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ON NORMAL GLOBALLY FRAMED f-MANIFOLDS

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1. Introduction. Consider an m-dimensional differentiable manifold of class C^∞ on which there is given a framed f-structure [6]. This structure is said to be normal if a particular almost complex structure of a certain subbundle of the tangent bundle of M is integrable [3]. In a previous paper, the authors showed that if m is even an almost complex structure may be defined on M in terms of the structure tensors of the framed f-structure, and if m is odd, an almost contact structure may similarly be defined [2]. In sections 3 and 4 it is shown, if the framed structure on M is normal, that in the former case, the induced almost complex structure is integrable, and in the latter case, the induced almost contact structure is normal. Thus, the product manifold of any two normal globally framed manifolds whose dimensions have the same parity has a complex structure. Sufficient conditions for the normality of the framed structure in terms of the integrability (normality) of the induced almost complex (almost contact) structure are also given.

It is well known that the group of automorphisms of a compact complex manifold is a Lie group [1]. Using this fact, it is easily shown that the group of automorphisms of a compact normal framed f-structure is a Lie group [5].

2. Globally framed f-manifolds. Let M be an m-dimensional differentiable manifold of class C^∞ on which there is given a linear transformation field f of class C^∞ satisfying the algebraic condition

$$f^3 + f = 0. \tag{2.1}$$

Such a structure on M is called an *f-structure* of rank r if the rank r of f is constant on M, and in this case, M is called an *f-manifold* [6].

If M is an f-manifold we put

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$$s = -f^2, \quad t = f^2 + I,$$

where I is the identity transformation field. Then,

$$s + t = I,$$

$$s^2 = s, \qquad t^2 = t, \qquad st = ts = 0,$$

$$f^2 s = -s, \quad ft = 0.$$

Thus, the operators s and t acting in the tangent space at each point of M are complementary projection operators defining distributions S and T in M corresponding to s and t, respectively. The distribution S is r-dimensional and $\dim T = m-r$. The set of all tangent vectors belonging to the distribution T has a bundle structure, denoted by $V(M)$, which is a subbundle of the tangent bundle of M of dimension $2m-r$. If there are $m-r$ vector fields E_a spanning the distribution T at each point of M, and if in addition, there are $m-r$ linear differential forms η^a satisfying

$$\eta^a(E_b) = \delta^a_b, \tag{2.2}$$

$$f^2 = -I + \eta^a \otimes E_a, \tag{2.3}$$

where a and b range over the set $\{1, \cdots, m-r\}$, the summation convention being employed here and in the sequel, M is said to be *globally framed* or have a *framed f-structure* and M is then called a *globally framed f-manifold* or a *framed f-manifold*. From (2.2) and (2.3), one easily obtains

$$fE_a = 0, \qquad \eta^a \circ f = 0. \tag{2.4}$$

LEMMA 1. *Let $M(f, E_a, \eta^a)$ be a globally framed f-manifold. Then,*

(a) $$X(\eta^a(Y)) = (L_X\eta^a)(Y) + \eta^a([X, Y]),$$

(b) $$d\eta^a(E_b, X) = (L_{E_b}\eta^a)(X),$$

(c) $$d\eta^a(fX, Y) = (L_{fX}\eta^a)(Y),$$

where L_X is the operator of Lie derivation with respect to the vector field X and $[X, Y]$ is the Lie bracket of the vector fields X and Y.

When the subbundle $V(M)$ is endowed with an affine connection γ, it admits a natural almost complex structure. If this almost complex structure is integrable the framed f-structure on M is said to be *normal* with respect to the affine connection γ [3].

For any linear transformation field h let $[h, h]$ denote the tensor field of type (1, 2) given by

$$(2.5) \qquad [h, h](X, Y) = [hX, hY] - h[hX, Y] - h[X, hY] + h^2[X, Y].$$

Since the f-structure is framed, there exists a connection γ of zero curvature in $V(M)$, that is, a connection whose components all vanish with respect to a local coordinate system in $V(M)$. Following [3], a framed f-structure is normal if the tensor field S of type (1, 2) given by

$$(2.6) \qquad S = [f, f] + d\eta^a \otimes E_a$$

vanishes.

