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PRODUCTS IN SHEAF-COHOMOLOGY

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Introduction. The natural setting for a theory of sheaves is a *site*, i. e. a category C topologized in the sense of Grothendieck (cf. VERDIER [1963]). We shall consider a sheaf A of rings on a site C and the category $\mathcal{A}$ of sheaves of A-modules. When we speak of sheaf-cohomology, we shall mean the right-derived functor of $\Gamma = \mathrm{Hom}_{\mathcal{A}}(A, -)$, that is: $\mathrm{Ext}_{\mathcal{A}}(A, -)$.

In $\mathcal{A}$ one has a tensor-product and a local Hom (denoted: $\mathcal{H}om$) with the familiar exactness properties and the usual adjointness. Our problem is to associate to a pairing

$$(1) \qquad F \otimes F' \to G$$

or, equivalently, to a morphism

$$(2) \qquad F \to \mathcal{H}om(F', G)$$

of objects of $\mathcal{A}$ a canonical cohomology product

$$(3) \qquad H^p(F) \otimes H^q(F') \to H^{p+q}(G)$$

which respects coboundary operators in the usual way. Universality of the cohomology functor does not help, since $H^p(F \otimes -)$ does not, in general, yield a connected sequence of functors on any useful subcategory of short exact sequences. We shall exhibit two constructions for a product (3) which arise in different natural habitats but coincide in the context described above.

An obvious thing to try is the conversion of the first factor $H^p(F)$ into $\mathrm{Ext}^p(F', G)$ via (2) followed by the application of the ever-available Yoneda-product

$$\mathrm{Ext}^p(F', G) \times H^q(F') \to H^{p+q}(G).$$

More precisely, the transition from $H^p(F)$ to $\mathrm{Ext}^p(F', G)$ is accomplished by applying H^p to (2) and then using the edge-morphism

$$H^p(\mathcal{H}om(F', G)) \to \operatorname{Ext}^p(F', G)$$

whose existence is due to the fact that $\operatorname{Hom} = \Gamma \circ \mathcal{H}om$ and that $\mathcal{H}om(F', I)$ is acyclic for Γ if I is injective. This construction, which hinges on certain properties of $\mathcal{H}om$, is discussed in §3.

In the absence of a tensor-product, the construction just described is hopelessly asymmetric. In §2, we provide another one, based on properties of $\otimes$, by showing the existence of an external product for Ext with flat first variable. Since $\mathcal{A}$ does not, in general, have enough projectives (though it does have enough flats), we cannot work with projective resolutions and therefore find it convenient to interpret elements of Ext-groups as morphisms in the derived category of $\mathcal{A}$ and to define their products as tensor-products of these morphisms. The required facts about derived categories are recalled in §1. Finally, in §4, we show that in the presence of both $\mathcal{H}om$ and $\otimes$, the two products are the same.

Care has been taken not to engage the reader in speculations about sheaves: these were mentioned only as motivation and area of application; all the rest of this paper is strictly categorical. A possible way of reading it is to keep in mind the category $\mathcal{A}$ of modules over a group G with $\Gamma = \operatorname{Hom}_G(\boldsymbol{Z}, -)$ and $\mathcal{H}om = \operatorname{Hom}_{\boldsymbol{Z}}$, but to remember that the game is to be played without using projectives as such.

1. Review of derived categories. We recall the relevant properties of derived categories referring to Hartshorne [1966] for details and indication of proofs.

Let $\mathcal{A}$ be an abelian category. We denote by $C(\mathcal{A})$ the category of all complexes over $\mathcal{A}$ (with differentials augmenting degrees) and by $K(\mathcal{A})$ the homotopy category thereof. Furthermore, there are various full subcategories of $C(\mathcal{A})$ whose respective objects are the complexes which are bounded below, bounded above, bounded in both directions. These will be denoted by $C^+(\mathcal{A})$, $C^-(\mathcal{A})$, $C^b(\mathcal{A})$, respectively, and similarly for the corresponding homotopy categories.

For brevity, we call a morphism in one of these categories a *quiso*, if it induces isomorphisms on cohomology. Quisos will be denoted by double arrows $\Rightarrow$. The derived category $D(\mathcal{A})$ of $\mathcal{A}$, then, is constructed from $K(\mathcal{A})$ by formally inverting all quisos. We call a morphism in $D(\mathcal{A})$ a *quasi-morphism* of $C(\mathcal{A})$, and denote quasi-morphisms by broken arrows $\dashrightarrow$. Since the set of all quisos in $K(\mathcal{A})$ admits both a calculus of left and of right fractions, a quasi-morphism $X^* \dashrightarrow Y^*$ can be given either by a diagram

(1)
$$\begin{tikzcd} X^* \arrow[dr] & & Y^* \arrow[dl, Rightarrow] \\ & Z^* & \end{tikzcd}$$

in $C(\mathcal{A})$, where σ is a quiso, or by a diagram

$$
(2)\qquad
\begin{tikzcd}
X^* & & Y^* \\
& Z^* \arrow[ul, Rightarrow, "\tau"'] \arrow[ur] &
\end{tikzcd}
$$

in $C(\mathcal{A})$, where τ is a quiso.

In the same way, one gets categories $D^+(\mathcal{A})$, $D^-(\mathcal{A})$, $D^b(\mathcal{A})$ from the corresponding homotopy categories. These are full subcategories of $D(\mathcal{A})$; moreover, the inclusion of $D^+(\mathcal{A})$ in $D(\mathcal{A})$ is compatible with the calculus of left fractions, i. e. a morphism in $D^+(\mathcal{A})$ can be given by a diagram (1) in $C^+(\mathcal{A})$. Dually, a morphism in $D^-(\mathcal{A})$ can be given by a diagram (2) in $C^-(\mathcal{A})$. Finally, we note that $\mathcal{A}$ may be considered as a full subcategory of each of the categories introduced so far.

If $\mathcal{A}$ has enough injectives, there is a canonical equivalence of categories:

$$
(3)\qquad K^+(\mathcal{J}) \xrightarrow{\sim} D^+(\mathcal{A}),
$$

where $K^+(\mathcal{J})$ denotes the full subcategory of $K^+(\mathcal{A})$ whose objects are the injective complexes (i. e. all of whose components are injective). More precisely, for each Y^* in $C^+(\mathcal{A})$ there exists an injective resolution, i. e. a mono $Y^* \to J^*$ with J^* injective, which is a quiso, and any injective resolution $Y^* \Rightarrow J^*$ induces a bijection

$$
(4)\qquad \mathrm{Hom}_{D(\mathcal{A})}(X^*, Y^*) \xrightarrow{\sim} \mathrm{Hom}_{K(\mathcal{A})}(X^*, J^*)
$$

for arbitrary X^* in $C(\mathcal{A})$.

