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HOMOGENEOUS HYPERSURFACES IN A SPHERE WITH THE TYPE NUMBER 2

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0. Introduction. There is a problem of giving a complete classification of homogeneous hypersurfaces M^n in a sphere S^{n+1} of dimension $n+1 (n \geqq 2)$. This problem can be naturally divided into three cases:

(i) The rank of the second fundamental form (which is called the type number) is not smaller than 3 at some point.
(ii) The type number is equal to 2 at some point.
(iii) The type number is equal to 0 or 1 at some point.

In the case (i), it is known by a theorem of Ryan [9] that the full isometry group of every homogeneous hypersurface M^n can be considered as a subgroup of the orthogonal group $O(n+2)$, in other words, M^n is an orbit of a suitable subgroup of $O(n+2)$. Hsiang and Lawson [5] gave a complete list of compact minimal hypersurfaces in S^{n+1} each of which is an orbit of a subgroup of $O(n+2)$.

The condition "minimal" is not essential because among all homogeneous hypersurfaces obtained as orbits of a compact subgroup of $O(n+2)$ there is a minimal one ([5]). Thus our problem is solved in the case (i) if the hypersurfaces are compact.

The purpose of this paper is to determine all hypersurfaces in S^{n+1} in the case (ii). To describe our results, we begin with an example of homogeneous hypersurface in S^4. Let $S^n(c)$ denote an n-dimensional sphere in Euclidean $(n+1)$-space R^{n+1} with curvature c. We consider the hypersurface in $S^4 = S^4(1)$ defined by the equations

$$(1) \qquad \begin{cases} 2x_2^3 + 3(x_1^2 + x_2^2)x_5 - 6(x_3^2 + x_4^2)x_5 + 3\sqrt{3}(x_1^2 - x_2^2)x_4 \\ \qquad\qquad + 3\sqrt{3}\,x_1 x_2 x_3 = 2, \\ x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 = 1. \end{cases}$$

E. Cartan [2] proved that this space is a homogeneous Riemannian manifold $SO(3)/(\mathbf{Z}_2 \times \mathbf{Z}_2)$ and its principal curvatures are equal to $\sqrt{3}$, 0, and $-\sqrt{3}$

everywhere. We shall denote this hypersurface by CM^3.

Our main results are the following

THEOREM 1. *The manifold CM^3 is the only connected homogeneous hypersurface in S^4 whose type number is equal to 2 at some point.*

THEOREM 2. *Let M be a 2-dimensional connected complete Riemannian manifold of constant curvature $c(\neq 1)$. If M admits an isometric immersion in S^3, then either $c>1$ and M is isometric to $S^2(c)$, or $c=0$, that is, M is flat.*

A theorem of Takahashi [11] asserts that there are no homogeneous hypersurfaces in $S^n(n\geqq 5)$ whose type number is equal to 2 at some point. Therefore Theorem 1 and 2 give a solution to the case (ii), which will be proved in §1 and §2. Finally, the case (iii) is solved by a theorem of O'Neill [8], which will be stated in §3.

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1. A proof of Theorem 1. In this section, we shall adopt the notations of Takahashi [11] and refer to it for detail. For a moment, for later use we suppose M is an n-dimensional Riemannian submanifold of S^{n+1}. Let $F(S^{n+1})$ denote the bundle of orthonormal frames of S^{n+1} and $\theta_i, \theta_{ij}(i, j=1, \cdots, n)$ denote the canonical 1-forms, the connection 1-forms respectively. Then the structure equations for $F(S^{n+1})$ is given by

$$(2) \qquad d\theta_i = -\sum_j \theta_{ij}\wedge\theta_j,\ \theta_{ij}+\theta_{ji}=0$$

$$(3) \qquad d\theta_{ij} = -\sum_k \theta_{ik}\wedge\theta_{kj}+\theta_i\wedge\theta_j,\quad i, j, k=1, \cdots, n+1\,.$$

