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PROJECTIONS IN HILBERT SPACE*

HORST BEHNCKE

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In [1] C. Davis showed that there exist three projections on a separable Hilbert space H, which generate the ring of all bounded operators $B(H)$. By a different method we show the following generalization.

THEOREM: *On a separable Hilbert space H one can find a continuous family of triples of projectons $\{P_\lambda, Q_\lambda, R_\lambda\}$, $0 < \lambda < 2\pi$, such that each triple generates $B(H)$ in the sense of W^*-algebras, such that $P_\lambda Q_\lambda = 0$ and such that these triples are unitarily inequivalent for different λ.*

PROOF. The method of group representations makes this problem almost trivial. Let G be the free product of the cyclic group of order two and of the cyclic group of order three. In terms of its generators a, b G is defined by $a^3 = b^2 = e$. Let U be an infinite dimensional unitary irreducible representation of G. Then $U_a^3 = U_b^2 = 1$ shows by the spectral theorem the existence of three projections P, Q, R with $P \cdot Q = 0$ such that $U_a = P + Qe^{i(2\pi/3)} + (1-P-Q) \cdot e^{i(4\pi/3)}$ and $U_b = 1 - 2R$. Since the representation is irreducible U_a and U_b generate $B(H)$. The same applies then obviously also for P, Q and R.

Thus it remains to show the existence of a family $U_\lambda, 0 < \lambda < 2\pi$ of unitary, irreducible, infinite dimensional representations of G, which are pair-wise inequivalent. It is easy to see that all nontrivial conjugacy classes in G are infinite. Thus G is not a type I group [2], and the general theory of non-type I C^*-algebras would show the existence of an infinite family of such representations. However we prefer to give a direct proof. This is done by the method of induced representations. Let $c = ab$ and denote by F the infinite cyclic subgroup of G generated by c. Let $G/F = \{xF | x \in G\}$ be the collection of all left cosets of F in G. By $s(x)$ we shall denote a particular representative of xF, which is chosen once and for all. We shall also identify G/F and the set $S = \{s\}$ of representatives of the cosets. $l^2(G/F)$ will stand for the infinite dimensional Hilbert space with orthonormal basis $\{\varepsilon_s | s \in S\}$. Since F is an infinite cyclic group all its irreducible

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representations are determined by characters χ^λ with $\chi^\lambda(c^n)=e^{i\lambda n}$, $0\leqq\lambda<2\pi$. Then the representation U^λ of G on $l^2(G/F)$ induced by χ^λ is given by

$$U^\lambda_x\varepsilon_s = \chi^\lambda(x[s]^{-1}\cdot x\cdot s)\varepsilon_{x[s]}$$

Here $x[s]$ denotes the representative of $x\cdot s\cdot F$. A simple computation shows that U^λ is indeed an infinite dimensional representation of G. If U^λ, $0<\lambda<2\pi$ were not irreducible we could find a nontrivial operator $T \in B(l^2(G/F))$ with $TU^\lambda_x=U^\lambda_x T$ for all $x\in G$. In particular we have

$$TU^\lambda_{c^n}\varepsilon_{s_0} = \chi^\lambda(c^n)\,T\varepsilon_{s_0} = U^\lambda_{c^n}T\varepsilon_{s_0} \quad c^n\in F.$$

Here s_0 is the representative of F. An easy computation shows that the only eigenvectors of $U^\lambda_{c^n}$ are of the form constant $\cdot\,\varepsilon_{s_0}$. Therefore $T\varepsilon_{s_0}=k\varepsilon_{s_0}$. Then $T\varepsilon_s=TU^\lambda_s\varepsilon_{s_0}=U^\lambda_s T\varepsilon_{s_0}=k\varepsilon_{s_0}$ for all s finally gives $T=kI$, and the irreducibility of U^λ is shown.

Assume T intertwines U^λ and $U^{\lambda'}$, then $TU^\lambda_x=U^{\lambda'}_x T$ and in particular $TU^\lambda_{c^n}\varepsilon_{s_0}=\chi^\lambda(c^n)\,T\,\varepsilon_{s_0}=U^{\lambda'}_{c^n}T\,\varepsilon_{s_0}$ Thus $T\varepsilon_{s_0}$ is an eigenvector for $U^{\lambda'}_{c^n}$ for all n. This shows by our above remarks

$$\chi^\lambda(c^n)k\varepsilon_{s_0} = \chi^{\lambda'}(c^n)k\varepsilon_{s_0}$$

This however is only possible for $\lambda=\lambda'$.

In a similar fashion we can characterize the W^*-algebra generated by two arbitrary projections P, Q on a separable Hilbert space H [3]. In this case the operators $U_a=1-2P$ and $U_b=1-2Q$ determine a unitary representation of the group D, with generators a and b and relations $a^2=b^2=e$. Since D is an extension of $F=\{(ab)^n=c^n\,|\,n$ all integers$\}$ by a group of order two, D is a group of Type I and all its irreducible representations are finite dimensional of uniformly bounded dimension [2]. Let U be an arbitrary irreducible representation of D. Then one sees easily that $A=U_aU_b+U_bU_a$ is a self adjoint operator commuting witn U_a and U_b. By irreducibility $A=kI$ with $-2\leqq k\leqq 2$. Then $U_aU_b=e^{i\varphi}1(-R)+e^{i\varphi}R$, for $2\cos\varphi=k\in(-2,2)$ and R a projection. A simple computation shows now that U is either two or one dimensional. Thus the W^*-algebra generated by two projectionsis of type $\mathrm{I}_{\leqq 2}$. Since the oprator A is a unitary invariant for representations U of D with no one dimensional parts, it is also a unitary invariant for the projections on the part of type I_2.

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INSTITUT FÜR ANGEWANDTE MATHEMATIK
UNIVERSITÄT HEIDELBERG
HEIDELBERG, DEUTSCHLAND