The familiar construction of a mapping cylinder can be used to pass from short exact sequences of complexes to quasi-morphisms. To explain this we recall that, for a morphism $f: X^* \to Y^*$ in $C(\mathcal{A})$, the mapping cylinder C_f^* of f gives a short exact sequence

$$
(5)\qquad 0 \to Y^* \xrightarrow{j} C_f^* \xrightarrow{q} X^*[1] \to 0
$$

in $C(\mathcal{A})$, where $X^*[1]$ denotes the complex X^* shifted one place to the left: $(X^*[1])^n = X^{n+1}$, $\partial_{X^*[1]} = -\partial_{X^*}$. Given a short exact sequence

$$
(6)\qquad 0 \to X_1^* \xrightarrow{i} X_2^* \xrightarrow{p} X_3^* \to 0
$$

in $C(\mathcal{A})$, one has an obvious morphism $\sigma: C_{-i}^* \to X_3^*$ such that the diagram

$$(7)\qquad \begin{tikzcd} X_1^* \arrow[r, "i"] & X_2^* \arrow[r, "p"] & X_3^* \\ X_1^* \arrow[u, "-1"] \arrow[r, "-i"] & X_2^* \arrow[u, "1"] \arrow[r] & C_{-i}^* \arrow[u, "\sigma"'] \end{tikzcd}$$

is commutative. Since the coboundary operator of the exact sequence of the form (5) belonging to $-i$ is the map induced by $-i$ on cohomology, σ is a quiso by the 5-Lemma. Hence the diagram

$$(8)\qquad \begin{tikzcd} X_3^* & & X_1^*[1] \\ & C_{-i}^* \arrow[ul, Rightarrow, "\sigma"] \arrow[ur, "q"'] & \end{tikzcd}$$

defines a quasi-morphism $X_3^* \dashrightarrow X_1^*[1]$, which is said to correspond to (6). It is clear that the passage from (6) to (8) is functorial.

In case $\mathcal{A}$ has enough injectives, there is, for short exact sequences in $C^+(\mathcal{A})$, a second method of constructing the corresponding quasi-morphism: one takes an injective resolution

$$(9)\qquad 0 \to J_1^* \to J_2^* \to J_3^* \to 0$$

of a short exact sequence (6) in $C^+(\mathcal{A})$, and one notes that J_2^* is, in fact, the mapping cylinder of a $\varphi: J_3^*[-1] \to J_1^*$, which is determined by the differential of J_2^*. Hence (9) gives a quasi-morphism $X_3^* \dashrightarrow X_1^*[1]$ by the diagram

$$(10)\qquad \begin{tikzcd} J_3^* \arrow[r] & J_1^*[1] \\ X_3^* \arrow[u, Rightarrow] & X_1^*[1] \arrow[u, Rightarrow] \end{tikzcd}$$

which can be verified to be the same as the one obtained by the previous construction.

In terms of derived categories, the right-derived functor of an additive functor $T: \mathcal{A} \to \mathcal{B}$ between abelian categories is a functor $\mathcal{R}T: D^+(\mathcal{A}) \to D^+(\mathcal{B})$, together with a canonical morphism $\xi: T \to \mathcal{R}T \mid \mathcal{A}$.

To construct $\mathcal{R}T$, one has to assume the existence of enough T-acyclic objects, that is, of a class I_T of objects in $\mathcal{A}$ such that:

(I) I_T is closed both under the formation of direct summands and of cokernels of monos.

(II) T takes short exact sequences of objects of I_T to short exact sequences.
(III) For each object in $\mathcal{A}$ there exists a mono into an object of I_T.

Such a class contains the injectives of $\mathcal{A}$, and in case $\mathcal{A}$ has enough injectives I_T may be taken to be the class of all injectives of $\mathcal{A}$.

(I) and (II) imply that T takes short exact sequences of I_T-complexes in $C^+(\mathcal{A})$ (i.e. of complexes all of whose components are in I_T) into exact sequences. This is equivalent to saying that T preserves quisos between I_T-complexes; hence T induces a functor $T': D^+(I_T)\to D^+(\mathcal{B})$, where $D^+(I_T)$ is the category constructed from the full subcategory of $K^+(\mathcal{A})$ whose objects are the I_T-complexes by formally inverting all quisos.

It follows from (III) that, for each complex in $C^+(\mathcal{A})$, there exists an I_T-acyclic resolution, i.e. a mono into an I_T-complex of $C^+(\mathcal{A})$, which is a quiso. Hence the obvious functor $D^+(I_T)\to D^+(\mathcal{A})$ is an equivalence of categories, so that $\mathfrak{R}T$ and ξ can be constructed by applying T to T-acyclic resolutions.

Passage to cohomology yields functors $R^pT: D^+(\mathcal{A})\to\mathcal{B}$. By means of the quasi-morphisms corresponding to short exact sequences in $C^+(\mathcal{A})$, one also gets the usual long exact sequences. Obviously, one recovers the old definition of right-derived functors in case $\mathcal{A}$ has enough injectives. Finally, we note that if the right-derived functor of T can be constructed, each class of T-acyclic objects is contained in the class of all T-acyclic objects, i.e. the class of all X in $\mathcal{A}$ with $R^pT(X)=0$ for $p>0$.

To conclude this review we indicate how the Ext-functors work out in the language of derived categories. First, we observe that the Hom-functor of $\mathcal{A}$ can be extended to a functor

$$\mathrm{Hom}^*: C(\mathcal{A})^0\times C(\mathcal{A})\to C(\mathcal{A}b) \tag{11}$$

in such a way that one has natural isomorphisms

$$Z^p(\mathrm{Hom}^*(X^*,Y^*))\xrightarrow{\sim}\mathrm{Hom}_{C(\mathcal{A})}(X^*,Y^*[p]) \tag{12a}$$

$$H^p(\mathrm{Hom}^*(X^*,Y^*))\xrightarrow{\sim}\mathrm{Hom}_{K(\mathcal{A})}(X^*,Y^*[p]) \tag{12b}$$

for p-cocycles and p-cohomology, respectively. By (12b), for each injective J^* in $C^+(\mathcal{A})$, the functor $X^*\mapsto\mathrm{Hom}^*(X^*,J^*)$ preserves quisos in $C(\mathcal{A})$. Therefore, assuming that $\mathcal{A}$ has enough injectives, one defines the Ext-groups by

$$\mathrm{Ext}^p(X^*,Y^*)=H^p(\mathrm{Hom}^*(X^*,J^*)) \tag{13}$$

for X^* in $C(\mathcal{A})$ and an injective resolution $Y^*\to J^*$ in $C^+(\mathcal{A})$. Using (12b) again one gets important natural isomorphisms:

(14) $$\mathrm{Ext}^p(X^*, Y^*) \cong \mathrm{Hom}_{D(\mathcal{A})}(X^*, Y^*[p])$$

for X^* in $C(\mathcal{A})$, Y^* in $C^+(\mathcal{A})$.