The bundle $F(M)$ of orthonormal frames of M can be considered as a subbundle of $F(S^{n+1})$ such that the restriction $\theta_{n+1}|F(M)$ of θ_{n+1} to $F(M)$ vanishes. Then putting $\omega_i=\theta_i|F(M)$ and $\omega_{ij}=\theta_{ij}|F(M)$ we have the following structure equations for $F(M)$:

$$(4) \qquad d\omega_i = -\sum_j \omega_{ij}\wedge\omega_j,\ \omega_{ij}+\omega_{ji}=0\,,$$

$$(5) \qquad d\omega_{ij} = -\sum_k \omega_{ik}\wedge\omega_{kj}+\Omega_{ij},\quad i, j=1, \cdots, n$$

where Ω_{ij} are the curvature forms of M. The equation $\omega_{n+1}=0$ implies that $\phi_i=\omega_{n+1i}(i=1, \cdots, n)$ is written as

(6) $$\phi_i = \sum_j H_{ij}\omega_j, \quad H_{ij} = H_{ji}.$$

Then it follows from (2) and (3) that

(7) $$d\phi_i = -\sum_j \omega_{ij} \wedge \phi_j ,$$

(8) $$\Omega_{ij} = \omega_i \wedge \omega_j + \phi_i \wedge \phi_j .$$

Let G be the full isometry group of M and H be the isotropy subgroup at a fixed point $O \in M$. If M is homogeneous, the orbit $G(u_0)$ of a frame u_0 at O under the natural action of G on $F(M)$ is a principal fibre bundle over M with structure group H. The restriction of the differential forms ω_i, ω_{ij}, and $\Omega_{ij}(i, j = 1, \cdots, n)$ are invariant under the action of G on $G(u_0)$.

Now in order to prove Theorem 1, we assume that M is a connected homogeneous hypersurface in S^4. By means of Lemma 3.1 and 3.5 in [11] we may set

(9) $$\phi_1 = H_{11}\omega_1 + H_{12}\omega_2 ,$$

(10) $$\phi_2 = H_{21}\omega_1 + H_{22}\omega_2 ,$$

(11) $$\phi_3 = 0 ,$$

(12) $$\omega_{31} = b\,\omega_2 ,$$

(13) $$\omega_{32} = c\,\omega_1 ,$$

where $H_{11}H_{22} - H_{12}^2$ is a non zero constant and b, c are also constant on $G(u_0)$. Taking the exterior differentiation of (12) and (13), we have

$$\{(b+c)\omega_{12} - (1+bc)\omega_3\} \wedge \omega_1 = 0 ,$$

$$\{(b+c)\omega_{12} + (1+bc)\omega_3\} \wedge \omega_2 = 0 ,$$

from which we find that (A) $1+bc=0$, $b+c=0$ or (B) $1+bc=0$, $\omega_{12}=0$. In the case (A), taking exterior differentiation of (11), we have

$$(cH_{22} - bH_{11})\omega_1 \wedge \omega_2 = 0$$

and hence

$$H_{11} + H_{22} = 0 .$$

Denoting then by λ any principal curvature, we see that λ is equal to one of 0, $\sqrt{H_{11}^2+H_{12}^2}$, and $-\sqrt{H_{11}^2+H_{12}^2}$. Therefore λ is constant on $G(u_0)$. However, E. Cartan [2] proved that the manifold CM^3 is the only complete minimal hypersurface in S^4 with three distinct constant principal curvatures up to congruences in S^4.

In the sequel we want to show that the case (B) can not occur, and for it assume the contrary. Then $\omega_1, \omega_2, \omega_3$ form a basis for $G(u_0)$. Taking exterior differentiation of (9) and (10), we have

$$dH_{11}\wedge\omega_1+bH_{11}\omega_2\wedge\omega_3+dH_{12}\wedge\omega_2+cH_{12}\omega_1\wedge\omega_3=0, \tag{14}$$

$$dH_{12}\wedge\omega_1+bH_{12}\omega_2\wedge\omega_3+dH_{22}\wedge\omega_2+cH_{22}\omega_1\wedge\omega_3=0. \tag{15}$$

