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# ON THE LOCAL BEHAVIOR OF FUNCTION ALGEBRAS

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**1.** In this paper we shall give complete proofs of theorems announced recently in [8]. These results are concerned with properties of interpolation sets and essential set with respect to a function algebra, and some of them are regarded as generalization of those results established in several literatures [4], [7] and [9]. Our main results are Theorem 1 and Theorem 3, which state that a $w$-interpolation set is not compatible with the essential set of the function algebra. By using these results, in some cases we can determine the essential set of the function algebra from those essential sets of the restriction algebras of countable closed partitions (Theorem 4). Theorem 3, together with Theorem 4, has been previously treated by Mullins [9] under the assumptions that the representing space $X$ of the function algebra coincides with $M_A$ and $M_A$ is metrizable.

Let $X$ be a compact Hausdorff space and $C(X)$ the algebra of all complex-valued continuous functions on $X$. We consider a function algebra $A$ on $X$, that is, $A$ is closed subalgebra of $C(X)$ which separates the points of $X$ and contains the constants. In this case $X$ is sometimes called a representing space of $A$. Throughout this paper $M_A$ will indicate the maximal ideal space of $A$. The Šilov boundary of $A$ will be denoted by $\partial A$. For a subset $F$ in $X$, we shall denote by $A|F$ the restricted algebra of $A$ to $F$. Let $F$ be a closed subset of $X$. If $A|F$ is closed in $C(F)$, $A|F$ is regarded as a function algebra on $F$. Then $F$ is called an interpolation set for $A$ when $A|F=C(F)$. Let $G$ be an open subset in $X$. $G$ is called a $w$-interpolation set for $A$ in $X$ if any compact subset of $G$ is an interpolation set for $A$ in $X$. Following Bear [2] we define the essential set $E$ of $A$ in $X$ as the set which is the hull of the largest ideal of $C(X)$ contained in $A$.

**2.** Bishop [3] and Glicksberg [6] have proved that a function algebra $A$ is characterized by the restriction algebra on the disjoint partition of its maximal antisymmetric sets and Tomiyama [12] has shown that among these maximal antisymmetric sets the set $P$ of points, each of which makes itself a maximal antisymmetric set, is free from the representing space $X$ and plays a special rôle in determining the essential set of $A$; in fact

$$E = X \sim \operatorname{int}(P),$$

where int($P$) is again determined freely from the representing space $X$. It is to be noticed that since any continuous function vanishing on $E$ belongs necessarily to $A$ one easily sees that int($P$) is a $w$-interpolation set for $A$ in $X$. We recall that a subset $K$ of $X$ is said to be an antisymmetric set of $A$ if, for $f$ in $A$, $f$ real valued on $K$ implies $f$ is constant on $K$. The following Lemma 1 is basic in our forthcoming discussions.

LEMMA 1. *Let $A$ be a function algebra on a compact Hausdorff space $X$ and let $p$ be a point of the Choquet boundary of $A$. If there exists a $w$-interpolation set containing $p$, then $p$ belongs to* int($P$).