2. **External and internal products for Ext**. By means of a tensor product in an abelian category $\mathcal{A}$, we can define a product for suitable pairs of quasi-morphisms, which will yield an external product

$$\mathrm{Ext}^p(A, F) \times \mathrm{Ext}^q(A', F') \to \mathrm{Ext}^{p+q}(A \otimes A', F \otimes F')$$

under certain circumstances.

We assume, then, that $\mathcal{A}$ has a "tensor-product", which for the moment, shall mean any bi-additive bifunctor $\mathcal{A} \times \mathcal{A} \to \mathcal{A}$ which is right-exact in both variables and symmetric and associative up to natural isomorphisms. It will be denoted by the usual $\otimes$.

An object P of $\mathcal{A}$ is called *flat*, if the functor $F \mapsto F \otimes P$ is exact; a complex is called flat if all its constituents are. $\mathcal{A}$ will always be assumed to have enough flats, i.e. an epimorphism $P \to X$ with flat source for every X in $\mathcal{A}$. This implies that for every complex X^* in $C^-(\mathcal{A})$ there is a flat P^* in $C^-(\mathcal{A})$ and an epimorphism $P^* \to X^*$ which is a quiso. Hence every quasi-morphism $X^* \dashrightarrow Y^*$ in $C^-(\mathcal{A})$ can be given by a diagram

(1) $$X^* \Leftarrow P^* \longrightarrow Y^*$$

with flat P^* in $C^-(\mathcal{A})$.

Under these assumptions we could proceed to construct the left-derived functor of our tensor product. However, we need only the first stage of this construction, namely a bi-additive bifunctor

(2) $$C^-(\mathcal{A}) \times P^-(\mathcal{A}) \to D^-(\mathcal{A})$$

where $P^-(\mathcal{A})$ denotes the full subcategory of flat complexes in $D^-(\mathcal{A})$. For this we take the usual extension of $\otimes$ to a bifunctor on $K^-(\mathcal{A})$, which will extend further to the categories indicated in (2) iff the following two conditions hold:

(i) Tensoring with $P^* \in P^-(\mathcal{A})$ preserves quisos in $C^-(\mathcal{A})$.

(ii) Tensoring with $X^* \in C^-(\mathcal{A})$ preserves quisos between flat complexes in $C^-(\mathcal{A})$.

Using mapping-cylinders one reduces the proof of both conditions to

(iii) For X^*, P^* in $C^-(\mathcal{A})$ and P^* flat, the complex $X^* \otimes P^*$ is exact if either X^* or P^* is.

Now (iii) follows from the simple fact that a bicomplex which lives in the third quadrant and has either exact rows or exact columns has an exact total complex.

We can now define the product of quasi-morphisms $f: P^* \dashrightarrow X^*$ and $g: Q^* \dashrightarrow Y^*$ in $C^-(\mathcal{A})$ with flat sources P^* and Q^* via the commutative diagram

$$(3)\qquad \begin{tikzcd} & X^* \otimes Q^* \arrow[dr, dashed, "1 \otimes g"] & \\ P^* \otimes Q^* \arrow[ur, dashed, "f \otimes 1"] \arrow[dr, dashed, "1 \otimes g"'] & & X^* \otimes Y^* \\ & P^* \otimes Y^* \arrow[ur, dashed, "f \otimes 1"'] & \end{tikzcd}$$

whose arrows result from the bifunctor (2). Under our assumptions, it is obviously natural, bi-additive, associative, and symmetric up to the usual signs.

In terms of the representations (1):

$P^* \overset{\sigma}{\Longleftarrow} Z^* \longrightarrow X^*$ and $Q^* \overset{\tau}{\Longleftarrow} Z'^* \longrightarrow Y^*$ for f and g, the product $f \otimes g$

is given by

$$P^* \otimes Q^* \underset{\sigma\otimes 1}{\Longleftarrow} Z^* \otimes Q^* \longrightarrow X^* \otimes Q^* \underset{1\otimes\tau}{\Longleftarrow} X^* \otimes Z'^* \longrightarrow X^* \otimes Y^*,$$

where $\sigma \otimes 1$ and $1 \otimes \tau$ are quisos by (iii).

In order to make this into a product for Ext, we assume furthermore, that $\mathcal{A}$ has enough injectives. Then it is known that

$$(4)\qquad \mathrm{Ext}^p(X^*, Y^*) = \mathrm{Hom}_{D(\mathcal{A})}(X^*, Y^*[p])$$

for X^* in $C(\mathcal{A})$ and Y^* in $C^+(\mathcal{A})$. Thus by (3) we obtain a product for Ext if the first variable is restricted to flat complexes in $C^-(\mathcal{A})$, the second to arbitrary complexes in $C^b(\mathcal{A})$. To sum up, we have

PROPOSITION 2.1. *Let $\mathcal{A}$ be an abelian category with tensor product. If $\mathcal{A}$ has both enough flats and enough injectives, this tensor product induces a product*

$$(5)\qquad \mathrm{Ext}^p(P^*, F^*) \times \mathrm{Ext}^q(P'^*, F'^*) \to \mathrm{Ext}^{p+q}(P^* \otimes P'^*, F^* \otimes F'^*)$$

for flat complexes P^*, P'^* in $C^-(\mathcal{A})$ and arbitrary F^*, F'^* in $C^b(\mathcal{A})$. This product is bilinear, natural, associative, and symmetric (up to the usual signs).

This product will be called the *external product for* Ext. Given a pairing $F^* \otimes F'^* \to G^*$, it is clear that a corresponding *internal product* will arise. We shall show that the latter is compatible with coboundary maps.

