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ALMOST CONVERGENT VECTOR SEQUENCES

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1.1. Introduction. In his paper [4] Lorentz defined the concept of almost convergence for bounded sequences of (real or complex) scalars in terms of Banach limits, and characterized the almost convergent sequences as those whose translates are uniformly $(C, 1)$-summable, i.e., $\lim_{p\to\infty}\frac{1}{p}(x_{n+1}+\cdots+x_{n+p})=x$ uniformly in n. More recently Deeds [2] extended the notion to bounded sequences of vectors in a separable Hilbert space and used the expansion in terms of a complete orthonormal set to obtain component-wise criteria for almost convergence. It is the purpose of this paper to extend the results of Lorentz and Deeds to bounded sequences of vectors in an arbitrary (real or complex) Banach space, avoiding the assumptions of separability, reflexivity, and the existence of a Schauder basis. We will show that almost convergence is characteristically a generalization of weak convergence, but that for a special class of sequences it generalizes norm convergence.

1.2. Definitions and notation. X will denote an arbitrary Banach space over the real or complex numbers with dual X^*, and $\hat{X}$ will denote the cononical image of X in X^{**}. Let m denote the space of bounded scalar sequences and $m(X)$ the space of bounded X-valued sequences with $\|u\|=\sup_n\|x_n\|$ for $u=\{x_n\}\in m(X)$. L will denote a Banach limit, i.e., a positive linear functional in m^* which preserves ordinary limits and is translation invariant. $\mathcal{L}$ will denote the collection of all Banach limits.

1.3. Banach operators. Take any $L\in\mathcal{L}$ and a fixed element $u=\{x_n\}\in m(X)$. For each $x^*\in X^*$ define $\bar{L}(u)(x^*)=L(\{x^*(x_n)\})$. Then $\bar{L}(u)$ is a linear functional on X^* and

$$\|\bar{L}(u)\|_{X^{**}}=\sup_{\|x^*\|\leq 1}|\bar{L}(u)(x^*)|=\sup_{\|x^*\|\leq 1}|L(\{x^*(x_n)\})|$$
$$\leqq\sup_{\|x^*\|\leq 1}\sup_n|x^*(x_n)|\leqq\sup_n\|x_n\|=\|u\|_{m(X)}.$$

Thus $\overline{L}(u) \in X^{**}$ and $\overline{L} \in B[m(X), X^{**}]$ with $\|\overline{L}\| \leqq 1$. If $u = \{x_n\} \in m(X)$ such that $\lim_{n\to\infty} x_n = x$ then $\overline{L}(u)(x^*) = x^*(x) = \hat{x}(x^*)$ for each $x^* \in X^*$, so $\overline{L}(u) = \hat{x}$, and $\overline{L}$ preserves ordinary limits relative to the canonical embedding. $\overline{L}$ is clearly translation invariant. In the case where $\overline{L}(u) \in \hat{X}$ we will often consider it as an element of x, and in the case where X is reflexive we think of $\overline{L}$ as an element of $B[m(X), X]$.

2.1. Almost convergence. Suppose $u = \{x_n\} \in m(X)$. If $\overline{L}(u) = x^{**}$ for every $L \in \mathcal{L}$ we say that $\{\hat{x}_n\}$ is almost convergent to x^{**} and write $\hat{x}_n \dot{\to} x^{**}$. If $x^{**} = \hat{x}$ then we write $x_n \dot{\to} x$. We note that any weakly convergent sequence is almost convergent to its weak limit. In general we have the following

THEOREM 2.1.1. *If $u = \{x_n\} \in m(X)$, then $\hat{x}_n \dot{\to} x^{**}$ if and only if $\hat{x}_n(x^*) \dot{\to} x^{**}(x^*)$ for every $x^* \in X^*$.*

PROOF. If $\hat{x}_n \dot{\to} x^{**}$ then by definition $\overline{L}(u)(x^*) = L(\{x^*(x_n)\}) = x^{**}(x^*)$ for all $L \in \mathcal{L}$, $x^* \in X^*$, i.e., $\hat{x}_n(x^*) \dot{\to} x^{**}(x^*)$ for each $x^* \in X^*$. The converse follows similarly.

We will be particularly interested in the question, in the case X is not reflexive, when $x_n \dot{\to} x \in X$. The following theorem suggests an answer to this question.

THEOREM 2.1.2. *If $u = \{x_n\} \in m(X)$ and $\{x_n\}$ converges weakly to x, then $x_n \to x$ if and only if u has range in a compact subset of X.*

PROOF. If $x_n \to x$ then the set $\{x_n\} \cup \{x\}$ is clearly totally bounded, hence compact. Conversely, if $x_n \to x$ then there exists a neighborhood $N(x; \varepsilon)$ of x and a subsequence $\{x_{n_k}\}$ such that $x_{n_k} \notin N(x; \varepsilon)$ for all k. But then there exists a subsequence $\{x_{n_{k_i}}\}$ of $\{x_{n_k}\}$ such that $x_{n_{k_i}} \to \bar{x} \in X (i \to \infty)$. Since $x_n \overset{\omega}{\to} x$ we have $\bar{x} = x$, which yields a contradiction.

