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NORMAL FAMILIES OF COUSIN I AND II DATA

YUM-TONG SIU*)

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It is well-known that on Stein manifolds the solution of the Cousin I problem is always possible and the obstruction to the solution of the Cousin II problem is only topological. Here we use Hörmander's L^2-estimates for the $\bar{\partial}$ operator to investigate the problem of finding normal families of solutions for given normal families of Cousin I and II data on Stein manifolds. In [8] Stoll proved that, if $M=\boldsymbol{C}^p$ with $p>1$, then the following two theorems hold:

(1) Let N be a normal set of non-negative divisors on M and $x_0 \in M$. Suppose x_0 dose not belong to the support of any non-negative divisor which is the limit of a net in N. Then there exists a continuous map γ from N to the set of all holomorphic functions on M such that for $\nu \in N$ $\gamma(\nu)$ defines the divisor ν and $\gamma(\nu)(x_0) = 1$.

(2) Let N be a normal set of non-negative divisors on M and G be an open subset of M. Then there is a normal family $\{f_\nu\}_{\nu \in N}$ of holomorphic functions on M such that for $\nu \in N$ f_ν defines the divisor ν and $f_\nu(c_\nu) = 1$ for some $c_\nu \in G$.

Later (1) and (2) were proved for $M=$ a polydisc by McGrath in [6]. We prove in this paper that on a Stein manifold M a normal family of solutions for a given normal family of Cousin I data can always be found and, if $H^1(M, \boldsymbol{R}/\boldsymbol{Z}) = H^2(M, \boldsymbol{R}) = 0$, then a normal family of solutions for a given normal family of Cousin II data can always be found. As a consequence, (2) holds for Stein manifolds M satisfying $H^1(M, \boldsymbol{R}/\boldsymbol{Z}) = H^2(M, \boldsymbol{R}) = 0$ and thus a problem proposed by Stoll (problem 17, p. 307, [1]) is solved for such Stein manifolds. The method we use also enables us to prove (1) for $M=$ the product of $\boldsymbol{C}^q$ and the unit ball of $\boldsymbol{C}^p$ and a slightly weaker version of (1) for Stein manifolds M satisfying $\pi_1(M) = H^2(M, \boldsymbol{R}) = 0$.

In what follows M is a connected positive-dimensional Stein manifold.

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$\mathcal{O}, \mathcal{O}^*, \mathcal{M}$, and $\mathcal{M}^*$ denote respectively the sheaves of germs of holomorphic, nowhere zero holomorphic, meromorphic, and non-identically-zero meromorphic functions on M. ${}_n\mathcal{O}$ = the structure sheaf of $\boldsymbol{C}^n$. If G is an open subset of M (or $\boldsymbol{C}^n$), then $\Gamma(G, \mathcal{O})$ (or $\Gamma(G, {}_n\mathcal{O})$), is given the topology of uniform convergence on compact subsets. A normal family in $\Gamma(G, \mathcal{O})$ (or $\Gamma(G, {}_n\mathcal{O})$) is a relatively compact subset of $\Gamma(G, \mathcal{O})$ (or $\Gamma(G, {}_n\mathcal{O})$). $\boldsymbol{N}$ = the set of all natural numbers. If $\boldsymbol{U} = \{U_\lambda\}_{\lambda \in I}$ is an open covering of a topological space, then $U_{\lambda_0 \cdots \lambda_p}$ denotes $U_{\lambda_0} \cap \ldots \cap U_{\lambda_p}$, $\{f_{\lambda_0 \cdots \lambda_p}\}'_{\lambda_0, \cdots, \lambda_p \in I} = \{f_{\lambda_0 \cdots \lambda_p} \mid \lambda_0, \cdots, \lambda_p \in I, U_{\lambda_0 \cdots \lambda_p} \neq \emptyset\}$, and $\sum'_{\lambda_0, \cdots, \lambda_p \in I} = \sum_{\lambda_0, \cdots, \lambda_p \in I, U_{\lambda_0 \cdots \lambda_p} \neq \phi}$.

DEFINITION 1. Suppose G is an open subset of M.

(i) A net $\{f_\sigma\}_{\sigma \in S} \subset \Gamma(G, \mathcal{M})$ is said to converge to $f \in \Gamma(G, \mathcal{M})$ if for every $x \in G$ there exist an open neighborhood U of x in G and $g, h, g_\sigma, h_\sigma \in \Gamma(U, \mathcal{O})$, $\sigma \in S$, such that $f_\sigma \mid U = g_\sigma (h_\sigma)^{-1}$, $f \mid U = gh^{-1}$, and g_σ and h_σ converge respectively to g and h in $\Gamma(U, \mathcal{O})$.