Put $dH_{11}=\sum_i \alpha_i\omega_i$, $dH_{12}=\sum_i \gamma_i\omega_i$ and $dH_{22}=\sum_i \beta_i\omega_i$ on $G(u_0)$. Then (14) and (15) amount to

$$\left\{\begin{aligned} \alpha_2 &= \gamma_1 \\ \alpha_3 &= cH_{12} \\ bH_{11} &= \gamma_3 \quad , \end{aligned}\right. \qquad \left\{\begin{aligned} \beta_1 &= \gamma_2 \\ \beta_3 &= bH_{12} \\ cH_{22} &= \gamma_3 \quad . \end{aligned}\right. \tag{16}$$

Taking exterior differentiation of $\omega_{12}=0$, we find

$$H_{11}H_{22}-H_{12}^2-bc+1=0. \tag{17}$$

Substituting $H_{11}=\gamma_3/b$, $H_{22}=\gamma_3/c$ obtained from (16) into (17), we have the following differential equation

$$(\partial H_{12}/\partial x_3)^2+H_{12}^2-2=0\,,$$

where (x_1, x_2, x_3) be a local coordinate system on a neighbourhood U of $G(u_0)$ such that $dx_3=\omega_3$. Then the above equation has the solution $H_{12}=\sqrt{2}\sin f$, where f is a function on U of the form $f(x_1, x_2, x_3)=x_3+a(x_1, x_2)$. Thus from (16) we get on U

$$H_{11} = -\sqrt{2}\; c\cos f\,,$$
$$H_{22} = -\sqrt{2}\; b\cos f\,.$$

Then putting $df=\omega_3+f_1\omega_1+f_2\omega_2$ we have from (14) and (15)

$$f_1\, b\sin f-f_2\cos f=0\,,$$

$$f_1 \cos f - f_2\, c \sin f = 0$$

which imply that $f_1 \equiv 0$ and $f_2 \equiv 0$ on U, namely, $df = \omega_3$. Thus we see

$$0 = d(df) = d\omega_3 = (b-c)\omega_1 \wedge \omega_2$$

and so $b - c = 0$, which contradicts the fact that $1 + bc = 0$. This completes the proof of Theorem 1. q. e. d.

REMARK. The manifold CM^3 appears in the list due to Hsiang and Lawson (table II, [6]) since it is a minimal orbit of a suitable compact subgroup of $O(5)$ which is isometric to $SO(3)$.

2. A proof of Theorem 2. We shall prove the following theorems containing Theorem 2 as a special case.

THEOREM 3. *Let $M^n(c)$ denote an n-dimensional connected complete Riemannian manifold of constant sectional curvature c. If $c_1 < c_2$ and $c_1 \neq 0$, then $M^2(c_1)$ can not be isometrically immersed in $M^3(c_2)$.*

THEOREM 4. *Let $c_1 > c_2$ and $c_1 > 0$. If $M^2(c_1)$ is a surface isometrically immersed in $M^3(c_2)$, then $M^2(c_1)$ is totally umbilic, i. e., it is a standard sphere $S^2(c_1)$ in $M^3(c_2)$.*

The case $c_1 < 0$ and $c_2 = 0$ in Theorem 3 is the well-known Hilbert's theorem [4]. Theorem 3 can be proved by the method similar to Hilbert's one. In the following we shall check that the formulas he employed remain valid for our situation. Assume $M^2(c_1)$ is isometrically immersed in $M^3(c_2)$ with the property $c_1 < c_2$. For a local coordinate system (x_1, x_2) of $M^2(c_1)$ we denote the first fundamental form I and the second fundamental form II of $M^2(c_1)$ by

$$\mathrm{I} = E dx_1^2 + 2F dx_1 dx_2 + G dx_2^2 ,$$

$$\mathrm{II} = L dx_1^2 + 2M dx_1 dx_2 + N dx_2^2 .$$

From the Gauss equation, we have

$$c_1 = c_2 + (LN - M^2)/g , \tag{18}$$

where we put $g = EG - F^2$. Our assumption implies that

$$LN - M^2 < 0.$$

It follows that in each tangent plane of $M^2(c_1)$ there are two real asymptotic directions which are defined by the differential equation

$$\mathrm{II} = Ldx_1^2 + 2Mdx_1 dx_2 + Ndx_2^2 = 0\,.$$

A curve is called asymptotic if it is a differentiable curve each of whose velocity vector belongs to one of asymptotic directions. Choose here as (x_1, x_2) the following special one. First draw an asymptotic curve a through a fixed point 0 on $M^2(c_1)$ and denote by p the point on a with parameter x_1 after parametrizing a by arc length from 0. Next draw another asymptotic curve b through p and denote by q the point on b with parameter x_2 after parametrizing b by arc length from p. Then the obtained mapping $(x_1, x_2) \to q$ is a local diffeomorphism. About such local coordinate system (x_1, x_2) we find