Now let U be the class of all sequences $u \in m(X)$ which have conditionally compact range. The next lemma is important in what follows.

LEMMA 2.1.1. *If $K = \{x_\lambda : \lambda \in \Lambda\}$ is compact, then for each $\varepsilon > 0$ there exists a finite set $\{x^*_{\lambda_1}, \cdots, x^*_{\lambda_r}\} \subseteq X^*$ such that $\|x^*_{\lambda_i}\| = 1$ $(1 \leqq i \leqq r)$ and*

$$\sup_{1 \leqq i \leqq r} |x^*_{\lambda_i}(x_\lambda)| > \|x_\lambda\| - \varepsilon$$

for all $\lambda \in \Lambda$.

PROOF. Let $\varepsilon > 0$ be given. For each $\lambda \in \Lambda$ there exists an $x_\lambda^* \in X^*$, $\|x_\lambda^*\| = 1$, such that

$$|x_\lambda^*(x_\lambda)| > \|x_\lambda\| - \varepsilon .$$

Set $U_\lambda = \{x \in X : |x_\lambda^*(x)| > \|x\| - \varepsilon\}$. Then each U_λ is open and $K \subseteq \bigcup_{i=1}^{r} U_{\lambda_i}$. The result is now immediate.

We can now prove the following

THEOREM 2. 1. 3. *If* $u = \{x_n\} \in U$, *then* $\hat{x}_n \dot{\to} x^{**}$ *if and only if* $\|\widehat{S_n^p(u)} - x^{**}\| \to 0 (p \to \infty)$ *uniformly in* n, *in which case* $x^{**} = \hat{x}$ *for some* $x \in X$ *and this is equivalent to* $\|S_n^p(u) - x\| \to 0 (p \to \infty)$ *uniformly in* n. *Here we use the notation* $S_n^p(u) = \dfrac{1}{p}(x_{n+1} + \cdots + x_{n+p})$.

PROOF. If $\hat{x}_n \dot{\to} x^{**}$ then by definition $x^*(x_n) \dot{\to} x^{**}(x^*)$ for each $x^* \in X^*$, which by Lorentz theorem is equivalent to

$$S_n^p(\{x^*(x_n)\}) \to x^{**}(x^*) (p \to \infty)$$

uniformly in n, for each $x^* \in X^*$. By Mazur's theorem [3; page 416], for fixed n, $\{S_n^p(u)\}_{p=1}^{\infty}$ is conditionally compact, so there exists a subsequence $\{S_n^{p_k}(u)\}$ convergent to $x \in X$. But then for each $x^* \in X^*$ we have

$$x^*(S_n^{p_k}(u)) \to x^*(x) = x^{**}(x^*) \qquad (k \to \infty)$$

and thus $x^{**} = \hat{x}$. Now let K be the closure of $\{S_n^p(u) - x\}_{n,p=1}^{\infty}$. By lemma 2. 1. 1 there exist $\{x_1^*, \cdots, x_r^*\}, \|x_i^*\| = 1 (1 \leqq i \leqq r)$, such that

$$\sup_{1 \leqq i \leqq r} |x_i^*(y)| > \|y\| - \varepsilon \quad \text{for all } y \in K.$$

Choose $N > 0$ such that for all $p > N$, $1 \leqq i \leqq r$,

$$|x_i^*(S_n^p(u)) - x_i^*(x)| > \varepsilon \text{ for all } n.$$

Then $p > N \Rightarrow \|S_n^p(u) - x\| < 2\varepsilon$ uniformly in n. For the converse, suppose $\|S_n^p(u) - x^{**}\| \to 0 (p \to \infty)$ uniformly in n. Then $x^{**} = \hat{x}$ and we have $S_n^p(\{x^*(x_n)\}) \to x^*(x)$ $(p \to \infty)$ uniformly in n and $\|x^*\| \leqq 1$. By Lorentz theorem we have

$x^*(x_n) \xrightarrow{\cdot} x^*(x)$ for each $x^* \in X$, so $\hat{x}_n \xrightarrow{\cdot} \hat{x}(=x^{**})$ by Theorem 2. 1. 1.