PROOF. Since $p$ is contained in a $w$-interpolation set, there exists a closed neighborhood $U$ of $p$ such that $A|U = C(U)$. This implies there is no antisymmetric set contained in $U$ which consists of more than two points. Now, since $p$ is a point of the Choquet boundary of $A$, for any open neighborhood $V$ of $p$ we can find a function $f$ in $A$ such that $f(p) = 1 = \|f\|$ and $|f| < 1$ outside of $V$, and we may assume that $V \subset U$. Then one verifies easily that we can choose real numbers $s$ and $r$ with $0 < s < r < 1$ and an open neighborhood $W$ of $p$ so that $\mathrm{Re} f > r$ in $W$ and $\mathrm{Re} f \leqq s$ in $X \sim V$. Thus $W$ is contained in $V$ and the sets $f(\overline{W})$, $f(X \sim V)$ are contained in compact sets $F_1$, $F_2$ repsectively such that $F_1 \cap F_2 = \emptyset$ and the complement of $F_1 \cup F_2$ is connected in the complex plane. By Mergelyan's theorem, there exists a sequence $\{p_n\}$ of polynomials converging uniformly in $F_1 \cup F_2$ to the function which is one in $F_1$ and zero in $F_2$. Next, choose a function $g$ in $A$ such that $0 \leqq g(x) \leqq 1$ in $U$, $g(p) = 1$ and $g = 0$ in $U \sim W$. We can see that the sequence $g\cdot(p_n \circ f)$ converges uniformly on $X$ to a continuous function, which necessarily belongs to $A$. Thus we have found a function $h$ of $A$ whose support is contained in $\overline{W}$ and $h(p) = 1$. Moreover, from the condition $A|U = C(U)$, we may suppose, without loss of generality, that $h$ is a positive function. Let $W_0$ be the open neighborhood of $p$ in which $h(x) > 0$. Then as it is easily seen, there exist no antisymmetric sets of $A$ which consist of both inside and outside points of $W_0$. Besides, we already know that an antisymmetric set of $A$ contained in $U$ necessarily consists of single point. Hence each point in $W_0$ is a maximal antisymmetric set of $A$ and $p$ is an interior point of $P$.

Let $E$ be the essential set of $A$ in $X$, then it is known (cf. [2]) that $A|E$ is a function algebra on $E$. Now let $P_{A|E}$ be the set of all maximal antisymmetric sets of $A|E$ in $E$ which consist of single point.

LEMMA 2. *The interior of $P_{A|E}$ is empty.*

PROOF. By the above cited result in Tomiyama [12] we have only to show that the interior of $P_{A|E}$ in $E$ is empty. At first we notice that $P_{A|E} \subset P$, because of the relation $X \sim E = \mathrm{int}(P)$ and the definition of maximal antisymmetric set. Now since $E$ is closed in $X$, $E \sim \mathrm{int}(P_{A|E})$ is again closed in $X$. Thus $\mathrm{int}(P) \cup \mathrm{int}(P_{A|E}) = X \sim (E \sim \mathrm{int}(P_{A|E}))$ is open in $X$ and is contained in $P$. Hence $\mathrm{int}(P) \cup \mathrm{int}(P_{A|E}) = \mathrm{int}(P)$. As $\mathrm{int}(P_{A|E})$ is disjoint from $\mathrm{int}(P)$, this implies $\mathrm{int}(P_{A|E}) = \emptyset$.

With these preparation we can state our

THEOREM 1. *Let $A$ be a function algebra on a compact Hausdorff space $X$. If $G$ is a $w$-interpolation set for $A$, then $G \cap \partial_{A|E} = \emptyset$, where $E$ is the essential set of $A$ in $X$.*

PROOF. At first, we notice that $G \cap E$ is a $w$-interpolation set for $A|E$ in $E$. Suppose $G \cap \partial_{A|E} \neq \emptyset$, then $G$ contains a point which belongs to the Choquet boundary of $A|E$ because the latter is dense in $\partial_{A|E}$. However Lemma 1 and Lemma 2 show that this yields a contradiction.

There is an another proof of the above Theorem 1 without using the concept of maximal antisymmetric sets. We shall show the idea and the sketch of the proof.

ANOTHER PROOF OF THEOREM 1. Without loss of generality we may assume that $A$ is an essential algebra. Let $G$ be a $w$-interpolation set and let $F_0 = X \sim G$. We assert that $F_0 \supset \partial A$. Let $\widehat{F_0}$ be the $A$-convex hull of $F_0$ which is defined as the set of all elements $\varphi$ in $M_A$ such that

$$|\varphi(f)| \leqq \|f\|_{F_0} \qquad \text{for all } f \in A.$$

Then, it suffices to show that $\widehat{F_0} \supset X$. Let there exists a point $x_0$ in $X$ which does not belong to $\widehat{F_0}$, then separating $x_0$ and $F_0$ by a function in $A$ we can find a function $f$ in $A$ and open neighborhoods $V$ of $x_0$ and $W$ of $F_0$ such that $|1-f(x)| < 1/2$ in $V$ and $|f(x)| < 1/2$ in $W$. Since $X \sim W$ is an interpolation set the same arguments as in Lemma 1 show that there exists a function $\psi$ in $A$ such that $\psi(F_0) = 0$ and $\psi(x_0) = 1$.