LEMMA 5. *Two curves $x_1 = const.$ and $x_2 = const.$ are asymptotic, that is, $L \equiv 0$, $N \equiv 0$, and $M \not\equiv 0$.*

PROOF. By definition, it is evident that $x_1 = \text{const.}$ is asymptotic. Thus II must have dx_1 as a factor and so we have $N = 0$. Then the Codazzi's formula amounts to

$$(19)\qquad \begin{cases} \partial M/\partial x_1 + \left\{ {1 \atop 12} \right\} L + \left\{ {2 \atop 12} \right\} M = \partial L/\partial x_2 + \left\{ {1 \atop 11} \right\} M \\ \left\{ {1 \atop 22} \right\} L + \left\{ {2 \atop 22} \right\} M = \partial M/\partial x_2 + \left\{ {1 \atop 21} \right\} M \end{cases}$$

where $\left\{ {i \atop jk} \right\}$ denote the Christoffel's symols*). Now substituting $g = M^2/(c_2 - c_1)$ obtained from (18) into the formula

$$\frac{\partial \log \sqrt{g}}{\partial x_i} = \sum_j \left\{ {j \atop ij} \right\}$$

we have

$$(20)\qquad \frac{\partial M}{\partial x_i} = \sum_j \left\{ {j \atop ij} \right\} M\,.$$

Noting that $G \equiv 1$, we can easily see by (19) and (20) that

*) In the remainder of this section the indices i, j, k stand for 1 or 2.

(21) $$\partial L/\partial x_2 = (c_2 - c_1)(L - 2MF)(\partial E/\partial x_2)/2M^2,$$

(22) $$\partial E/\partial x_2 = L(\partial F/\partial x_2)/M,$$

from which we have the differential equation on L

(23) $$\partial L/\partial x_2 = (c_2 - c_1)(L - 2MF)L(\partial F/\partial x_2)/2M^3.$$

For any fixed x_1, this equation has a special solution $L(x_1, x_2) \equiv 0$. But $L(x_1, 0) = 0$ holds along the asymptotic curve $x_2 = 0$. Thus by uniqueness we see $L(x_1, x_2) \equiv 0$ whenever (x_1, x_2) is defined, which implies that $x_2 = \text{const.}$ is asymptotic. q. e. d,

From (22) it turned out that $\dfrac{\partial E}{\partial x_2} = 0$, that is, $E \equiv 1$. Now the first and second fundamental forms can be written as

$$\mathrm{I} = dx_1^2 + 2F dx_1 dx_2 + dx_2^2,$$
$$\mathrm{II} = 2M dx_1 dx_2 .$$

Then the egregium theorem says

(24) $$c_1 g^2 = \frac{\partial^2 F}{\partial x_1 \partial x_2} g + F \frac{\partial F}{\partial x_1} \frac{\partial F}{\partial x_2} .$$

We denote by φ the angle between two vectors $\partial/\partial x_1$ and $\partial/\partial x_2$. Then (24) means

(25) $$\frac{\partial^2 \varphi}{\partial x_1 \partial x_2} = -c_1 \sin \varphi.$$

If $c_1 \neq 0$, from (25) we have a generalization of a classical result:

THEOREM 6. *Let Γ be a quadrilateral on $M^2(c_1)$ whose edges consist of asymptotic curves. Let S denote the area of Γ and $\alpha, \beta, \gamma, \delta$ denote the four interior angles of Γ. Then*

$$S = -(\alpha + \beta + \gamma + \delta - 2\pi)/c_1.$$

Making use of Lemma 5 and Theorem 6 essentially, Hilbert [4] proved

THEOREM 7. *With respect to the above coordinate system (x_1, x_2), $M^2(c_1)$*

is diffeomorphic to 2-plane.