LEMMA 2. *Let $M(f, E_a, \eta^a)$ be a normal globally framed f-manifold. Then,*

(i) $$L_{E_a}\eta^b = 0.$$

(ii) $$[E_a, E_b] = 0,$$

(iii) $$L_{E_a}f = 0,$$

(iv) $$d\eta^a(fX, Y) + d\eta^a(X, fY) = 0$$

for any vector fields X and Y and $a, b = 1, \cdots, m-r$.

PROOF. Since the structure (f, E_a, η^a) is normal

$$(2.7) \qquad [f, f](X, Y) + d\eta^a(X, Y)E_a = 0.$$

Putting $Y = E_b$ in (2.7), we find

$$-f[fX, E_b] + f^2[X, E_b] + d\eta^a(X, E_b)E_a = 0,$$

that is,

$$(2.8) \qquad f(L_{E_b}f)X + d\eta^a(X, E_b)E_a = 0.$$

Taking the interior product of both sides of (2.8) by η^c, we obtain $d\eta^c(X, E_b) = 0$ which is equivalent to formula (i) by (b) of Lemma 1.

Substituting $X = E_a$ and $Y = E_b$ in (2.7), we have

$$f^2[E_a, E_b] + d\eta^c(E_a, E_b)E_c = 0,$$

so by (i), $f^2[E_a, E_b] = 0$. Applying (2.1), we obtain $f[E_a, E_b] = 0$. Thus, $[E_a, E_b] = \lambda^c_{ab} E_c$ for some functions λ^c_{ab} on M. From (2.2), we get $\eta^c([E_a, E_b]) = \lambda^c_{ab}$, so by (i), the λ^c_{ab} vanish.

From (2.8), $fL_{E_a}f = 0$, so $(L_{E_a}f)X = \mu^b_a(X)E_b$ for some functions $\mu^b_a(X)$. Consequently, $\eta^b((L_{E_a}f)X) = \mu^b_a(X)$. Thus, since

$$0 = (L_{E_a}(\eta^b \circ f))X = ((L_{E_a}\eta^b)f)X + \eta^b((L_{E_a}f)X),$$

the $\mu^b_a(X)$ vanish by (i), that is $L_{E_a}f=0$, $a=1, \cdots, m-r$.

From (2.7), we find

$$\eta^a([fX, fY]) + d\eta^a(X, Y) = 0,$$

so that

$$\begin{aligned} 0 &= \eta^a([fX, f^2Y]) + d\eta^a(X, fY) \\ &= -\eta^a([fX, Y]) + fX(\eta^a(Y)) + d\eta^a(X, fY). \end{aligned}$$

On the other hand, $fX(\eta^a(Y)) - Y(\eta^a(fX)) - \eta^a([fX, Y]) = d\eta^a(fX, Y)$. This proves (iv).

Formula (iv) says that the $d\eta^a$, $a=1, \cdots, m-r$ are of bidegree (1, 1) with respect to f.

3. Even dimensional globally framed manifolds. In a recent paper [2] it was shown that by putting

$$\tilde{f} = f + \eta^{2i} \otimes E_{2i-1} - \eta^{2i-1} \otimes E_{2i}, \qquad i = 1, \cdots, \left[\frac{m-r}{2}\right], \tag{3.1}$$

an almost complex structure tensor $\tilde{f}$ is defined on M if $m = 2n$ and an almost contact structure $(\tilde{f}, E_{2n-r-1}, \eta^{2n-r-1})$ is given if $m = 2n-1$. In this section we show that if $m = 2n$ and the framed f-structure (f, E_a, η^a) on M is normal, then $\tilde{f}$ is integrable.

