Tôhoku Math. Journ.
22(1970), 325-332.

ON ALMOST CONTACT 3-STRUCTURE

YING-YAN KUO

(Received May 14, 1969; Revised June 20, 1970)

Introduction. An almost contact structure (ϕ, ξ, η)[1] on a differentiable manifold is an aggregate consisting of a tensor field ϕ of type $(1,1)$, a contravariant vector field ξ and a covariant vector field η which satisfy

$$\eta(\xi) = 1, \qquad \phi\phi = -I + \xi \otimes \eta,$$

where $\otimes$ means the tensor product, I is the identity tensor.

It is known that almost contact structure has many similarities to almost complex one. The main purpose of this paper is to discuss a structure of contact type similar to almost quarternion structure[2].

I would like to thank Professor S. Tachibana sincerely for his kind guidance and suggestions throughout writing the paper. I also express my hearty thanks to Professor S. Sasaki whose advice resulted in many improvements of the formulation.

1. Almost contact 3-structure. Suppose a differentiable manifold admits 3 almost contact structures (ϕ_i, ξ_i, η_i), $i = 1, 2, 3$, satisfying

$$\eta_i(\xi_j) = \eta_j(\xi_i) = 0,$$

$$\phi_i\xi_j = -\phi_j\xi_i = \xi_k,$$

$$\eta_i\circ\phi_j = -\eta_j\circ\phi_i = \eta_k,$$

$$\phi_i\phi_j - \xi_i\otimes\eta_j = -\phi_j\phi_i + \xi_j\otimes\eta_i = \phi_k,$$

for any cyclic permutation (i, j, k) of $(1, 2, 3)$. We shall call such a structure an almost contact 3-structure. First we have the following

THEOREM 1. *If a differentiable manifold admits 2 almost contact structures (ϕ_i, ξ_i, η_i), $i = 1,2$, satisfying*

1) S. Sasaki, [2], S. Sasaki and Y. Hatakeyama, [3].
2) H. Wakakuwa, [5], M. Obata, [6].

(1.1) $$\eta_1(\xi_2) = \eta_2(\xi_1) = 0\,,$$

(1.2) $$\phi_1\xi_2 = -\phi_2\xi_1\,,$$

(1.3) $$\eta_1\circ\phi_2 = -\eta_2\circ\phi_1\,,$$

(1.4) $$\phi_1\phi_2 - \xi_1\otimes\eta_2 = -\phi_2\phi_1 + \xi_2\otimes\eta_1\,,$$

it admits an almost contact 3-structure.

In fact, we can get the third structure by putting

$$\xi_3 = \phi_1\xi_2\,, \qquad \eta_3 = \eta_1\circ\phi_2\,,$$
$$\phi_3 = \phi_1\phi_2 - \xi_1\otimes\eta_2\,.$$

2. Associated metric. Let (ϕ, ξ, η) be an almost contact structure on a differentiable manifold M^n. A Riemannian metric (positive definite) g is called to be associated to the structure if it satisfies

$$g(\xi, X) = \eta(X), \qquad g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y)$$

for any vectors X and Y.

The following theorem is easily obtained.

THEOREM 2. *Suppose a differentiable manifold M^n admits 2 almost contact structure (ϕ_i, ξ_i, η_i), $i = 1, 2$. If there is a Riemannian metric associated to both of the structures, then* (1.1), (1.2) *and* (1.3) *follows from* (1.4).

REMARK. If a Riemannian metric is associated to 2 structures of an almost contact 3-structure, it is associated to the third one.

When a Riemannian metric is associated to each structure of an almost contact 3-structure, it will be called an associated metric of the 3-structure. We shall discuss the existence of such metrics.

LEMMA. *For an almost contact 3-structure (ϕ_i, ξ_i, η_i), $i = 1, 2, 3$, there exists a Riemannian metric h such that*

$$h(\xi_i, X) = \eta_i(X)\,, \qquad i = 1, 2, 3\,,$$

hold good for any vector X.