(ii) A normal family F in $\Gamma(G, \mathcal{M})$ is a subset of $\Gamma(G, \mathcal{M})$ such that, if $\{f_\sigma\}_{\sigma \in S}$ is a net in F, then there is a subnet $\{f_{\sigma(\tau)}\}_{\tau \in T}$ converging to some $f \in \Gamma(G, \mathcal{M})$.

(iii) A normal family F in $\Gamma(G, \mathcal{M}^*)$ is a subset of $\Gamma(G, \mathcal{M}^*)$ such that, if $\{f_\sigma\}_{\sigma \in S}$ is a net in F, then there are a subnet $\{f_{\sigma(\tau)}\}_{\tau \in T}$ and an $f \in \Gamma(G, \mathcal{M}^*)$ with $f_{\sigma(\tau)} \to f$ in $\Gamma(G, \mathcal{M})$.

REMARKS.

(i) If $f \in \Gamma(G, \mathcal{O})$ and $\{f_\sigma\}_{\sigma \in S}$ is a net in $\Gamma(G, \mathcal{O})$, then $f_\sigma \to f$ in $\Gamma(G, \mathcal{O})$ if and only if $f_\sigma \to f$ in $\Gamma(G, \mathcal{M})$ (Lemma 1 below).

(ii) The convergence defined for $\Gamma(G, \mathcal{M})$ in Def. 1 (i) does not define a topology for $\Gamma(G, \mathcal{M})$ corresponding to it as is seen in the following counter-example: On $\boldsymbol{C}$ define meromorphic functions $f(z) \equiv 1$,

$$f_n(z) \equiv 1, \qquad f_{n,m}(z) = \left(z + \frac{1}{m}\right)^n \left(z - \frac{1}{m}\right)^{-n},$$

$m, n \in \boldsymbol{N}$. Let $S = \{(n, h) \mid n \in \boldsymbol{N}, h$ is a map from $\boldsymbol{N}$ to $\boldsymbol{N}\}$ be directed by the following ordering: $(n, h) \leqq (n', h')$ if and only if $n \leqq n'$ and $h(m) \leqq h'(m)$ for all $m \in \boldsymbol{N}$. Let $g_{(n,h)} = f_{n,h(n)}$ for $(n, h) \in S$. Then $f_n \to f$ as $n \to \infty$ and for fixed $n \in \boldsymbol{N}$ $f_{n,m} \to f_n$ as $m \to \infty$. However, the net $\{g_{(n,h)}\}_{(n,h) \in S}$ does not converge to f.

DEFINITION 2.

(i) A Cousin I datum f on M is an element of $\Gamma(M, \mathcal{M}/\mathcal{O})$, where $\mathcal{O}$ is regarded as a subsheaf of the additive sheaf $\mathcal{M}$.

(ii) A solution of a Cousin I datum f on M is an element of $\Gamma(M, \mathcal{M})$ which is mapped to f under the quotient map $\mathcal{M} \to \mathcal{M}/\mathcal{O}$.

(iii) A normal family of Cousin I data $\{f^{(\alpha)}\}_{\alpha \in A}$ is a set of Cousin I data on M with the following property: for every point x of M there exist an open neighborhood U of x in M and a normal family $\{g^{(\alpha)}\}_{\alpha \in A}$ in $\Gamma(U, \mathcal{M})$ such that $g^{(\alpha)}$ is a solution of $f^{(\alpha)} \mid U$ for $\alpha \in A$.

DEFINITION 3.

(i) A Cousin II datum (or a divisor) f on M is an element of $\Gamma(M, \mathcal{M}^*/\mathcal{O}^*)$, where $\mathcal{O}^*$ is regarded as a subsheaf of the multiplicative sheaf $\mathcal{M}^*$. f is called non-negative if every point of M has a connected open neighborhood U such that $f \mid U$ is the image of some non-zero element cf $\Gamma(U, \mathcal{O})$ under the quotient map $\mathcal{M}^* \to \mathcal{M}^*/\mathcal{O}^*$.

(ii) A solution of a Cousin II datum f on M is an element of $\Gamma(M, \mathcal{M}^*)$ which is mapped to f under the quotient map $\mathcal{M}^* \to \mathcal{M}^*/\mathcal{O}^*$.