PROPOSITION 2.2. *Let $\mathcal{A}$ be as in* (2.1). *If*

$$0 \to F_1^* \to F_2^* \to F_3^* \to 0$$

and

$$0 \to G_1^* \to G_2^* \to G_3^* \to 0$$

are exact sequences in $C^b(\mathcal{A})$ and if

$$\begin{array}{ccccc} F_1^* \otimes F'^* & \longrightarrow & F_2^* \otimes F'^* & \longrightarrow & F_3^* \otimes F'^* \\ \downarrow & & \downarrow & & \downarrow \\ G_1^* & \longrightarrow & G_2^* & \longrightarrow & G_3^* \end{array} \tag{6}$$

is a commutative diagram of pairings, the coboundary operators of the two sequences are compatible with the internal products in the sense that the diagram

$$\begin{array}{ccc} \mathrm{Ext}^p(P^*, F_3^*) \times \mathrm{Ext}^q(P'^*, F'^*) & \to & \mathrm{Ext}^{p+q}(P^* \otimes P'^*, G_3^*) \\ \downarrow{\scriptstyle \delta_F \times 1} & & \downarrow{\scriptstyle \delta_G} \\ \mathrm{Ext}^{p+1}(P^*, F_1^*) \times \mathrm{Ext}^q(P'^*, F'^*) & \to & \mathrm{Ext}^{p+q+1}(P^* \otimes P'^*, G_1^*) \end{array} \tag{7}$$

is commutative.

PROOF. We consider elements $\alpha \in \mathrm{Ext}^p(P^*, F_3^*)$ and $\beta \in \mathrm{Ext}^q(P'^*, F'^*)$ as quasi-morphisms, and fix a representation

$$P'^* \overset{\sigma}{\Longleftarrow} Z'^* \xrightarrow{f} F'^*[q] \tag{8}$$

of β with a flat Z'^* in $C^-(\mathcal{A})$. What we must show is the commutativity of the diagram

$$\begin{array}{ccccccc} P^* \otimes P'^* & \xrightarrow{\alpha\otimes\sigma} & F_3^*[p] \otimes Z'^* & \xrightarrow{1\otimes f} & F_3^*[p] \otimes F'^*[q] & \to & G_3^*[p+q] \\ & & \downarrow{\scriptstyle \varphi\otimes 1} & & & & \downarrow{\scriptstyle \gamma} \\ & & F_1^*[p+1] \otimes Z'^* & \xrightarrow{1\otimes f} & F_1^*[p+1] \otimes F'^*[q] & \to & G_1^*[p+q+1], \end{array} \tag{9}$$

where $\varphi: F_3^* \dashrightarrow F_1^*[1]$ nd $\gamma: G_3^* \dashrightarrow G_1^*[1]$ are the quasi-morphisms arising from the given exact sequences. We are using the fact that coboundary operators for any right-derived functor are induced by these.

Using f to lift our pairings from $F'^*[q]$ to Z'^*, we have a commutative diagram

(10)
$$\begin{tikzcd} 0 \arrow[r] & F_1^*[p]\otimes Z'^* \arrow[r] \arrow[d] & F_2[p]\otimes Z'^* \arrow[r] \arrow[d] & F_3^*[p]\otimes Z'^* \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & G_1^*[p+q] \arrow[r] & G_2^*[p+q] \arrow[r] & G_3^*[p+q] \arrow[r] & 0 \end{tikzcd}$$

in which *both* rows are exact. Hence by the naturality of the passage from short exact sequences to quasi-morphisms, we have the commutative square

(11)
$$\begin{tikzcd} F_3^*[p]\otimes Z'^* \arrow[r] \arrow[d, dashed, "\psi"] & G_3^*[p+q] \arrow[d, dashed, "\gamma"] \\ (F_3^*[p]\otimes Z'^*)[1] \arrow[r] & G_1^*[p+q+1]\,, \end{tikzcd}$$

where ψ belongs to the top row of (10).

Hence it remains to show that $\psi=\varphi\otimes 1$, which is an immediate consequence of the following observation: if C^* is the mapping cylinder of a morphism $i: X_1^* \to X_2^*$ in $C^b(\mathcal{A})$, and if Z^* is any complex in $C^-(\mathcal{A})$, the mapping cylinder of $i\otimes 1_{Z^*}$ is canonically isomorphic to $C^*\otimes Z^*$.

By symmetry, i.e. the commutativity of

(12)
$$\begin{tikzcd} \mathrm{Ext}^p(P^*, F^*)\times \mathrm{Ext}^q(P'^*, F'^*) \arrow[d] \arrow[dr] & \\ & \mathrm{Ext}^{p+q}(P^*\otimes P'^*, G^*) \\ \mathrm{Ext}^q(P'^*, F'^*)\times \mathrm{Ext}^p(P^*, F^*) \arrow[ur] & \end{tikzcd}$$

up to the sign $(-1)^{pq}$, we get the

COROLLARY 2.3. *With notation and conditions as above, given two exact sequences*

$$0 \to F_1'^* \to F_2'^* \to F_3'^* \to 0$$

$$0 \to G_1^* \to G_2^* \to G_3^* \to 0\,,$$

and a pairing with F^* of the first of these into the second, the diagram

$$
\begin{array}{lccc}
(13) & \mathrm{Ext}^p(P^*, F^*) \times \mathrm{Ext}^q(P'^*, F_3'^*) & \to & \mathrm{Ext}^{p+q}(P^* \otimes P'^*, G_3^*) \\
& \downarrow 1\times\delta_{F'} & & \downarrow \delta_G \\
& \mathrm{Ext}^p(P^*, F^*) \times \mathrm{Ext}^{q+1}(P'^*, F_1'^*) & \to & \mathrm{Ext}^{p+q+1}(P^* \otimes P'^*, G_1^*)
\end{array}
$$

is commutative up to the sign $(-1)^p$.