THEOREM 1. *Let $M(f, E_a, \eta^a)$ be a $2n$-dimensional normal globally framed f-manifold of rank r, $a=1, \cdots, 2n-r$. Then the induced almost complex structure $\tilde{f} = f + \eta^{2i} \otimes E_{2i-1} - \eta^{2i-1} \otimes E_{2i}$ on M is integrable.*

PROOF. We evaluate the torsion $[\tilde{f}, \tilde{f}]$ of the induced almost complex structure $\tilde{f}$ in terms of the tensors f, E_a, η^a of the f-structure on M. First,

$$[\tilde{f}X, \tilde{f}Y] = [fX, fY] + ((L_{fX}\eta^{2i})(Y) + \eta^{2i}([fX, Y]))E_{2i-1} + \eta^{2i}(Y)[fX, E_{2i-1}]$$

$$- ((L_{fX}\eta^{2i-1})(Y) + \eta^{2i-1}([fX, Y]))E_{2i} - \eta^{2i-1}(Y)[fX, E_{2i}]$$
$$+ \eta^{2i}(X)\{[E_{2i-1}, fY] + ((L_{E_{2i-1}}\eta^{2i})(Y) + \eta^{2i}([E_{2i-1}, Y]))E_{2i-1}$$
$$- ((L_{E_{2i-1}}\eta^{2i-1})(Y) + \eta^{2i-1}([E_{2i-1}, Y]))E_{2i} - \eta^{2i-1}(Y)[E_{2i-1}, E_{2i}]\}$$
$$- \{(L_{fY}\eta^{2i})(X) + \eta^{2i}([fY, X]) + \eta^{2i}(Y)((L_{E_{2i-1}}\eta^{2i})(X) + \eta^{2i}([E_{2i-1}, X]))$$
$$- \eta^{2i-1}(Y)((L_{E_{2i}}\eta^{2i})(X) + \eta^{2i}([E_{2i}, X]))\}E_{2i-1}$$
$$- \eta^{2i-1}(X)\{[E_{2i}, fY] + ((L_{E_{2i}}\eta^{2i})(Y) + \eta^{2i}([E_{2i}, Y]))E_{2i-1}$$
$$+ \eta^{2i}(Y)[E_{2i}, E_{2i-1}] - ((L_{E_{2i}}\eta^{2i-1})(Y) + \eta^{2i-1}([E_{2i}, Y]))E_{2i}\}$$
$$+ \{(L_{fY}\eta^{2i-1})(X) + \eta^{2i-1}([fY, X]) + \eta^{2i}(Y)((L_{E_{2i-1}}\eta^{2i-1})(X) + \eta^{2i-1}([E_{2i-1}, X]))$$
$$- \eta^{2i-1}(Y)((L_{E_{2i}}\eta^{2i-1})(X) + \eta^{2i-1}([E_{2i}, X]))\}E_{2i}$$
$$= [fX, fY] + (L_{fX}\eta^{2i})(Y)E_{2i-1} + \eta^{2i}([fX, Y])E_{2i-1} - \eta^{2i}(Y)(f[E_{2i-1}, X] + (L_{E_{2i-1}}f)X)$$
$$- (L_{fX}\eta^{2i-1})(Y)E_{2i} - \eta^{2i-1}([fX, Y])E_{2i} + \eta^{2i-1}(Y)(f[E_{2i}, X] + (L_{E_{2i}}f)X)$$
$$+ \eta^{2i}(X)\{f[E_{2i-1}, Y] + (L_{E_{2i-1}}f)Y + ((L_{E_{2i-1}}\eta^{2i})(Y) + \eta^{2i}([E_{2i-1}, Y]))E_{2i-1}$$
$$- ((L_{E_{2i-1}}\eta^{2i-1})(Y) + \eta^{2i-1}([E_{2i-1}, Y]))E_{2i} + \eta^{2i-1}(Y)[E_{2i}, E_{2i-1}]\}$$
$$- \{(L_{fY}\eta^{2i})(X) + \eta^{2i}([fY, X]) + \eta^{2i}(Y)((L_{E_{2i-1}}\eta^{2i})(X)$$
$$+ \eta^{2i}([E_{2i-1}, X])) - \eta^{2i-1}(Y)((L_{E_{2i}}\eta^{2i})(X) + \eta^{2i}([E_{2i}, X]))\}E_{2i-1}$$
$$- \eta^{2i-1}(X)\{f[E_{2i}, Y] + (L_{E_{2i}}f)Y + ((L_{E_{2i}}\eta^{2i})(Y) + \eta^{2i}([E_{2i}, Y]))E_{2i-1}$$
$$+ \eta^{2i}(Y)[E_{2i}, E_{2i-1}] - ((L_{E_{2i}}\eta^{2i-1})(Y) + \eta^{2i-1}([E_{2i}, Y]))E_{2i}\}$$
$$+ \{(L_{fY}\eta^{2i-1})(X) + \eta^{2i-1}([fY, X]) + \eta^{2i}(Y)((L_{E_{2i-1}}\eta^{2i-1})(X)$$
$$+ \eta^{2i-1}([E_{2i-1}, X])) - \eta^{2i-1}(Y)((L_{E_{2i}}\eta^{2i-1})(X) + \eta^{2i-1}([E_{2i}, X]))\}E_{2i}.$$