(iii) A normal family of Cousin II data $\{f^{(\alpha)}\}_{\alpha \in A}$ is a set of Cousin II data on M with the following property: for every point x of M there exist an open neighborhood U of x in M and a normal family $\{g^{(\alpha)}\}_{\alpha \in A}$ in $\Gamma(U, \mathcal{M}^*)$ such that $g^{(\alpha)}$ is a solution of $f^{(\alpha)} \mid U$ for $\alpha \in A$.

REMARK. The definition given here of a normal family of non-negative Cousin II data agrees with the definition given in [8] of a normal family of non-negative divisors.

DEFINITION 4. A normal family of non-negative divisors F on M is called small if we can find an open covering $\boldsymbol{U} = \{U_\lambda\}_{\lambda \in I}$ of M and a subset K of M such that (1) intersections of finite subcollections of $\boldsymbol{U}$ are empty or contractible open subsets of M, (2) U_λ^- is a compact subset of an open subset of M which is biholomorphic to a ball in a complex number space, (3) $K \cap U_{\lambda\mu} \neq \emptyset$ if $U_\lambda \cap U_\mu \neq \emptyset$, (4) $K \cap U_\lambda$ is pathwise connected for $\lambda \in I$, and (5) if a non-negative divisor f on M is the limit of a net in F, then $K \cap \operatorname{Supp} f = \emptyset$ (where the set of non-negative divisors is given the topology defined in [8] and Supp f denotes the support of the section f of the sheaf $\mathcal{M}^*/\mathcal{O}^*$ on M).

The following three theorems are the main results:

THEOREM 1. *Every normal family of Cousin I data on* M *admits a family of solutions which is normal in* $\Gamma(M, \mathcal{M})$.

THEOREM 2. *If* $H^1(M, \boldsymbol{R}/\boldsymbol{Z}) = H^2(M, \boldsymbol{R}) = 0$, *then every normal family of Cousin II data on* M *admits a family of solutions which is normal in* $\Gamma(M, \mathcal{M}^*)$.

THEOREM 3. *If* $\pi_1(M) = H^2(M, \boldsymbol{R}) = 0$ *and* F *is a small normal family of non-negative divisors on* M, *then there is a continuous map* $\gamma: F \to \Gamma(M, \mathcal{O})$ *such that* $\gamma(f)$ *is a solution of* f *for every* $f \in F$.

REMARKS. (i) Suppose G is an open subset of M. Th. 2 implies that every normal family of non-negative divisors admits a normal family of solutions F such that for every $f \in F$ there exists $x \in G$ with $f(x) = 1$. The reason is the following: For the given family of non-negative divisors we first find a normal family of solutions H. For $x \in G$ and $n \in \boldsymbol{N}$ let $D_{x,n} = \{f \in \Gamma(M, \mathcal{O}) \mid |f(x)| > 1/n\}$. Since the closure H^- of H in $\Gamma(M, \mathcal{O})$ is compact and $H^- \subset \cup \{D_{x,n} \mid x \in G,\ n \in \boldsymbol{N}\}$, $H^- \subset \cup_{i=1}^k D_{x_i, n_i}$ for some $x_1, \cdots, x_k \in G$ and $n_1, \cdots, n_k \in \boldsymbol{N}$. For $h \in H$ $h \in D_{x_i, n_i}$ for some i. Define $f_h = h(x_i)^{-1} h$. The normal family of solutions $F = \{f_h\}_{h \in H}$ satisfies the requirement.

(ii) Under the assumptions of Th. 3, if $x_0 \in M$ such that $x_0 \notin \operatorname{Supp} f$ for $f \in F^-$, then we can choose γ such that $\gamma(f)(x_0) = 1$. The reason is the following: We first find a continuous map $\tilde{\gamma}: F^- \to \Gamma(M, \mathcal{O})$ such that $\tilde{\gamma}(f)$ is a solution of f. Then define $\gamma: F \to \Gamma(M, \mathcal{O})$ by $\gamma(f) = f(x_0)^{-1} \tilde{\gamma}(f)$.

LEMMA 1. *Suppose* G *is a connected open subset of* M. *Suppose* $f, g \not\equiv 0 \in \Gamma(G, \mathcal{O})$ *and* $\{f_\sigma\}_{\sigma \in S}$, $\{g_\sigma\}_{\sigma \in S}$ *are nets in* $\Gamma(G, \mathcal{O})$ *such that* (1) $f_\sigma (g_\sigma)^{-1} \in \Gamma(G, \mathcal{O}), \sigma \in S$, and (2) $f_\sigma \to f$ *and* $g_\sigma \to g$ *in* $\Gamma(G, \mathcal{O})$. *Then* $fg^{-1} \in \Gamma(G, \mathcal{O})$ *and* $f_\sigma (g_\sigma)^{-1} \to fg^{-1}$ *in* $\Gamma(G, \mathcal{O})$.