PROOF. Let j be an associated metric to (ϕ_1, ξ_1, η_1) and define a new metric f by

$$f(X, Y) = j(X - \eta_2(X)\xi_2, Y - \eta_2(Y)\xi_2) + \eta_2(X)\eta_2(Y).$$

Next define h by

$$h(X, Y) = f(X - \eta_3(X)\xi_3, Y - \eta_3(Y)\xi_3) + \eta_3(X)\eta_3(Y),$$

and we can see that it has the required property. Q. E. D.

If we define a new metric g from h by

$$4g(X, Y) = h(X, Y) + \sum_{i=1}^{3} \{h(\phi_i(X), \phi_i(Y)) + \eta_i(X)\eta_i(Y)\},$$

we can check that g is associated to (ϕ_i, ξ_i, η_i), $i = 1, 2, 3$. Thus we have

THEOREM 3. *In a differentiable manifold of almost contact 3-structure, there is an associated metric of the structure.*

3. **Dimension**. It is well known[3] that the dimension of an almost contact manifold is odd. We shall study the corresponding problem in this section.

Consider M^n with an almost contact 3-structure (ϕ_i, ξ_i, η_i), $i = 1, 2, 3$. Let $M^{n+1} = M^n \times R$ be the product manifold of M^n with the real line R. If we define a tensor field Φ_i of type (1, 1) on M^{n+1} by

$$\Phi_i = \begin{pmatrix} \phi_i & \xi_i \\ -\eta_i & 0 \end{pmatrix}, \qquad i = 1, 2, 3,$$

they define an almost quarternion structure, i. e., they satisfy

$$\Phi_1\Phi_1 = \Phi_2\Phi_2 = \Phi_3\Phi_3 = -I,$$

$$\Phi_i\Phi_j = -\Phi_j\Phi_i = \Phi_k,$$

where I denotes the unit tensor and (i, j, k) is any cyclic permutation of (1,2,3).

On the other hand, it is known[4] that the dimension of a manifold with an almost quarternion structure is of the form $4(m+1)$. Thus we get

3) As to the proof, see S. Sasaki, [2].
4) M. Obata, [6].

THEOREM 4. *The dimension of a manifold with almost contact 3-structure is* $4m+3$ *for a non-negative integer* m.

Let g be an associated metric of an almost contact 3-structure and define a metric $\widetilde{g}$ on M^{n+1} by

$$\widetilde{g}=\begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix}.$$

Then $\widetilde{g}$ satisfies

$$\widetilde{g}(\widetilde{X},\widetilde{Y})=\widetilde{g}(\Phi_i\widetilde{X},\Phi_i\widetilde{Y}),\qquad i=1,2,3,$$

for any vectors $\widetilde{X}$ and $\widetilde{Y}$ on M^{n+1}.

4. Structure group of tangent bundle. Let g be an associated metric to an almost contact 3-structure (ϕ_i,ξ_i,η_i), $i=1,2,3$. Put $e_{4m+3}=\xi_1$, $e_{4m+2}=\xi_2$, $e_{4m+1}=\xi_3$, then they are an orthonormal field over M^{4m+3} globally. Let e_{4m} be a unit vector field defined locally in a neighborhood U such that $e_{4m},\cdots,e_{4m+3}$ are orthonormal. If we put

$$e_{4m-1}=\phi_1 e_{4m},\quad e_{4m-2}=\phi_2 e_{4m},\quad e_{4m-3}=\phi_3 e_{4m},$$

these 7 vectors $e_{4m-3},\cdots,e_{4m+3}$ are orthonormal. In this way we finally have an orthonormal frame field in U consisting of

$$e_{4\lambda},\qquad \lambda=1,\cdots,m,$$
$$e_{4\lambda-i}=\phi_i e_{4\lambda},\qquad i=1,2,3,$$
$$e_{4m+1},\qquad e_{4m+2},\qquad e_{4m+3}.$$

The product manifold $M^{4m+4}=M^{4m+3}\times R$ has the almost quarternion structure Φ_i and the metric $\widetilde{g}$ stated in §3. Let e_{4m+4} be the unit vector on R. Moreover let $E_1,\cdots,E_{4m+4}$ be the orthonormal frame in $U\times R$ which are obtained by natural extension from $e_1,\cdots,e_{4m+4}$. We have easily

$$\Phi_i E_{4\lambda}=E_{4\lambda-i},\qquad \lambda=1,\cdots,m+1,\qquad i=1,2,3. \tag{4.1}$$