Expressing the second and third terms on the right hand side of (2.5) in a similar manner, we see that

$$[\tilde{f}, \tilde{f}](X, Y) = [f, f](X, Y) + d\eta^a(X, Y)E_a$$
$$+ ((L_{fX}\eta^{2i})(Y) - (L_{fY}\eta^{2i})(X))E_{2i-1}$$
$$- ((L_{fX}\eta^{2i-1})(Y) - (L_{fY}\eta^{2i-1}(X))E_{2i}$$
$$- \eta^{2i}(Y)(L_{E_{2i-1}}f)X - \eta^{2i-1}(X)(L_{E_{2i}}f)Y$$
$$+ \eta^{2i}(X)(L_{E_{2i-1}}f)Y + \eta^{2i-1}(Y)(L_{E_{2i}}f)X$$

$$
\begin{aligned}
&+ \{\eta^{2i}(X)(L_{E_{2i-1}}\eta^{2i})(Y) - \eta^{2i}(Y)(L_{E_{2i-1}}\eta^{2i})(X)\}E_{2i-1} \\
&- \{\eta^{2i}(X)(L_{E_{2i-1}}\eta^{2i-1})(Y) - \eta^{2i}(Y)(L_{E_{2i-1}}\eta^{2i-1})(X)\}E_{2i} \\
&+ \{\eta^{2i-1}(X)(L_{E_{2i}}\eta^{2i-1})(Y) - \eta^{2i-1}(Y)(L_{E_{2i}}\eta^{2i-1})(X)\}E_{2i} \\
&- \{\eta^{2i-1}(X)(L_{E_{2i}}\eta^{2i})(Y) - \eta^{2i-1}(Y)(L_{E_{2i}}\eta^{2i})(X)\}E_{2i-1} \\
&+ \eta^{2i}(X)\eta^{2i-1}(Y) - \eta^{2i-1}(X)\eta^{2i}(Y))[E_{2i}, E_{2i-1}] \\
= {}& [f, f](X, Y) + d\eta^a(X, Y)E_a \\
&+ (d\eta^{2i}(fX, Y) + d\eta^{2i}(X, fY))E_{2i-1} \\
&- (d\eta^{2i-1}(fX, Y) + d\eta^{2i-1}(X, fY))E_{2i} \\
&+ \eta^{2i}(X)(L_{E_{2i-1}}\tilde{f})Y - \eta^{2i}(Y)(L_{E_{2i-1}}\tilde{f})X \\
&- \eta^{2i-1}(X)(L_{E_{2i}}\tilde{f})Y + \eta^{2i-1}(Y)(L_{E_{2i}}\tilde{f})X \\
&+ 2(\eta^{2i}\wedge\eta^{2i-1})(X, Y)[E_{2i}, E_{2i-1}].
\end{aligned}
$$

Theorem 1 is now a consequence of Lemma 2.

We state the following converse of Theorem 1.