PROOF. Let Z be the zero-set of g. We have to prove that every point of G has an open neighborhood in which fg^{-1} is holomorphic and on which $f_\sigma (g_\sigma)^{-1}$ converges uniformly to fg^{-1}. Fix $z^0 \in G$. If $z^0 \notin Z$, then $f_\sigma (g_\sigma)^{-1}$ converges uniformly to fg^{-1} on any compact neighborhood of z^0 in G disjoint from Z. Suppose $z^0 \in Z$. We can assume w. l. o. g. that (1) G is an open subset of $\boldsymbol{C}^n$, (2) $z^0 = 0$, (3) $K = \{(z_1, \cdots, z_n) \in \boldsymbol{C}^n \mid |z_i| \leqq 1,\ 1 \leqq i \leqq n\} \subset G$, and (4) $L = \{(z_1, \cdots, z_n) \in K \mid |z_n| = 1\}$ is disjoint from Z. Then $f_\sigma (g_\sigma)^{-1}$ converges uniformly to fg^{-1} on L. By considering the coefficients of the negative powers of z_n in the Laurent series of fg^{-1} with respect to z_n, we conclude that

fg^{-1} is holomorphic in the interior of K. Hence $f_\sigma(g_\sigma)^{-1}$ converges to fg^{-1} uniformly on the interior of K. q. e. d.

LEMMA 2. *Suppose G is an open subset of M and K is a compact subset of G. If F is a normal family in $\Gamma(G, \mathcal{O})$, then F is uniformly bounded on K.*

PROOF. This follows from the fact that the map $\Phi: \Gamma(G, \mathcal{O}) \to [0, \infty)$ defined by $\Phi(f) = |f(K)|$ for $f \in \Gamma(G, \mathcal{O})$ is upper semi-continuous. q. e. d.

LEMMA 3. *Suppose $\{a_n\}_{n\in N}$ and $\{b_n\}_{n\in N}$ are two non-decreasing sequences of points in $[0, \infty)$ such that $a_n \to \infty$. Then there exists a C^2 real-valued function ρ on $[0, \infty)$ such that $\rho' \geqq 0$, $\rho'' \geqq 0$, and $\rho(a_n) \geqq b_n$ for $n \in \boldsymbol{N}$ (where ρ' and ρ'' are respectively the first and second derivatives of ρ).*

PROOF. Define ρ_0 on $[0, \infty)$ by $\rho_0(x) = b_n$ for $a_{n-1} \leqq x < a_n$ (where $a_{-1} = 0$). Define $\rho_{i+1}(x) = \int_0^{x+1} \rho_i(t)\, dt$ for $0 \leqq i \leqq 2$. Then $\rho = \rho_3$ satisfies the requirement. q. e. d.

LEMMA 4. *Suppose Ω is a connected Stein open subset of $\boldsymbol{C}^N$ and $\{K_n\}_{n\in N}$ is a locally finite sequence of compact subsets of Ω. Suppose $\{c_n\}_{n\in N} \subset [0, \infty)$. Then there exists a C^2 plurisubharmonic function φ on Ω such that* inf $\{\varphi(z) |\ z \in K_n\} \geqq c_n$, $n \in \boldsymbol{N}$.

PROOF. Let $f: \Omega \to \boldsymbol{C}^{N'}$ be an embedding of Ω as a closed complex submanifold of $\boldsymbol{C}^{N'}$. For $n \in \boldsymbol{N}$ let a_n be the largest integer such that $f\left(\bigcup_{m=n}^{\infty} K_m\right)$ is disjoint from the ball of $\boldsymbol{C}^{N'}$ centered at 0 and with radius a_n. Since $\{K_n\}_{n\in N}$ is locally finite, $a_n \to \infty$. Let $b_n = \sup_{1\leqq m \leqq n} c_m$, $n \in \boldsymbol{N}$. By Lemma 3 there exists a C^2 real-valued function ρ on $[0, \infty)$ such that $\rho' \geqq 0$, $\rho'' \geqq 0$, and $\rho(a_n^2) \geqq b_n$, $n \in \boldsymbol{N}$. Define $\tilde{\varphi}(z) = \rho(|z|^2)$ for $z \in \boldsymbol{C}^{N'}$. Since $\sum_{i,j=1}^{N'} \frac{\partial^2 \tilde{\varphi}}{\partial z_i \partial \bar{z}_j}(z)\, \xi_i \bar{\xi}_j = \rho''(|z|^2)| \sum_{i=1}^{N'} \xi_i \bar{z}_i|^2 + \rho'(|z|^2)\left(\sum_{i=1}^{N'} |\xi_i|^2\right) \geqq 0$ for $\xi_1, \cdots, \xi_{N'} \in \boldsymbol{C}$, $\tilde{\varphi}$ is plurisubharmonic on $\boldsymbol{C}^{N'}$. $\varphi = \tilde{\varphi} \circ f$ satisfies the requirement. q. e. d.