From this theorem we may conclude that $M^2(c_1)$ can not be isometrically immersed in $M^3(c_2)$ if $c_1>0$. In the case $c_1<0$ the same argument as Hilbert's one induces a contradiction. Thus Theorem 3 is proved. q. e. d.

PROOF OF THEOREM 4. Let λ, μ denote the principal curvatures of $M^2(c_1)$. Whether $M^2(c_1)$ is orientable or not, we may assume that λ^2, μ^2 are both continuous function on $M^2(c_1)$ with $\lambda^2 \geqq \mu^2$. Then an analogous argument to one in [4] implies that λ^2 can not attain a maximum at a point such that $\lambda^2>\mu^2$. Thus we have $\lambda \equiv \mu$ since $\lambda\mu>0$ by the relation $c_1=c_2+\lambda\mu$. q. e. d

REMARK. In Theorem 2 the author could not clarify the manner of the isometric immersion of a flat Riemannian manifold in S^3. It seems that a flat hypersurface in S^3 is congruent to a Clifford torus $S^1(r)\times S^1(s)$ with $1/r+1/s=1$.

3. The case (iii). In this section we shall give another proof of the following theorem due to O'Neill [8]. From this proof we obtain new results as a by-product.

THEOREM 8. *If $M^n(c)(c>0)$ is a hypersurface isometrically immersed in $S^{n+1}(c)$, then $M^n(c)$ is isometric to a great sphere $M^n(c)$.*

First we shall establish

PROPOSITION 9. *Let M^n be an n-dimensional compact Riemannian manifold such that there exists a tangent 2-plane at each point of M whose sectional curvature is not greater than $c>0$. Then M can not be isometrically immersed in any open hemisphere in $S^{n+1}(c)$.*

PROOF OF PROPOSITION 9. Suppose M is isometrically immersed in $S^{n+1}(c)$. Let σ be a local cross section of M to the bundle $F(M)$ defined in § 1. We denote the 1-forms $\sigma^*\omega_i$, $\sigma^*\omega_{ij}$ and $\sigma^*\phi_i$ pulled back to M by σ by the same letters ω_i, ω_{ij} and ϕ_i respectively*). We can consider σ as a locally defined orthonormal frame field $(x, e_1, \cdots, e_{n+1})$, $x\in M^n$, with e_{n+1} normal to M and $\omega_1, \cdots, \omega_n$ as a locally defined coframe field dual to $e_1, \cdots, e_n$. Then we have the vectorial equations

$$d_{e_i}x = e_i\ ,$$

$$d_{e_i}e_j = \sum_k \omega_{kj}(e_i)e_k + \phi_j(e_i)e_{n+1} - \omega_j(e_i)x\,,$$

*) In the following the indices i, j, k run from 1 to n.

where d_{e_i} denotes the derivative in the direction of e_i. For any point p of $S^{n+1}(c)$ consider the mapping $f_p = f: M \to R$ which sends $x \in M$ to $f(x) = <p, x>$, where $< \ , \ >$ is the canonical inner product of R^{n+2}. Since M is compact, f attains a minimum at some point of M, say x_0. If $x_0 = -p$, there is nothing to prove. Thus we assume that $x_0 \neq -p$. For each i we obtain at x_0

$$d_{e_i} f = <p, d_{e_i} x> = <p, e_i> = 0 \tag{26}$$

and

$$\begin{aligned} d^2_{e_i} f &= <p, d_{e_i} e_i> \\ &= <p, \sum_j \omega_{ji}(e_i) e_j + \phi_i(e_i) e_{n+1} - x_0> \\ &= <p, H_{ii} e_{n+1} - x_0> \geqq 0. \end{aligned}$$