Let $F_1 = \{x \in X : f(F_0) = 0 \text{ implies } f(x) = 0 \text{ for } f \in A\}$ and consider the factor space $Y$ of $X$ obtained by identifying the points of $F_1$. The algebra $B = \{f \in A : f \text{ constant on } F_1\}$ is directly transfered to the function algebra $B^* = \{f^* : f \in B\}$ on $Y$ whose elements $f^*$ are defined as $f^*(x) = f(x)$ for

$x \in X \sim F_1$ and $f^*(p) = f(y)$ for $y \in F_1$ where $p$ denotes the point of $Y$ corresponding to $F_1$.

Now let $F$ be a compact subset in $X \sim F_1$. We can separate $F_1$ from any point of $F$ by function in $A$ and since $X \sim F_1(\subset G)$ is a $w$-interpolation set we can find, to each point $x$, an open neighborhood $U(x)$ of $x$ and a function $f_x$ in $A$ such that $f_x = 1$ in $U(x)$ and $f_x(F_1) = 0$. Hence by the compactness of $F$ an usual device shows that we obtain a function $h$ with $h(F)=1$ and $h(F_1)=0$. Take an arbitrary continuous function $g$ on $F$. There exists a function $\bar{g}$ in $A$ such as $g = \bar{g}$ on $F$. Then $\bar{g}h \in B$ and $\bar{g}h|F = g$. Hence $B^*|F = C(F)$. Therefore by the Lemma cited below which is essentially due to Y. Katznelson (cf. Bade and Cartis [1; p. 92]) we can conclude that $B^* = C(Y)$. Thus any continuous function on $X$ vanishing on $F_1$ belongs to $B$ and a-posteriori to $A$. Hence $X = E \subset F_1$. This contradicts to the existence of the non-zero function $\psi$ constructed above. Hence $\hat{F}_0 \supset X$.

The above cited Lemma is: Suppose that $A$ is a function algebra on a compact Hausdorff space $Y$ and $p$ is a point in $Y$. If any compact subset $F$ in $Y \sim \{p\}$ is an interpolation set, then $A = C(Y)$.

The first consequence of Theorem 1 is the following

COROLLARY. *Let $A$ be a function algebra on a compact Hausdorff space $X$ and suppose $E = \partial_{A|E}$. Then the set $X \sim E$ is the largest $w$-interpolation set for $A$ in $X$.*

The proof is immediate from Theorem 1 and the remark given in the first part of this section.

The hypothesis of the corollary is necessary. In fact, let $X$ be the set consisting of the unit circle and the origin $O$ in the unit disc and let $A$ be the restriction of $A_0$ to $X$, where $A_0$ denotes the function algebra of all continuous functions on the unit disc which are analytic in the open disc. Then $E = X$. But we here see that $G = \{0\}$ is a $w$-interpolation set and $G \supsetneqq X \sim E = \emptyset$. Hence we can not expect the conclusion of this kind for an arbitrary representing space $X$.

A function algebra $A$ on a compact Hausdorff space $X$ is said to be $\varepsilon$-regular for some (fixed) number $\varepsilon > 0$ if for each closed set $F$ and a point $x$ which does not belong to $F$, there exists a function $f$ in $A$ with $|1 - f(x)| < \varepsilon$ and $|f| < \varepsilon$ on $F$. If for each disjoint closed sets $F_1$ and $F_2$, there exists a function $f$ in $A$ such that $|f| < \varepsilon$ in $F_1$ and $|1 - f| < \varepsilon$ in $F_2$, we call $A$ an $\varepsilon$-normal function algebra. Then, for these specialized function algebras we can get the condition in the above Corollary.