LEMMA 5. *Suppose H_1 and H_2 are Hilbert spaces and $\varphi: H_1 \to H_2$ is a continuous linear surjection. Then there exists a continuous linear map ψ: $H_2 \to H_1$ such that $\varphi \circ \psi =$ the identity map on H_2.*

PROOF. Let L be the orthogonal complement of Ker φ in H_1. Define $\psi: H_2 \to H_1$ as follows: for $x \in H_2$ set $\psi(x)$ to be the unique element in L such that $\varphi(\psi(x)) = x$. The open mapping theorem implies that ψ is continuous. q. e. d.

LEMMA 6. *Suppose E and F are Fréchet spaces and $\varphi: E \to F$ is a continuous linear surjection. If K is a compact subset of F, then there exists a compact subset L of E such that $\varphi(L) = K$.*

PROOF. This follows from § 22. 2 (7), p. 281, [5] and the open mapping theorem. q. e. d.

LEMMA 7. *Suppose $H^2(M, \boldsymbol{R}) = 0$ and $\boldsymbol{U} = \{U_\lambda\}_{\lambda \in N}$ is an open covering of M such that intersections of finite subcollections of $\boldsymbol{U}$ are empty or contractible open subsets of M. Suppose K is a compact subset of $Z^2(\boldsymbol{U}, \boldsymbol{R})$.*

(i) *There exists a continuous linear map $\varphi: K \to C^1(\boldsymbol{U}, \boldsymbol{R})$ such that $\delta \circ \varphi =$ the identity map on K.*

(ii) *If in addition $K \subset Z^2(\boldsymbol{U}, \boldsymbol{Z})$ and $H^1(M, \boldsymbol{R}/\boldsymbol{Z}) = 0$, then there exists a compact subset L of $C^1(\boldsymbol{U}, \boldsymbol{Z})$ such that $\delta(L) = K$.*

PROOF. (i) There exists $\xi_{\lambda\mu\nu} > 0$ for $U_{\lambda\mu\nu} \neq \emptyset$ such that

$$(3) \qquad |a_{\lambda\mu\nu}| \leqq \xi_{\lambda\mu\nu} \text{ for } \{a_{\lambda\mu\nu}\}'_{\lambda,\mu,\nu \in N} \in K.$$