Let U' be a neighborhood on M^{4m+3} such that $U\cap U'\neq 0$. Let $e'_1,\cdots,e'_{4m+4}$, $E'_1,\cdots,E'_{4m+4}$ be the corresponding vectors. There exists an orthogonal matrix $A=(a_{\alpha\beta})$ such that

$$\sum_{\beta=1}^{4m+4} a_{\alpha\beta}E_\beta = E'_\alpha, \qquad \alpha = 1, \cdots, 4m+4 \tag{4.2}$$

and it is known[5] that $A \in Sp(m+1) =$ the real representation of the symplectic group, by taking account of the choice (4.1) of frames.

As $E_{4m+i} = E'_{4m+i}$ for $i = 1, 2, 3, 4$, we know that

$$A = \begin{pmatrix} B & 0 \\ 0 & I_4 \end{pmatrix}, \quad B \in Sp(m).$$

If we restrict our attention to M^{4m+3}, we get from (4.2)

$$\sum_{\beta=1}^{4m+3} c_{\alpha\beta}e_\beta = e'_\alpha, \qquad \alpha = 1, \cdots, 4m+3,$$

$$C = (c_{\alpha\beta}) = \begin{pmatrix} B & 0 \\ 0 & I_3 \end{pmatrix}, \qquad B \in Sp(m).$$

Thus we have

THEOREM 5. *Let M^{4m+3} be a differentiable manifold of almost contact 3-structure. Then the structure group of the tangent bundle is reducible to $Sp(m) \times I_3$.*

5. Sasakian structure and Betti numbers. Consider a Riemannian manifold with metric g. For a vector field ξ we shall define a covariant vector η by

$$g(\xi, X) = \eta(X), \tag{5.1}$$

where X is any vector. The condition for ξ to be a Killing vector field is that

$$(\nabla_X\eta)Y + (\nabla_Y\eta)X = 0$$

holds good for any vectors X and Y.

A Sasakian manifold is a Riemannian manifold M^n with metric g such that it admits a unit Killing vector field ξ satisfying

$$(\nabla_Y\phi)X = \eta(X)Y - g(Y, X)\xi,$$

where ϕ is defined by

5) M. Obata, [6], [7]. H. Wakakuwa, [5].

(5.2) $$\phi X = \nabla_X \xi$$

and X and Y are any vectors. In this case ξ is called a Sasakian structure. For a Sasakian structure ξ, (ϕ, ξ, η) defined by (5.1) and (5.2) is an almost contact structure, and g is an associated metric.

The following theorem is easily obtained.

THEOREM 6. *Let ξ_1 and ξ_2 be Sasakian structures orthogonal to each other. Then $\xi_3 = \phi_1 \xi_2$ is a Sasakian structure orthogonal to ξ_1 and ξ_2. In this case, (ϕ_i, ξ_i, η_i), $i = 1, 2, 3$, is an almost contact 3-structure.*

We shall call Sasakian structure stated in Theorem 6 a Sasakian 3-structure.

In a Sasakian manifold M^n with Sasakian structure ξ, let Ψ be an operator defined by

(5.3) $$(\Psi u)(X_1, \cdots, X_p) = u(\phi X_1, \cdots, \phi X_p)$$

for differential p-form u, where $X_1, \cdots, X_p$ are vectors.

The following theorems are known[6].