Hence

$$<p, x_0> \leqq H_{ii} <p, e_{n+1}>, \ i = 1, \cdots, n. \tag{27}$$

Now retake a cross section σ so that $\lambda_i = H_{ii}, i = 1, \cdots, n$ are all eigenvalues of the second fundamental form at x_0. Let $u = \sum_i a_i e_i, v = \sum_i b_i e_i$ be an orthonormal basis for a tangent 2-plane whose sectional curvature $K(u, v)$ is not greater than c. Then it is easily seen from the Gauss equation that

$$K(u, v) = c + \sum_{i<j} (a_i b_j - a_j b_i)^2 \lambda_i \lambda_j.$$

Since $K(u, v) \leqq c$ and $a_i b_j - a_j b_i (i<j)$ don't all vanish, there exist indices i and j with $\lambda_i \lambda_j \leqq 0$. Thus one of $\lambda_i <p, e_{n+1}>$ and $\lambda_j <p, e_{n+1}>$ is non-positive, and hence from (27) we have

$$<p, x_0> \leqq 0 \tag{28}$$

which shows that M is not contained in the hemisphere with pole p. Since p is arbitrary, Proposition 9 is proved. q. e. d.

COROLLARY 10. *Let M be as in Proposition 9. If M admits an isometric immersion $\iota: M \to S^{n+1}(c)$, then the diameter ρ of M is greater than $\pi/2\sqrt{c}$.*

PROOF OF COROLLARY 10. Let d denote the distance function on M. Choose two point x_1, x_2 in M with $d(x_1, x_2) = \rho$. Let p_0 be a point of $\iota(M^n)$ where $f_{\iota(x_1)}$ attains a minimum and $x_0 \in \iota^{-1}(p_0)$. If γ denotes a shortest geodesic segment from x_1 to x_0, we have from (28)

$$\rho = d(x_1, x_2) \geqq d(x_1, x_0) = \text{length of } \gamma = \text{length of } \iota \circ \gamma \text{ in } \iota(M)$$
$$\geqq \text{distance between } \iota(x_1) \text{ and } \iota(x_0) \text{ in } S^{n+1}(c) \geqq \pi/2\sqrt{c}\,.$$

But all equalities don't hold simultaneously. In fact, if not so, then $\iota(\gamma)$ must be a geodesic segment of $S^{n+1}(c)$ contained in $\iota(M)$, which contradicts (26). q. e. d.

PROOF OF THEOREM 8. The diameter of M is greater than $\pi/2\sqrt{c}$ since M satisfies the condition of Corollary 10. Theorem 8 now follows from the following theorem of Shiohama [10].

THEOREM 11. *Let M be a complete Riemnanian manifold whose sectional curvature K satisfies*

$$0 < \delta c \leqq K \leqq c.$$

If the diameter of M is greater than $\pi/2\sqrt{c}$, then M is symply connected.

REMARK. Proposition 8 is a slight generalization of a theorem of Myers (Theorem 4, [7]).

REFERENCES

[1] E. CARTAN, La déformations des hypersurfaces dans l'espaces euclidean réel à n dimensions, Oeuvres completes, Part. III, vol. 1, 185–219.
[2] E. CARTAN, Familles de surfaces isoparamétriques dans les espaces à curbure constante, ibid. vol. 2, 1431–1445.
[3] S. S. CHERN, Som new characterization of the Euclidean sphere, Duke Math. J., 12(1945), 279–290.
[4] D. HILBERT, Ueber Flächen von konstanter Gausscher Krümmung, Trans. Amer. Math. Soc., 2(1901), 87–99.
[5] W. Y. HSIANG, On the compact homogeneous minimal submanifolds, Proc. Nat. Acad. Sci. U. S. A., 56(1966), 5–6.
[6] W. Y. HSIANG AND H. B. LAWSON, JR., Minimal submanifolds of low cohomogeneity, to appear.
[7] S. B. MYERS, Curvature of closed hypersurfaces and non-existence of closed minimal hypersurfaces, Trans. Amer. Math. Soc., 71(1951), 211–217.
[8] B. O'NEILL, Isometric immersions which preserve curvature operators, Proc. Amer. Math. Soc., 13(1962), 759–763.
[9] P. J. RYAN, Homogeneity and some curvature conditions for hypersurfaces, Tôhoku Math. J., 21(1969), 363–388.
[10] K. SHIOHAMA, On the diameter of δ-pinched manifolds, to appear.
[11] T. TAKAHASHI, Homogeneous hypersurfaces in spaces of constant curvature, J. Math. Soc. Japan, 22(1970).

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