Since $\delta: C^1(\boldsymbol{U}, \boldsymbol{R}) \to Z^2(\boldsymbol{U}, \boldsymbol{R})$ is surjective, by Lemma 6 there exists $\eta_{\lambda\mu} > 0$ for $U_{\lambda\mu} \neq \emptyset$ such that, if $a = \{a_{\lambda\mu\nu}\}'_{\lambda,\mu,\nu \in N} \in Z^2(\boldsymbol{U}, \boldsymbol{R})$ and $|a_{\lambda\mu\nu}| \leqq 2^{\frac{\lambda+\mu+\nu}{2}} \xi_{\lambda\mu\nu}$, then there exists $b = \{b_{\lambda\mu}\}'_{\lambda,\mu \in N} \in C^1(\boldsymbol{U}, \boldsymbol{R})$ with $\delta b = a$ and $|b_{\lambda\mu}| \leqq \eta_{\lambda\mu}$. Let $H_1 = \{b = \{b_{\lambda\mu}\}'_{\lambda,\mu \in N} \in C^1(\boldsymbol{U}, \boldsymbol{R}) \mid \Sigma'_{\lambda,\mu \in N} 2^{-\lambda-\mu} \eta_{\lambda\mu}^{-2} |b_{\lambda\mu}|^2 + \Sigma'_{\lambda,\mu,\nu \in N} 2^{-\lambda-\mu-\nu} \xi_{\lambda\mu\nu}^{-2} |(\delta b)_{\lambda\mu\nu}|^2 < \infty\}$ and $H_2 = \{a = \{a_{\lambda\mu\nu}\}'_{\lambda,\mu,\nu \in N} \in Z^2(\boldsymbol{U}, \boldsymbol{R}) \mid \|a\|^2 = \Sigma'_{\lambda,\mu,\nu \in N} 2^{-\lambda-\mu-\nu} \xi_{\lambda\mu\nu}^{-2} |a_{\lambda\mu\nu}|^2 < \infty\}$. H_1 and H_2 are Hilbert spaces. We claim that the map $\varphi_1: H_1 \to H_2$ induced by δ is surjective. Fix $a = \{a_{\lambda\mu\nu}\}'_{\lambda,\mu,\nu \in N} \in H_2$ and we want to find $b \in H_1$ such that $\varphi_1(b) = a$. We can assume w. l. o. g that $\|a\| \leqq 1$. Then $|a_{\lambda\mu\nu}| \leqq 2^{\frac{\lambda+\mu+\nu}{2}} \xi_{\lambda\mu\nu}$. There exists $b = \{b_{\lambda\mu}\}'_{\lambda,\mu \in N} \in C^1(\boldsymbol{U}, \boldsymbol{R})$ with $\delta b = a$ and $|b_{\lambda\mu}| \leqq \eta_{\lambda\mu}$. Hence $b \in H_1$ and $\varphi_1(b) = a$. By Lemma 5 there exists a continuous linear map $\varphi_2: H_2 \to H_1$ such that $\varphi_1 \varphi_2 =$ the identity map on H_2. Let $i_1: K \to H_2$ and $i_2: H_1 \to C^1(\boldsymbol{U}, \boldsymbol{R})$ be inclusion maps. i_1 is continuous because of (3). $\varphi = i_2 \varphi_2 i_1$ satisfies the requirement.

(ii) Let G be the multiplicative group $\{z \in \boldsymbol{C} \mid |z| = 1\}$. Since $G \approx \boldsymbol{R}/\boldsymbol{Z}$, $H^1(M, G) = 0$. Let $K = \{a^{(\alpha)}\}_{\alpha \in A}$. Let $\varphi(a^{(\alpha)}) = b^{(\alpha)} = \{b_{\lambda\mu}^{(\alpha)}\}'_{\lambda,\mu \in N}$. $\exp(2\pi i\, b_{\lambda\mu}^{(\alpha)}) \exp(2\pi i\, b_{\mu\nu}^{(\alpha)}) \exp(2\pi i\, b_{\lambda\nu}^{(\alpha)}) = 1$ for $U_{\lambda\mu\nu} \neq \emptyset$. Hence there exists $0 \leqq b_\lambda^{(\alpha)} < 1$ for U_λ

$\neq \emptyset$ and $\alpha \in A$, such that for $U_{\lambda\mu} \neq \emptyset$ $\exp(2\pi i\, b_\mu^{(\alpha)}) \exp(-2\pi i\, b_\lambda^{(\alpha)}) = \exp(2\pi i b_{\lambda\mu}^{(\alpha)})$. Let $c_{\lambda\mu}^{(\alpha)} = b_{\lambda\mu}^{(\alpha)} + b_\lambda^{(\alpha)} - b_\mu^{(\alpha)}$. Then $c^{(\alpha)} = \{c_{\lambda\mu}^{(\alpha)}\}'_{\lambda,\mu \in N} \in C^1(\boldsymbol{U}, \boldsymbol{Z})$, $\alpha \in A$. The closure L of $\{c^{(\alpha)}\}_{\alpha \in A}$ in $C^1(\boldsymbol{U}, \boldsymbol{Z})$ satisfies the requirement. q. e. d.

LEMMA 8. *Suppose Ω is a Stein open subset of $\boldsymbol{C}^N$. Suppose $\boldsymbol{U} = \{U_\lambda\}_{\lambda \in I}$ is a locally finite open covering of Ω such that U_λ is relatively compact in $\Omega, \lambda \in I$. Suppose $F \subset Z^1(\boldsymbol{U}, \mathcal{O})$ and $a_{\lambda\mu} > 0$, $\lambda, \mu \in I$, such that $|f_{\lambda\mu}(U_{\lambda\mu})| \leqq a_{\lambda\mu}$ for $\{f_{\lambda\mu}\}'_{\lambda,\mu \in I} \in F$. Then there exists a continuous map $\gamma: F \rightarrow C^0(\boldsymbol{U}, \mathcal{O})$ such that $\delta(\gamma(f)) = f$ for $f \in F$.*

