is appropriate to use two additional notations: $\tau_n = (n+1)^\gamma t_n$ and $\sigma_n = \left(\sum_{k=1}^{n} s_k\right)/(n+1)$.

To prove the "if"-part, we have to show that $\sigma_n \to s$ as $n \to \infty$ implies $\tau_n \to s$ as $n \to \infty$. Indeed, by partial summation of the sum on the right-hand side of (10) we have

$$\tau_n = (n+1)^\gamma \sum_{k=n+1}^{\infty} (k+1)\left\{\frac{1}{k^\gamma} - \frac{2}{(k+1)^\gamma} + \frac{1}{(k+2)^\gamma}\right\}\sigma_k \tag{11}$$

$$-(n+1)^{1+\gamma}\left\{\frac{1}{(n+1)^\gamma} - \frac{1}{(n+2)^\gamma}\right\}\sigma_n,$$

and the result follows by taking into account that

$$1 = (n+1)^\gamma \sum_{k=n+1}^{\infty} (k+1)\left\{\frac{1}{k^\gamma} - \frac{2}{(k+1)^\gamma} + \frac{1}{(k+2)^\gamma}\right\}$$

$$-(n+1)^{1+\gamma}\left\{\frac{1}{(n+1)^\gamma} - \frac{1}{(n+2)^\gamma}\right\}$$

identically in $n = 0, 1, 2, \cdots$ and that

$$\lim_{n\to\infty} (n+1)^{1+\gamma}\left\{\frac{1}{(n+1)^\gamma} - \frac{1}{(n+2)^\gamma}\right\} = \gamma.$$

On the other hand, by (10)

$$t_{n-1} - t_n = \left\{\frac{1}{n^\gamma} - \frac{1}{(n+1)^\gamma}\right\}s_n := \frac{1}{C_n}s_n \quad (n = 1, 2, \cdots).$$

Setting s_0 as well as the constant C_0 equal to zero, we obtain again by partial summation

$$\sum_{k=0}^{n} s_k = s_0 + \sum_{k=1}^{n} C_k(t_{k-1} - t_k)$$

$$= s_0 + \sum_{k=0}^{n} (C_{k+1} - C_k)t_k - C_{n+1}t_n,$$

or

(12)
$$\sigma_n = \frac{1}{n+1}\sum_{k=0}^{n}\frac{C_{k+1}-C_k}{(k+1)^\gamma}\tau_k - \frac{C_{n+1}}{(n+1)^{1+\gamma}}\tau_n .$$

Since

$$1 = \frac{1}{n+1} + \frac{1}{n+1}\sum_{k=0}^{n}\frac{C_{k+1}-C_k}{(k+1)^\gamma} - \frac{C_{n+1}}{(n+1)^{1+\gamma}}$$

identically in $n = 0, 1, 2, \cdots$, and since $\lim_{n\to\infty} C_{n+1}/(n+1)^{1+\gamma} = \frac{1}{\gamma}$ (see above), it is easy to conclude that $\tau_n \to s$ as $n \to \infty$ implies $\sigma_n \to s$ as $n \to \infty$. This proves the "only if"-part.

The proof of Theorem B now follows by setting the coefficients $A_k(x)$ in Lemma 1 equal to $(-k^\gamma)A_k(f; x)$, $k = 1, 2, \cdots$, i.e., $\sigma_n(x) = -\sigma_n^{\{\gamma\}}(f; x)$.

With respect to the conditions upon $R_{n,\gamma}(f; x)$ at the points x in the exceptional set E, we have to mention that for $0 < \gamma < 1$ (4a) is equivalent to (7), for $\gamma = 1$ (1) implies (7), and for $\gamma > 1$ again (4b) is equivalent (7).

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