THEOREM 2. *Let $M(f, E_a, \eta^a)$ be an even dimensional framed f-structure of rank r, $a=1,\cdots,m-r$, whose induced almost complex structure $\tilde{f}=f+\eta^{2i}\otimes E_{2i-1}-\eta^{2i-1}\otimes E_{2i}$, $i=1,\cdots,\left[\frac{m-r}{2}\right]$, is integrable. Then, if* (a) *the $d\eta^a$ are of bidegree* $(1,1)$ *with respect to f,* (b) *the vector fields E_a are holomorphic and* (c) $[E_{2i-1}, E_{2i}]=0$, *the f-structure is normal.*

If $M(f, E_a, \eta^a)$ is an even dimensional normal globally framed manifold, then the $d\eta^a$ are of bidegree $(1,1)$ with respect to the induced almost complex structure $\tilde{f}$. For, $d\eta^a(fX,Y)=d\eta^a(\tilde{f}X,Y)-\eta^{2i}(X)d\eta^a(E_{2i-1},Y)+\eta^{2i-1}(X)d\eta^a(E_{2i},Y)$. But, by Lemma 2,

$$
\begin{aligned}
d\eta^a(E_b, Y) &= E_b(\eta^a(Y)) - \eta^a([E_b, Y]) \\
&= E_b(\eta^a(Y)) - \eta^a(L_{E_b}Y) \\
&= E_b(\eta^a(Y)) - L_{E_b}(\eta^a(Y)) \\
&= 0.
\end{aligned}
$$

4. Odd dimensional globally framed manifolds. An almost contact manifold $M(\phi, E, \eta)$ is said to be *normal* if the almost complex structure J on $M\times R$ given by $J\left(X, g\frac{d}{dt}\right)=\left(\phi X-gE, \eta(X)\frac{d}{dt}\right)$, where g is a C^∞ real valued function and X is a vector field on M, gives rise to a complex structure on M. In this case, the tensor field $[\phi, \phi]+d\eta\otimes E$ vanishes. Conversely, if this tensor field vanishes, M is normal. In this section, we show that if $m=2n-1$ and the framed f-structure (f, E_a, η^a) on M is normal, then $\tilde{f}$ is also normal.

THEOREM 3. *Let $M(f, E_a, \eta^a)$ be a $(2n-1)$-dimensional normal globally framed f-manifold of rank r, $a=1,\cdots,2n-r-1$. Then, the induced almost contact structure $\tilde{f}=f+\eta^{2i}\otimes E_{2i-1}-\eta^{2i-1}\otimes E_{2i}$ on M is normal.*

PROOF. We evaluate the torsion $\tilde{S}$ of the induced almost contact structure $\tilde{f}$ given by (3.1) in terms of the tensors f, E_a, η^a of the f-structure on M. An examination of the computation of the previous section then yields

$$
\begin{aligned}
(4.1)\quad \tilde{S}(X,Y) &= [\tilde{f}X,\tilde{f}Y]-\tilde{f}[\tilde{f}X,Y]-\tilde{f}[X,\tilde{f}Y]-[X,Y]\\
&\quad + d\eta^{2n-r-1}(X,Y)E_{2n-r-1}+\eta^{2n-r-1}([X,Y])E_{2n-r-1}\\
&= [fX,fY]-f[fX,Y]-f[X,fY]-[X,Y]\\
&\quad +(\eta^a([X,Y])+d\eta^a(X,Y))E_a-(\eta^{2n-r-1}([X,Y])\\
&\quad +d\eta^{2n-r-1}(X,Y))E_{2n-r-1}+(\eta^{2n-r-1}([X,Y])+d\eta^{2n-r-1}(X,Y))E_{2n-r-1}\\
&= [fX,fY]-f[fX,Y]-f[X,fY]+f^2[X,Y]+d\eta^a(X,Y)E_a\\
&= [f,f](X,Y)+d\eta^a(X,Y)E_a\\
&= 0,
\end{aligned}
$$

that is, $(\tilde{f}, E_{2n-r-1}, \eta^{2n-r-1})$ is normal.

We also have the following converse.