PROOF. Let $\{\rho_\lambda\}_{\lambda \in I}$ be a partition of unity subordinate to $\boldsymbol{U}$. Let $b_\lambda = \sup_{1 \leqq i \leqq N} \left| \frac{\partial \rho_\lambda}{\partial \bar{z}_i}(\Omega) \right|$ and $c_\lambda = \Sigma \{b_\mu a_{\mu\lambda} | U_\lambda \cap U_\mu \neq \emptyset\}$, $\lambda \in I$. Let k be a C^∞ positive-valued function on Ω such that $e = \int_\Omega k dx < \infty$ (where dx is the Euclidean volume element of $\boldsymbol{C}^N$). Since Ω is σ-compact, $\{\lambda \in I | U_\lambda \neq \emptyset\}$ is countable. By Lemma 4 there is a C^2 plurisubharmonic function φ on Ω such that inf $\{\varphi(z) | z \in U_\lambda^-\} \geqq \log(N c_\lambda^2) + |(\log k^{-1})(U_\lambda^-)|$ for $U_\lambda \neq \emptyset$. Let $\psi(z) = \varphi(z) + 2\log(1 + |z|^2)$ on Ω. Let $H_1 = \{\eta | \eta$ is a locally square integrable function on Ω and $\bar{\partial}\eta$ is an $(0,1)$-form on Ω with locally square integrable functions as coefficients such that $\|\eta\|^2_\psi + \|\bar{\partial}\eta\|^2_\varphi < \infty\}$ and let $H_2 = \{\omega | \omega$ is a $(0,1)$-form on Ω with locally square integrable functions as coefficients such that $\bar{\partial}\omega = 0$ and $\|\omega\|_\varphi < \infty\}$, where $\bar{\partial}$ is in the distribution sense and the norms $\|\cdot\|_\varphi$ and $\|\cdot\|_\psi$ are as defined on pp. 77-78 of [3]. H_1 and H_2 are Hilbert spaces. We are going to define a map $\gamma_1: F \rightarrow H_2$. Take $f = \{f_{\mu\nu}\}'_{\mu,\nu \in I} \in F$. Let $h_\lambda^{(f)} = \Sigma_\mu \sigma_\mu$, where σ_μ is the trivial extension of $\rho_\mu f_{\mu\lambda}$ on U_μ. Then $f_{\lambda\mu} = h_\mu^{(f)} - h_\lambda^{(f)}$ on $U_{\lambda\mu}$. $0 = \bar{\partial} h_\mu^{(f)} - \bar{\partial} h_\lambda^{(f)}$ on $U_{\lambda\mu}$. There is a unique C^∞ $(0,1)$-form ω on Ω such that $\omega = \bar{\partial} h_\lambda^{(f)}$ on U_λ. Then $\bar{\partial}\omega = 0$ and $\|\omega\|_\varphi \leqq \sqrt{e}$. Hence $\omega \in H_2$. Set $\gamma_1(f) = \omega$. γ_1 is continuous.

Let $\theta: H_1 \rightarrow H_2$ be induced by $\bar{\partial}$.

Then θ is a continuous linear surjection (Lemma 4, p. 945, [4]). By Lemma 5 there exists a continuous linear map $\gamma_2: H_2 \rightarrow H_1$ such that $\theta \circ \gamma_2 =$ the identity map on H_2. Define $\gamma: F \rightarrow C^0(\boldsymbol{U}, \mathcal{O})$ as follows: Take $f \in F$. Let $g_\lambda = h_\lambda^{(f)} - \gamma_2(\gamma_1(f)) | U_\lambda$, where $h_\lambda^{(f)}$ is as defined above. Set $\gamma(f) = \{g_\lambda\}'_{\lambda \in I}$. Then γ satisfies the requirement. q. e. d.