THEOREM 2. *Let $A$ be an $\varepsilon$-regular function algebra on a compact Hausdorff space $X$ for $0<\varepsilon\leqq 1/2$. Then $X\sim E$ is the largest $w$-interpolation set for $A$ in $X$ where $E$ is the essential set of $A$ in $X$.*

PROOF. Suppose there exists a point $p$ in $E$ which does not belong to $\partial_{A|E}$. Then there exists a function $f$ in $A$ such that $|1-f(p)|<\varepsilon\leqq 1/2$ and $|f|<\varepsilon\leqq 1/2$ in $\partial_{A|E}$. But this contradicts to the fact that $\partial_{A|E}$ is a boundary for $A|E$. Hence $\partial_{A|E}=E$, and $X\sim E$ is the largest $w$-interpolation set.

It seems to be interesting that in the case $X=M_A$, the above corollary holds without the restriction $E=\partial_{A|E}$. We state the result in the following

THEOREM 3. *Let $A$ be a function algebra on a compact Hausdorff space $X$ and suppose $X=M_A$. Let $E$ be the essential set of $A$ in $X$. Then $X\sim E$ is the largest $w$-interpolation set for $A$ in $X$.*

*In other words, a point $p$ in $X$ belongs to* $\mathrm{int}(P)=X\sim E$ *if and only if there exists a closed neighborhood $U$ of $p$ such that $A|U=C(U)$.*

PROOF. We have already noticed that $\mathrm{int}(P)$ is a $w$-interpolation set. Let $G$ be a $w$-interpolation set for $A$ in $X$. We assert that $G\cap E=\emptyset$. Now suppose $G\cap E\neq\emptyset$, then $G\cap E$ is a (non-trivial) $w$-interpolation set for $A|E$ in $E$, and by Theorem 1 $G\cap\partial_{A|E}=\emptyset$. On the other hand, the assumption $X=M_A$ implies that $M_{A|E}=E$. Let $U$ be a non-empty relative open set in $E$ such that $\overline{U}\subset G\cap E$, then $A|\overline{U}=C(\overline{U})$ and $U\subset E\sim\partial_{A|E}$. By the local maximum modulus principle in [10; Theorem6. 1.], we have $\partial_{A|\overline{U}}\subset\overline{U}\sim U$. But $A|\overline{U}=C(\overline{U})$ implies $\partial_{A|\overline{U}}=\overline{U}$, a contradiction. Thus $G\subset X\sim E$.

3. Theorem 3 is the generalization of Mullins' result [9; Theorem 1] in which the result is stated assuming that $M_A$ is metrizable. Now the same idea that derived his Theorem 2 from his Theorem 1 would lead us to get the generalization of [9; Theorem 2] in a more precise form.

THEOREM 4. *Let $A$ be a function algebra on a compact Hausdorff space $X$ with the essential set $E$ and let $\{F_j\}_{j=1}$ be a cover of closed sets in $X$ such that $A|F_j$ is closed in $C(F_j)$ for each $j$. Then, if $X=M_A$ or $E=\partial_{A|E}$ the set $E$ coincides with the closure of $\bigcup_{j=1}^{\infty}E_j$, where $E_j$ denotes the essential set of $A|F_j$ in $F_j$.*