THEOREM A. *In a compact Sasakian manifold M^n, Ψu is harmonic for a harmonic p-form u, provided that $p < (n+1)/2$.*

THEOREM B. *In a compact Sasakian manifold M^n, a harmonic p-form u, $(p < (n+1)/2)$, is orthogonal to ξ, i.e., $u(\xi, X_2, \cdots, X_p) = 0$ holds good for any vectors $X_2, \cdots, X_p$.*

By making use of these theorems, we can get

THEOREM 7[7]. *For a compact manifold M^{4m+3} of Sasakian 3-structure ξ_1, ξ_2, ξ_3, the p-th Betti number is of the form $4q$ for a non-negative integer q, if p is odd and satisfies $p < (n+1)/2 = 2m+2$.*

PROOF. Let $\mathfrak{X}_p(M)$ be the vector space of all harmonic p-form over M^{4m+3}. For ξ_i we define Ψ_i similarly to (5.3). These operators Ψ_i define an almost quaternion structure in $\mathfrak{X}_p(M)$ and hence $\dim \mathfrak{X}_p(M) = 4q$. Q. E. D.

6. Examples. (i) R^{4m+3}. Consider the 3 dimensional Euclidean space R^3.

6) S. Tachibana, [4]. S. Tachibana and Y. Ogawa, [9]. T. Fujitani, [10].
7) As to Kählerian manifold with quarternion structure, see H. Wakakuwa, [5].

Let e_1, e_2, e_3 be the fundamental vectors and define

$$
\begin{aligned}
&\phi_1 = (-e_1, 0, e_3), \qquad \phi_2 = (0, e_3, -e_2), \qquad \phi_3 = (-e_2, e_1, 0), \\
(6.1) \qquad &\xi_1 = e_2, \qquad \xi_2 = e_1, \qquad \xi_3 = -e_3, \\
&\eta_1 = e_2^*, \qquad \eta_2 = e_1^*, \qquad \eta_3 = -e_3^*,
\end{aligned}
$$

where e_i^* means the dual of e_i. Then these structures give an almost contact 3-structure and $g = I_3$ is an associated metric.

Next consider the $4m$ dimensional Euclidean space with standard quarternion structure $\Phi_i (i = 1, 2, 3)$ and define

$$
\hat{\phi}_i = \begin{pmatrix} \phi_i & 0 \\ 0 & \Phi_i \end{pmatrix}, \qquad \hat{\xi}_i = \begin{pmatrix} \xi_i \\ 0 \end{pmatrix}, \qquad \hat{\eta}_i = (\eta_i, 0),
$$

where (ϕ_i, ξ_i, η_i) are given by (6.1). They are an almost contact 3-structure in R^{4m+3} with associated metric I_{4m+3}.

(ii) S^{4m+3}. Consider the Euclidean space R^{2n+2} with the standard complex structure Φ. Let N be the unit normal vector field on the unit sphere S^{2n+1} with the origin as its center. It is known[8)] that $\xi = \Phi N$ gives S^{2n+1} a Sasakian structure and the induced metric on S^{2n+1} is just an associated one. If we regard S^{2n+1} as a projective space P^{2n+1} by identifying anti-podal points, ξ defined above induces a Sasakian structure on P^{2n+1}.

Now condider the Euclidean space R^{4m+4} with standard quarternion structure Φ_i, $i = 1, 2, 3$. They satisfy

$$
\Phi_i \Phi_i = -I, \qquad {}^t\Phi_i = -\Phi_i, \qquad i = 1, 2, 3, \qquad \Phi_1 \Phi_2 = \Phi_3.
$$

Define ξ_i by $\xi_i = \Phi_i N$, $i = 1, 2, 3$. Then the inner product of ξ_1 and ξ_2 satisfies

$$
\begin{aligned}
I(\xi_1, \xi_2) &= I(\Phi_1 N, \Phi_2 N) = ({}^t\Phi_1 \Phi_2)(N, N) \\
&= -(\Phi_1 \Phi_2)(N, N) = -\Phi_3(N, N) = 0,
\end{aligned}
$$

where $I = I_{4m+4}$ is the Euclidean metric in R^{4m+4}. Thus ξ_1 and ξ_2 are orthogonal to each other, and hence S^{4m+3} and P^{4m+3} have a Sasakian 3-structure by virtue of Theorem 6.

8) Y. Tashiro and S. Tachibana, [11].

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DEPARTMENT OF MATHEMATICS
NATIONAL TAIWAN UNIVERSITY
TAIPEI, TAIWAN, REPUBLIC OF CHINA