3. The Yoneda-edge-product. To recall the definition of the edge morphism, let $S: \mathcal{A} \to \mathcal{B}$, $T: \mathcal{B} \to C$ be additive functors between abelian categories, where $\mathcal{A}$, $\mathcal{B}$ have enough injectives, and S takes injectives into T-acyclic objects. In this situation the canonical map $\gamma: \mathfrak{R}(T\circ S) \to \mathfrak{R}T \circ \mathfrak{R}S$ is an isomorphism, and one gets the edge morphism $e: \mathfrak{R}T\circ S \to \mathfrak{R}(T\circ S)|\mathcal{A}$ by means of the canonical map $\xi: S \to \mathfrak{R}S|\mathcal{A}$ via the commutative triangle

$$
\begin{tikzcd}
\mathfrak{R}T\circ S \arrow[r, "\mathfrak{R}T*\xi"] \arrow[dr, "e"'] & \mathfrak{R}T\circ\mathfrak{R}S|\mathcal{A} \\
& \mathfrak{R}(T\circ S)|\mathcal{A}. \arrow[u, "\gamma"']
\end{tikzcd}
$$

We use e to pass from morphisms of the form

$$(1) \qquad \pi: F \to S(G),$$

where F, G are objects of $\mathcal{B}$, $\mathcal{A}$ respectively to morphisms of the form

$$(2) \qquad d(\pi): \mathfrak{R}T(F) \to \mathfrak{R}(T\circ S)(G)$$

by putting

$$d(\pi) = e \circ \mathfrak{R}(T)(\pi).$$

In other words, d is a functor between comma-categories,

$$(3) \qquad d: (1_{\mathcal{B}}, S) \to (\mathfrak{R}T, \mathfrak{R}(T\circ S)),$$

whose objects are morphisms of the types (1) and (2) respectively and whose morphisms are the usual commutative squares. We observe that d is natural with respect to S in the following sense: a morphism $S_2 \to S_1$ will induce functors on the corresponding comma-categories which commute with d, thus:

$$
\begin{tikzcd}
(1_{\mathscr{B}}, S_2) \arrow[r, "d"] \arrow[d, dashed] & (\mathscr{R}T, \mathscr{R}(T\circ S_2)) \arrow[d, dashed] \\
(1_{\mathscr{B}}, S_1) \arrow[r, "d"] & (\mathscr{R}T, \mathscr{R}(T\circ S_1))
\end{tikzcd}
\tag{4}
$$

To have a more explicit description of d, we note that $d(\pi)$ is obtained from injective resolutions $F \Rightarrow J^*$, $G \Rightarrow I^*$ by application of T to the quasi-morphism λ in the diagram below:

$$
\begin{tikzcd}
F \arrow[r, "\pi"] \arrow[d, Rightarrow] & S(G) \arrow[d] \\
J^* \arrow[r, dashed, "\lambda"] & S(I^*)
\end{tikzcd}
\tag{5}
$$

In particular, if we start from a *short exact sequence* in $(1_{\mathscr{B}}, S)$, i. e., a diagram

$$
\begin{tikzcd}
F_1 \arrow[r, "\alpha"'] \arrow[d, dashed, "\pi_1"'] & F_2 \arrow[r, "\varphi"'] \arrow[d, dashed, "\pi_2"'] & F_3 \arrow[d, dashed, "\pi_3"'] \\
S(G_1) \arrow[r, "S(\beta)"'] & S(G_2) \arrow[r, "S(\psi)"'] & S(G_3)
\end{tikzcd}
\tag{6}
$$

in which (α, φ) and (β, ψ) form short exact sequences in $\mathscr{B}$ and $\mathscr{A}$ respectively, we can work with exact sequences of resolutions to obtain a corresponding diagram of quasi-morphisms

$$
\begin{tikzcd}
J_1^* \arrow[r] \arrow[d, dashed] & J_2^* \arrow[r] \arrow[d, dashed] & J_3^* \arrow[d, dashed] \\
S(I_1^*) \arrow[r] & S(I_2^*) \arrow[r] & S(I_3^*)
\end{tikzcd}
\tag{7}
$$

Applying T to (7) and passing to cohomology, we obtain a morphism of the cohomology sequences. To summarize, we state:

PROPOSITION 3.1. *Given two additive functors*

$$S: \mathscr{A} \to \mathscr{B} \quad \text{and} \quad T: \mathscr{B} \to C$$

between abelian categories, suppose that $\mathscr{A}$ and $\mathscr{B}$ have enough injectives *and S leaves these T-acyclic. Then the edge-morphism*

$$e:\ \mathscr{R}T\circ S \to \mathscr{R}(T\circ S)|\mathscr{A}$$

induces a functor

$$d:\ (1_{\mathscr{B}}, S) \to (\mathscr{R}T, \mathscr{R}(T\circ S)).$$

For every short exact sequence in $(1_{\mathscr{B}}, S)$, d induces a morphism of cohomology sequences.

Using this construction, we shall define a product on the cohomology of certain derived functors. We consider an abelian category $\mathscr{A}$ with enough injectives and assume the existence of a bi-additive functor $\mathscr{H}om:\ \mathscr{A}^0 \times \mathscr{A} \to \mathscr{A}$ such that $\mathrm{Hom}_{\mathscr{A}} = T\circ\mathscr{H}om$. We assume furthermore that $\mathscr{H}om$ and T are left exact and that $\mathscr{H}om\ (F', G)$ is T-acyclic whenever G is injective. Such a functor $\mathscr{H}om$ will be called an *internal Hom-functor*.

If we had the existence of a tensor-product $\otimes$ left adjoint to $\mathscr{H}om$ a pairing $F\otimes F' \to G$ would correspond to a morphism

$$(8) \qquad F \to \mathscr{H}om(F', G).$$

Accordingly, we call a morphism of the form (8) a *pre-pairing of F with F' into G*. A *morphism of pre-pairings*, then, is given by a triple (f, f', g), $f: F_1 \to F_2$, $f': F_1' \to F_2'$, $g: G_1 \to G_2$, such that the diagram

$$(9) \qquad \begin{tikzcd} F_1 \arrow[r] \arrow[d] & \mathscr{H}om(F_1', G_1) \arrow[dr] & \\ & & \mathscr{H}om(F_1', G_2) \\ F_2 \arrow[r] & \mathscr{H}om(F_2', G_2) \arrow[ur] & \end{tikzcd}$$

commutes.

Fixing F' and setting $S_{F'} = \mathscr{H}om(F, -)$, we can apply our functor d to transform any pre-pairing $F \to S_{F'}(G)$ into a sequence of morphisms

$$(10) \qquad R^pT(F) \to \mathrm{Ext}^p(F', G).$$

Thence, using the well-known operation of Ext on R^*T via bi-additive maps

$$(11) \qquad \mathrm{Ext}^p(F', G) \times R^qT(F') \to R^{p+q}T(G),$$

we obtain the *Yoneda-edge-product*, a family of bi-additive maps

$$R^pT(F) \times R^qT(F') \to R^{p+q}T(G). \tag{12}$$

By proposition 3.1, we immediately have

PROPOSITION 3.2. *Let $\mathcal{A}$ be an abelian category with enough injectives and an internal Hom-functor. Then*

a) *the Yoneda-edge-product is natural with respect to morphisms of pre-pairings of the form $(f, 1, g)$;*

b) *if* $0 \to F_1 \to F_2 \to F_3 \to 0$ *and* $0 \to G_1 \to G_2 \to G_3 \to 0$

are exact, and if we have a pre-pairing with F' of the first of these sequences into the second, each of the diagrams

$$\begin{tikzcd} R^pT(F_3) \times R^qT(F') \arrow[r] \arrow[d, "\delta_F \times 1"] & R^{p+q}T(G_3) \arrow[d, "\delta_G"] \\ R^{p+1}T(F_1) \times R^qT(F') \arrow[r] & R^{p+q+1}T(G_1) \end{tikzcd} \tag{13}$$

is commutative.