PROOF. From the definition of the essential set, one can easily see that

$E_j \subset E$ for each $j$. Hence $E$ contains the closure of $\bigcup_{j=1}^{\infty} E_j$. Thus it suffices to consider only the restriction algebra $A|E$ in $E$. Since $X=M_A$ implies $M_{A|E}=E$, in both cases we can assume, by the Corollary and Theorem 3, that in $X$ there is no non-empty $w$-interpolation set for $A$. Suppose $X \neq \overline{\bigcup_{j=1}^{\infty} E_j}$ and let $U = X \sim \overline{\bigcup_{j=1}^{\infty} E_j}$. Then $U$ is a non-empty open set and $U = \bigcup_{j=1}^{\infty} (U \cap F_j)$. Hence by the category theorem there exists one index $j_0$ such that $U \cap F_{j_0}$ contains an (non-empty) open set $V$ of $X$. Since $V \subset F_{j_0} \sim E_{j_0}$, $V$ is a $w$-interpolation set for $A|F_{j_0}$, hence for $A$ in $X$, a contradiction.

As a direct consequence of this theorem, a result proved recently by Gamelin and Wilken [5] follows.

COROLLARY 1. *Let $A$ be a function algebra on a compact Hausdorff space $X$. If $X$ is covered by a countable number of interpolation sets for $A$ then $A = C(X)$.*

PROOF The conclusion in the case where $X = M_A$ is derived immediately from the theorem since $E_j = \emptyset$ for all $j$ and $E = \emptyset$. In general case we have only to note that $X = \bigcup_{j=1}^{\infty} F_j$ implies $M_A = \bigcup_{j=1}^{\infty} \hat{F}_j$ ([5; Theorem 1]) where $\hat{F}_j$ is the $A$-convex hull of $F_j$ in $M_A$. In fact, in this case $\hat{F}_j = F_j$ for all $j$, and the case is reduced to the previous one.

The above corollary is also established by Chalice [4] and Mullins [9] under some restricted conditions (cf. next corollary).

Notice that in Theorem 4 one can not expect the same result for an arbitrary representing space $X$. In the example cited after the corollary of Theorem 1, if we consider the closed cover $\{F_1, F_2\}$ of $X$ specifying as $F_1$=unit circle and $F_2 = \{0\}$, then $E_1 = F_1$ and $E_2 = \emptyset$ whereas $E = X = E_1 \cup \{0\}$.

However if $A$ is $\varepsilon$-regular for $0 < \varepsilon \leqq 1/2$ we have known that the condition in Theorem 4 holds. Hence we get the following more precise formulation of Chalice's announced result [4].

COROLLARY 2. *If $A$ is an $\varepsilon$-regular function algebra on a compact Hausdorff space $X$ for $0 < \varepsilon \leqq 1/2$. Then, we get the same conclusions as in Theorem 4 and in Corollary 1.*

Therefore if $A$ is approximately regular, or approximately normal the consequence of Theorem 4 is also true.

Let $F$ be a closed subset in $X$. Then $F$ is called a closed restriction set for $A$ if $A|F$ is closed in $C(F)$. It is known by Glicksberg [7] that if any closed subset in $X$ is a closed restriction set for $A$, then $A=C(X)$. We moreover obtain the following theorem by Theorem 1.

THEOREM 5. *Let $A$ be a function algebra on a compact Hausdorff space $X$ and let $F_0$ be a closed set in $X$. If $A|F_0$ is dense in $C(F_0)$ and if any compact subset $F$ in $X\sim F_0$ is an interpolation set for $A$ (or a closed restriction set for $A$), then $A=C(X)$.*

PROOF. Since $X\sim F_0$ is a $w$-interpolation set, we have $F_0\supset\partial_{A|E}$ by Theorem 1. Consequently, $A|\partial_{A|E}$ is dense in $C(\partial_{A|E})$ and $A|\partial_{A|E}=C(\partial_{A|E})$. If $E$ is non-void, this is a contradiction ([2], p. 388), and so $A=C(X)$.

COROLLARY. *Let $A$ be a function algebra on a compact Hausdorff space $X$. If $F_0$ is a closed set in $X$ without perfect subsets (in particular, a closed countable set) and if any compact subset $F$ in $X\sim F_0$ is an interpolation set for $A$ (or closed restriction set for $A$), then $A=C(X)$.*

The proof is clear from the well-known Rudin's theorem [11].

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