2. 2. An example. One can quite generally give examples of sequences not in U which converge weakly to zero, hence are almost convergent to zero, but for which $\|S_n^p(\mathrm{u})\| \to 0 (p \to \infty)$. The following example is essentially due to Deeds [2]. Let X be a reflexive Banach space with a Schauder basis $\{e_i\}_1^\infty$. We may assume (renorming if necessary) that the basis is monotone [1]. Define a sequence of integers $\{n_k\}$ by taking $n_1 = 1$ and choosing n_2 such that

$$\left\| \frac{n_1 e_1 + n_2 e_2}{n_1 + n_2} \right\| > \frac{1}{2}.$$

Set $S_i = \sum_{k=1}^{i} n_k$. If $n_1, \cdots, n_i$ are chosen, choose n_{i+1} such that

$$\left\| \frac{n_1 e_1 + \cdots + n_{i+1} e_{i+1}}{s_{i+1}} \right\| > \frac{1}{2}.$$

This is possible since

$$\left\| \frac{n_1 e_1 + \cdots + n_{i+1} e_{i+1}}{s_{i+1}} \right\| \geqq \left\| \frac{n_{i+1} e_{i+1}}{s_{i+1}} \right\| - \left\| \frac{n_1 e_1 + \cdots + n_i e_i}{s_i + n_{i+1}} \right\| \to 1 \qquad (n_{i+1} \to \infty).$$

Set $x_1 = e_1$; $x_n = e_{i+1}$ if $s_i < n \leqq s_{i+1}$. Then

$$\left\| \frac{x_i + \cdots + x_{s_k}}{s_k} \right\| = \left\| \frac{n_1 e_1 + \cdots + n_k e_k}{s_k} \right\| > \frac{1}{2}.$$

Since X is reflexive, $\{e_i^*\}$ is a basis for X^*, where $e_i^*(e_j) = \delta_{ij}$ [1]. If $x^* \in X^*$ then

$$x^*(e_i) = \sum_{j=1}^{\infty} c_j e_j^*(e_i) = c_i \to 0 \quad (i \to \infty)$$

so that $e_i \xrightarrow{\omega} 0$ and also $x_n \xrightarrow{\omega} 0$. Thus we have $x_n \xrightarrow{\cdot} 0$.

3. 1. Periodic and almost periodic sequences. If $u = \{x_n\} \in m(X)$ is periodic then $u \in U$ since the range of u is finite dimensional. For such sequences we have the following.

THEOREM 3. 1. 1. *A sequence $u = \{x_n\}$ is periodic if and only if $\{x^*(x_n)\}$ is periodic for each $x^* \in X^*$.*

PROOF. The necessity is obvious. For the sufficiency we proceed as in [2]. Set

$$E_k = \{x^* \in X^* : x^*(x_{n+k}) = x^*(x_n) \quad \text{for all } n\}.$$

Then $X^* = \bigcup_{k=1}^{\infty} E_k$ and each E_k is closed. By the Baire category theorem some E_k contains a ball $S = \{x^* : \|x^* - x_0^*\| < \rho\}$. Then we have $y^*(x_{n+k} - x_n) = 0$ for all $\|y^*\| < \rho$ and all n, so that $x_{n+k} - x_n = 0$ for all n, and k is a period for $\{x_n\}$. By theorem 2.1.1 and 2.1.3 we see that if $\{x_n\}$ is periodic with period k, then $\{x_n\}$ is almost convergent to the mean value $\frac{1}{k}(x_1 + \cdots + x_k)$.

A sequence $\{x_n\}$ is almost periodic if for each $\varepsilon > 0$ there is a k such that each interval $[l, l+k]$ contains an integer p such that $\sup_n \|x_{n+p} - x_n\| \leqq \varepsilon$. Deeds [2] showed that such a sequence is in U and that the almost periodic sequences form a linear subspace of $m(X)$.

THEOREM 3.1.2. *If $\{x_n\} \in U$, then $\{x_n\}$ is almost periodic if and only if $\{x^*(x_n)\}$ is almost periodic for each $x^* \in X^*$.*

PROOF. The necessity is obvious. For the sufficiency, let K be the closure of $\{x_{n+p} - x_n\}_{n,p=1}^{\infty}$ and suppose $\varepsilon > 0$ is given. By lemma 2.1.1 there exists a finite set $\{x_1^*, \cdots, x_r^*\} \subseteq X^*, \|x_i^*\| = 1$, such that

$$\sup_{1 \leqq i \leqq r} |x_i^*(x_{n+p} - x_n)| > \|x_{n+p} - x_n\| - \varepsilon/2$$

for all $n, p \geqq 1$. Then we can find a k such that each interval $[l, l+k]$ contains an integer p such that

$$\sup_n \sup_{1 \leqq i \leqq r} |x_i^*(x_{n+p} - x_n)| \leqq \varepsilon/2 .$$

It follows that for this integer p

$$\sup_n \|x_{n+p} - x_n\| \leqq \varepsilon .$$

Since Lorentz [4] proved that every almost periodic scalar sequence is almost convergent, the result follows for vector sequences by theorem 2.1.1, 2.1.3 and 3.1.2.

Deeds [2] gave an example in Hilbert space which shows that theorem 3.1.2 no longer holds if we drop the assumption that $\{x_n\} \in U$.

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