We have used, of course, that the corresponding statements are known for the usual Yoneda-product (11).

To examine the behaviour of our product with respect to changes in the second variable, we consider an arrow $f' : F'_1 \to F'_2$, and the induced morphism $S_2 = \mathcal{H}om(F'_2, -) \to S_1 = \mathcal{H}om(F'_1, -)$. Using (4) we conclude that a commutative triangle like

$$\begin{tikzcd} F \arrow[r] \arrow[dr] & \mathcal{H}om(F'_1, G) \\ & \mathcal{H}om(F'_2, G) \arrow[u] \end{tikzcd}$$

is transformed by d into a collection of commutative triangles

$$\begin{tikzcd} RT^p(F) \arrow[r] \arrow[dr] & \mathrm{Ext}^p(F'_1, G) \\ & \mathrm{Ext}^p(F'_2, G) \arrow[u] \end{tikzcd} \tag{14}$$

Since every morphism (f, f', g) of pre-pairings can obviously be factored into $(f, 1, g)$ and $(1, f', 1)$, we deduce that the diagram (9) goes over into

(15)
$$\begin{tikzcd} R^pT(F_1) \arrow[r] \arrow[d] & \mathrm{Ext}^p(F_1', G_1) \arrow[dr] & \\ & & \mathrm{Ext}^p(F_1, G_2) \\ R^pT(F_2) \arrow[r] & \mathrm{Ext}^p(F_2', G_2) \arrow[ur] & \end{tikzcd}$$

PROPOSITION 3.3. *Let $\mathcal{A}$ be an abelian category with enough injectives and an internal Hom-functor. Then*

(a) *the Yoneda-edge-product is natural with respect to all morphisms of pre-pairings;*

(b) *if* $0 \to F_1' \xrightarrow[\alpha']{} F_2' \xrightarrow[\varphi']{} F_3' \to 0$ *and* $0 \to G_1 \xrightarrow[\beta]{} G_2 \xrightarrow[\psi]{} G_3 \to 0$

are exact and if we have a pre-pairing of F with the first of these sequences into the second (i.e.: $(1, \alpha', \beta)$ and $(1, \varphi', \psi)$ are morphisms of pre-pairings), each of the diagrams

(16)
$$\begin{tikzcd} R^pT(F) \times R^qT(F_3') \arrow[r] \arrow[d, "1 \times \delta_F"] & R^{p+q}T(G_3) \arrow[d, "\delta_G"] \\ R^pT(F) \times R^{p+1}T(F_1') \arrow[r] & R^{p+q+1}T(G_1) \end{tikzcd}$$

commutes up to the sign $(-1)^p$.

PROOF. a) In view of Proposition 3.2(a), it suffices to look at pre-pairings of type $(1, f', 1)$. For these our statement follows from (14) and naturality of the usual Yoneda-product.

b) By (15), our pre-pairing of sequences gives commutative diagrams

(17)
$$\begin{tikzcd} & \mathrm{Ext}^p(F_1', G_1) \arrow[dr] & \\ & & \mathrm{Ext}^p(F_1, G_2) \\ R^pT(F) \arrow[uur] \arrow[r] \arrow[ddr] & \mathrm{Ext}^p(F_2', G_2) \arrow[ur] \arrow[dr] & \\ & & \mathrm{Ext}^p(F_2', G_3) \\ & \mathrm{Ext}^p(F_3', G_3) \arrow[ur] & \end{tikzcd}$$

Consider the first of these squares and look at the images of a fixed $x \in R^pT(F)$ in the three Ext-groups involved, regarding them as exact sequences of p terms each. They are connected as follows:

$$
\begin{tikzcd}
G_1 \arrow[r] \arrow[d, "\beta"'] & \cdots \arrow[r] & F_1' \arrow[d, equal] \\
G_2 \arrow[r] \arrow[d, equal] & \cdots \arrow[r] & F_1' \arrow[d, "\alpha'"] \\
G_2 \arrow[r] & \cdots \arrow[r] & F_2'
\end{tikzcd}
$$

with the middle sequence produced from the other two by β and α' respectively. Applying this also to the second square of (17) and putting the two together yields a diagram

(18)
$$
\begin{tikzcd}
G_1 \arrow[r] \arrow[d] & \cdots \arrow[r] & F_1' \arrow[d] \\
G_2 \arrow[r] \arrow[d] & \cdots \arrow[r] & F_2 \arrow[d] \\
G_3 \arrow[r] & \cdots \arrow[r] & F_3'
\end{tikzcd}
$$

whose rows are the images of x and whose outer columns are the given exact sequences. Taking a $y \in R^qT(F_3)$ with x clockwise through (16) means feeding y through the bottom sequence of (18) and then through the first column. Traveling counter-clockwise in (16) corresponds to going through the last column of (18) and then through the top row. It is well-known (MacLane [1963], VIII. 3) that the two processes differ by the sign $(-1)^p$.

REMARK. We have defined a product with the usual properties of a cup-product. However, we did not succeed in showing its uniqueness, not even its symmetry. It is of interest to note, that our product is unique (up to isomorphism) for all pre-pairings

$$F \to \mathcal{H}om(F', G)$$

into objects G which are acyclic for $\mathcal{H}om$. For these, e yields an isomorphism

$$RT^p(\mathcal{H}om(F', G)) \xrightarrow{\sim} \mathrm{Ext}^p(F', G)$$

and the uniqueness of our construction follows from that of the Yoneda product.

4. Comparison of the two products. In this final paragraph we wish to show that the two products introduced above coincide under certain circumstances. Thus we consider a situation in which the constructions of both §2 and §3 can be performed.