THEOREM 4. *Let $M(f, E_a, \eta^a)$ be an odd dimensional framed f-structure of rank r whose induced almost contact structure $\tilde{f}=f+\eta^{2i}\otimes E_{2i-1}-\eta^{2i-1}\otimes E_{2i}$ is normal. Then, if* (a) *the $d\eta^a$ are of bidegree* $(1,1)$ *with respect to f,* (b) *$L_{E_a}\tilde{f}$ vanishes and* (c) $[E_{2i-1}, E_{2i}]=0$, $i=1,\cdots,\left[\frac{m-r}{2}\right]$, *the f-structure is normal.*

REMARK. It is known that the product manifold of two (normal) almost contact manifolds has an (integrable) almost complex structure [4]. It then follows easily from the paragraph preceding Theorem 1 that the product manifold of any two odd dimensional globally framed f-manifolds has an almost complex structure. Applying Theorems 1 and 3, we obtain the following generalization of Proposition 2.4 of [5].

THEOREM 5. *The direct product of any two normal globally framed manifolds whose dimensions have the same parity has a complex structure.*

5. Automorphisms. Let $M(f, E_a, \eta^a)$ and $M'(f', E'_a, \eta'^a)$ be globally framed f-structures of the same rank on the C^∞ manifolds M and M', respectively. A diffeomorphism μ of M onto M' is called an *isomorphism* of M onto M' if

$$\mu_* \circ f = f' \circ \mu_* \tag{5.1}$$

and

$$\mu_* E_a = E'_a , \tag{5.2}$$

where μ_* denotes the induced map on tangent spaces. If $M' = M$ and $f' = f$, $E'_a = E_a$, $\eta'^a = \eta^a$, $a = 1, \cdots, m-r$, then μ is said to be an *automorphism* of M. The set of all automorphisms of M clearly forms a group which we denote by $A(f, E_a, \eta^a)$.

LEMMA 3. *Let* $\mu \in A(f, E_a, \eta^a)$. *Then,*

$$\mu^* \eta^a = \eta^a ,$$

where μ^* *is the induced map on forms.*

LEMMA 4. *Let* $\mu \in A(f, E_a, \eta^a)$. *Then* $\mu \in A(\tilde{f})$, *that is*

$$\mu \circ \tilde{f} = \tilde{f} \circ \mu$$

where

$$\tilde{f} = f + \eta^{2i} \otimes E_{2i-1} - \eta^{2i-1} \otimes E_{2i} .$$

Thus, if $m = 2n$ an automorphism of $M(f, E_a, \eta^a)$ is also an automorphism of the induced almost complex structure and if $m = 2n-1$, an automorphism of $M(f, E_a, \eta^a)$ is also an automorphism of the induced almost contact structure.

It is well known that the group of automorphisms of a compact complex manifold is a Lie group [1]. If $M(f, E_a, \eta^a)$ is normal and $m = 2n$, the induced almost complex structure $\tilde{f}$ is integrable. Hence, if M is compact, $A(\tilde{f})$ is a Lie group. The group $A(f, E_a, \eta^a)$ is a closed subgroup of $A(\tilde{f})$, hence a Lie group with the induced topology; for, (f, E_a, η^a) is a fixed point of $A(f, E_a, \eta^a) \subset A(\tilde{f})$. The case $m = 2n-1$ is a consequence of Theorem 3 and the fact that the group of automorphisms of a compact normal almost contact manifold is a Lie group [4].

THEOREM 6. *The group of automorphisms of a compact normal globally framed f-structure is a Lie group.*

BIBLIOGRAPHY

[1] S. BOCHNER AND D. MONTGOMERY, Locally compact groups of differentiable transformations, Ann. of Math., 47(1946), 639-653.
[2] S. I. GOLDBERG AND K. YANO, Globally framed f-manifolds, to appear in Illinois J. Math.
[3] S. ISHIHARA, Normal structure f satisfying $f^3+f=0$, Kōdai Math. Sem. Rep., 18(1966), 36-47.
[4] A. MORIMOTO, On normal almost contact structures, J. Math. Soc. Japan, 15(1963), 420-436.
[5] H. NAKAGAWA, On framed f-manifolds, Kōdai Math. Sem. Rep., 18(1966), 293-306.
[6] K. YANO, On a structure defined by a tensor field f of type (1, 1) satisfying $f^3+f=0$, Tensor, N. S., 14(1963), 99-109.

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