Let $\mathcal{A}$ be an abelian category with an internal Hom-functor $\mathcal{H}om$ (cf. §3) having a left-adjont tensor-product (cf. §2) denoted by $\otimes$ or $\mathcal{T}en$ depending on the particular formulas at hand. Thus,

$$(1) \qquad \operatorname{Hom}(X, \mathcal{H}om(Y, Z)) \xrightarrow{\sim} \operatorname{Hom}(X \otimes Y, Z).$$

Moreover, we assume that the functor $T: \mathcal{A} \to \mathcal{A}$ which turns $\mathcal{H}om$ into Hom is representable; i. e. there is an object A such that

$$(2) \qquad T = \operatorname{Hom}(A, -).$$

By (1), $\mathcal{T}en(A, -)$ is isomorphic to the identity. In particular, A is flat. It will be important that the formula

$$(3) \qquad \operatorname{Hom}(A, \mathcal{H}om(X, Y)) = \operatorname{Hom}(X, Y)$$

is a special case of (1) via the natural identification of $A \otimes X$ with X.

Finally, we suppose that $\mathcal{A}$ has enough flats and injectives and that $\mathcal{H}om(X, I)$ is T-acyclic whenever I is injective.

Since the main examples of such categories are categories of modules over ringed sites, we shall call a category $\mathcal{A}$ with the properties listed above an *abelian quasi-topos*. The functor T will be referred to as the *section-functor*, the object A as the *structure-object* of $\mathcal{A}$.

To compare our products, we shall first extend the adjointness (1) to the derived category of $\mathcal{A}$ and then relate it to the edge-morphism used in §3.

Both $\mathcal{H}om$ and $\mathcal{T}en$ can be extended to functors

$$(4) \qquad \mathcal{H}om^*: C^-(\mathcal{A})^0 \times C^+(\mathcal{A}) \to C^+(\mathcal{A})$$

$$\mathcal{T}en^*: C^-(\mathcal{A}) \times C^-(\mathcal{A}) \to C^-(\mathcal{A})$$

with the usual sign conventions for differentials. (The indicated restrictions on the complexes are necessary, because we do not assume existence of infinite sums or products in $\mathcal{A}$).

PROPOSITION 4. 1. *Let $\mathcal{A}$ be an abelian quasi-topos. The adjunction-isomorphisms* (1) *extend to natural isomorphisms*

$$\mathrm{Hom}^*(X^*, \mathcal{H}om^*(Y^*, Z^*)) \xrightarrow{\sim} \mathrm{Hom}^*(X^* \otimes Y^*, Z^*) \tag{5}$$

for X^*, Y^* *in* $C^-(\mathcal{A})$ *and* Z^* *in* $C^+(\mathcal{A})$.

PROOF. The trick is to find the right sign-convention to obtain an isomorphism of complexes. Taking $\varphi^p \in \mathrm{Hom}^p(X^*, \mathcal{H}om^*(Y^*, Z^*))$, i. e. a family $\varphi_k^p : X^k \to \mathcal{H}om^{k+p}(Y^*, Z^*)$, regarded as a family

$$\varphi_{k,l}^p : X^k \to \mathcal{H}om(Y^l, Z^{l+k+p})$$

we define its image $\psi_n^p : (X^* \otimes Y^*)^n \to Z^{n+p}$ to be composed of maps

$$\psi_{k,l}^p : X^k \otimes Y^l \to Z^{l+k+p},$$

which correspond by adjointness to $(-1)^\varepsilon \varphi_{k,l}^p$, where $\varepsilon = \frac{1}{2}(p+k)(p+k+1)$.

Looking in (5) at 0-cocycles and 0-cohomology respectively, we get

COROLLARY 4.2. *$\mathcal{H}om^*$ and $\mathcal{T}en^*$ are adjoint to each other in $C(\mathcal{A})$ well as as in $K(\mathcal{A})$ (both restricted as in (4)).*

REMARK. The formula (5) shows up as (8.7) in Chapter VI of MacLane [1963], where it does not involve the sign $(-1)^\varepsilon$. The reason for its appearance here is that we follow a different sign-convention for the differentials of Hom*: our differential ∂^p differs from that used by MacLane by the sign$(-1)^{p+1}$. This is necessary for obtaining formulas (12) of §1.

Now to the derived functors. From §2 we recall that $X^* \otimes P^*$ is exact, whenever P^* is flat and one of X^* or P^* is exact. Thus we can define the total left-derived functor

$\mathcal{L}\,\mathcal{T}en^* : D^-(\mathcal{A}) \times D^-(\mathcal{A}) \to D^-(\mathcal{A})$ by putting $\mathcal{L}\,\mathcal{T}en^*(X^*, Y^*) = X^* \otimes P^*$ where P^* is a flat resolution of Y^*. To do the analogous thing for $\mathcal{H}om^*$, we need the

LEMMA. *For any injective I in an abelian quasi-topos $\mathcal{A}$, the functor $\mathcal{H}om(-, I)$ is exact.*

PROOF. Let $Y_1 \to Y_2$ be a mono. For flat P, Hom $(P, \mathcal{H}om(Y_2, I)) \to$ Hom $(P, \mathcal{H}om(Y_1, I))$ is surjective by adjointness. Looking at an epi $P \to \mathcal{H}om(Y_1, I)$, we see that $\mathcal{H}om(Y_2, I) \to \mathcal{H}om(Y_1, I)$ is epic.

From this Lemma, we conclude that $\mathcal{H}om^*(X^*, I^*)$ is exact whenever I^* is injective and X^* is exact. Hence we obtain the total right-derived functor

$$\mathcal{R}\ \mathcal{H}om^*: D^-(\mathcal{A})^0 \times D^+(\mathcal{A}) \to D^+(\mathcal{A})$$

by putting $\mathcal{R}\ \mathcal{H}om^*(X^*, Y^*) = \mathcal{H}om^*(X^*, I^*)$ for an injective resolution I^* of Y^*.

PROPOSITION 4.3. *The adjointness* (1) *extends to an adjointness*

$$(6) \qquad \mathrm{Hom}_{D(\mathcal{A})}(X^*, \mathcal{R}\ \mathcal{H}om^*(Y^*, Z^*)) \xrightarrow{\sim} \mathrm{Hom}_{D(\mathcal{A})}(\mathcal{L}\ \mathcal{T}en(X^*, Y^*), Z^*)$$

for X^*, Y^* in $D^-(\mathcal{A})$ and Z^* in $D^+(\mathcal{A})$.

PROOF. Taking a flat resolution P^* of Y^* and an injective one I^* of Z^*, we note that $\mathcal{H}om^*(P^*, I^*)$ is injective. Hence we are reduced to showing that

$$\mathrm{Hom}_{K(\mathcal{A})}(X^*, \mathcal{H}om^*(P^*, I^*)) \to \mathrm{Hom}_{K(\mathcal{A})}(X^* \otimes P^*, I^*)$$

is an isomorphism, which is guaranteed by Corollary 4.2.

REMARK. In particular, for $X^* = A$ (remember that $A \otimes Y \simeq Y$), (6) goes over into

$$(7) \qquad \mathrm{Hom}_{D(\mathcal{A})}(A, \mathcal{R}\ \mathrm{Hom}^*(Y^*, Z^*)) \xrightarrow{\sim} \mathrm{Hom}_{D(\mathcal{A})}(Y^*, Z^*).$$

Using the canonical map $\mathcal{H}om^* \to \mathcal{R}\ \mathcal{H}om^*$ (derived, say, from $t: \mathcal{H}om^*(Y^*, Z^*) \to \mathcal{H}om^*(Y^*, I^*)$) we obtain maps

$$(8) \qquad \mathrm{Hom}_{D(\mathcal{A})}(A, \mathcal{H}om^*(Y^*, Z^*)) \to \mathrm{Hom}_{D(\mathcal{A})}(Y^*, Z^*).$$

Applying this to objects Y, Z in $\mathcal{A}$ and remembering the interpretation of Ext in terms of $\mathrm{Hom}_{D(\mathcal{A})}$, we have

$$(9) \qquad \mathrm{Ext}^p(A, \mathcal{H}om(Y, Z)) \to \mathrm{Ext}^p(Y, Z).$$

On the other hand, we have a similar map from the edge-morphism

$$e: \mathcal{R}T(\mathcal{H}om(-,-)) \to \mathcal{R}\ \mathrm{Hom}^* | \mathcal{A} \text{ (cf. §3)}.$$

The main point of this paragraph is that these are the same.

PROPOSITION 4.4. *The map* (9) *obtained from adjointness in* $D(\mathcal{A})$ *coincides with the analogous map derived from the edge-morphism e.*

PROOF. With notation as in the proof of (4.3) consider the following diagram

$$\begin{tikzcd} \mathcal{H}om\,(Y,Z)[p] \arrow[r, Rightarrow, "\sigma"] \arrow[d, dashed, "t"'] & J^*[p] \arrow[d, "s"] \\ \mathcal{H}om^*(Y, I^*[p]) \arrow[r, Rightarrow, "\tau"] & \mathcal{H}om^*(P^*, I^*[p]), \end{tikzcd} \tag{10}$$

in which, for the moment, we ignore J^* and its arrows. The transition (9) is defined as follows:

A quasi-arrow $\alpha: A \dashrightarrow \mathcal{H}om(Y, Z)[p]$ is transformed into $\beta : A \dashrightarrow \mathcal{H}om^*(Y, I^*)[p]$ via t. Since τ is an injective resolution, β can be represented as $\tau^{-1}b$ with $b: A \to \mathcal{H}om^*(P^*, I^*[p])$ in $\mathcal{A}$. By adjointness b corresponds to some $c: P^* \to I^*[p]$ which gives the desired quasi-arrow $\gamma: Y \dashrightarrow I^*[p]$.

To compare this process with the edge-morphism, we introduce the injective resolution J^* of $\mathcal{H}om\,(Y, Z)$. By injectivity of $\mathcal{H}om^*\,(P^*, I^*[p])$, we get the right vertical arrow s making (10) commute up to homotopy. The morphism b found above could therefore have been defined by setting $\alpha = \sigma^{-1}a$ with $a: A \to J^*[p]$ and composing a with s.

The prescription for the edge-morphism, on the other hand, is to apply $\mathrm{Hom}(A, -)$ to s and watch what happens in cohomology. There we get

$$\mathrm{Hom}_{K(\mathcal{A})}(A, J^*[p]) \to \mathrm{Hom}_{K(\mathcal{A})}(A, \mathcal{H}om^*(P^*, I^*[p])$$

with the right hand side to be identified with

$$\mathrm{Hom}_{K(\mathcal{A})}(P^*, I^*[p]).$$

This is exactly what was done above.

We are now ready to deal with the problem which motivated these excursions. Given a pairing

$$\pi: F \otimes F \to G$$

in $\mathcal{A}$, we have two ways of constructing products

$$\mathrm{Ext}^p(A, F) \times \mathrm{Ext}^q(A, F') \to \mathrm{Ext}^{p+q}(A, G), \tag{11}$$

one by the Ext-product of §2, the other via the pre-pairing

$$\pi' : F \to \mathcal{H}om(F', G)$$

adjoint to π by the Yoneda-edge-product of §3.

PROPOSITION 4.5. *Let $\pi : F \otimes F' \to G$ be a pairing in an abelian quasi-topos with structure-object A. Then the internal product*

$$\mathrm{Ext}^p(A, F) \times \mathrm{Ext}^q(A, F') \to \mathrm{Ext}^{p+q}(A, G)$$

coincides with the Yoneda-edge-product

$$RT^p(F) \times RT^p(F') \to RT^{p+q}(G).$$

PROOF. Let elements of $\mathrm{Ext}^p(A, F)$ and $\mathrm{Ext}^q(A, F')$ be given as quasi-arrows

$$\alpha : A \dashrightarrow F[p] \quad \text{and} \quad \alpha' : A \dashrightarrow F[q].$$

Consider the commutative diagram

$$\begin{tikzcd} \mathrm{Hom}_{\mathcal{A}}(F[p], \mathcal{H}om(F', G)[p]) \arrow[r, "\xi"] \arrow[d, "\alpha"] & \mathrm{Hom}_{\mathcal{A}}(F \otimes F'[p], G[p]) \arrow[d, "\alpha \otimes F'"] \\ \mathrm{Hom}_{D(\mathcal{A})}(A, \mathcal{H}om(F', G)[p]) \arrow[r, "\eta"] & \mathrm{Hom}_{D(\mathcal{A})}(F', G[p]) \arrow[d, "\alpha'"] \\ & \mathrm{Hom}_{D(\mathcal{A})}(A[-q], G[p]). \end{tikzcd} \tag{12}$$

Starting with π in the upper right corner, the internal product is described by the right column. The Yoneda-edge-product is obtained in three steps:

(i) going over to $\pi' = \xi^{-1}(\pi)$,

(ii) composing π' with α and applying the "edge" η (cf. 4.4)

(iii) composing the result $\bar{\alpha} \in \mathrm{Hom}_{D(\mathcal{A})}(F', G[p])$ of (i) and (ii) with $\alpha' : A[-q] \dashrightarrow F'$.

By commutativity of (12) the two processes coincide.

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