THEOREM 4. *Suppose $\boldsymbol{U} = \{U_\lambda\}_{\lambda \in I}$* and *$\boldsymbol{V} = \{V_\lambda\}_{\lambda \in I}$ are open coverings of M such that $\boldsymbol{V}$ is locally finite and V_λ^- is a compact subset of U_λ, $\lambda \in I$. Suppose F is a compact subset of $Z^1(\boldsymbol{U}, \mathcal{O})$. Then there exists a continuous map γ: $F \rightarrow C^0(\boldsymbol{V}, \mathcal{O})$ such that $\delta(\gamma(f)) = f$ on $\boldsymbol{V}$.*

PROOF. M is a closed complex submanifold of $\boldsymbol{C}^N$ for some N. There exists an open neighborhood Ω_1 of M in C^N and a holomorphic retraction ρ_1 : $\Omega_1 \to M$(VIII. C. 8., p. 257, [2]). There exists a Stein open neighborhood Ω of M in Ω_1 (Th. 2, p. 380, [7]). Let $\rho = \rho_1 | \Omega$. Let $\widetilde{V} = \{\widetilde{V}_\lambda\}_{\lambda \in I}$, where $\widetilde{V}_\lambda = \rho^{-1}(V_\lambda)$. Let $\rho^* : Z^1(U, \mathcal{O}) \to Z^1(\widetilde{V}, {}_N\mathcal{O})$ be induced by ρ, i. e., if $\{f_{\lambda\mu}\}'_{\lambda,\mu \in I} \in Z^1(U, \mathcal{O})$, then $\rho^*(f) = \{\widetilde{f}_{\lambda\mu}\}'_{\lambda,\mu \in I} \in Z^1(\widetilde{V}, {}_N\mathcal{O})$, where $\widetilde{f}_{\lambda\mu} = (f_{\lambda\mu} \circ \rho) | \widetilde{V}_{\lambda\mu}$. ρ^* is linear and continuous. Let $F = \rho^*(\widetilde{F})$. By Lemma 2 there exists $a_{\lambda\mu} > 0$, $\lambda, \mu \in I$, such that $|f_{\lambda\mu}(V^-_{\lambda\mu})| \leqq a_{\lambda\mu}$ for $\{f_{\lambda\mu}\}'_{\lambda,\mu \in I} \in F$. Hence $|\widetilde{f}_{\lambda\mu}(\widetilde{V}_{\lambda\mu})| \leqq a_{\lambda\mu}$ for $\{\widetilde{f}_{\lambda\mu}\}'_{\lambda,\mu \in I} \in \widetilde{F}$. By Lemma 8 there exists a continuous map $\widetilde{\gamma}$: $\widetilde{F} \to C^0(\widetilde{V}, {}_N\mathcal{O})$ such that $\delta(\widetilde{\gamma}(\widetilde{f})) = \widetilde{f}$ for $\widetilde{f} \in \widetilde{F}$. Define γ: $F \to C^0(V, \mathcal{O})$ as follows: Take $f \in F$. Let $\widetilde{\gamma}(\rho^*(f)) = \{\widetilde{g}_\lambda\}'_{\lambda \in I}$. Let $g_\lambda = \widetilde{g}_\lambda | V_\lambda$. Set $\gamma(f) = \{g_\lambda\}'_{\lambda \in I}$. Then γ satisfies the requirement. q. e. d.

REMARK. The map γ constructed in the proofs of Lemma 8 and Th. 4 satisfies the following linearity conditions: $\gamma(f+f') = \gamma(f) + \gamma(f')$ and $\gamma(af) = a\gamma(f)$ if $a \in C$ and $f, f', f+f', af \in F$.

PROOF OF THEOREM 1. Suppose $F = \{f^{(\alpha)}\}_{\alpha \in A}$ is a normal family of Cousin I data on M. There exists an open covering $U = \{U_\lambda\}_{\lambda \in I}$ of M and $\{f_\lambda^{(\alpha)}\}'_{\lambda \in I} \in C^0(U, \mathcal{M})$, $\alpha \in A$, such that (1) $f_\lambda^{(\alpha)}$ is a solution of $f^{(\alpha)} | U_\lambda, \alpha \in A$, $\lambda \in I$, and (2) for fixed $\lambda \in I$ $\{f_\lambda^{(\alpha)}\}_{\alpha \in A}$ is a normal family in $\Gamma(U_\lambda, \mathcal{M})$. We can suppose w. l. o. g. that U is locally finite and there is an open covering $V = \{V_\lambda\}_{\lambda \in I}$ of M such that V_λ^- is a compact subset of U_λ, $\lambda \in I$.

Let $h_{\lambda\mu}^{(\alpha)} = (f_\mu^{(\alpha)} - f_\lambda^{(\alpha)}) | U_{\lambda\mu}$, $\alpha \in A$, $\lambda, \mu \in I$. $h^{(\alpha)} = \{h_{\lambda\mu}^{(\alpha)}\}'_{\lambda,\mu \in I} \in Z^1(U, \mathcal{O})$. By Lemma 1 for fixed $\lambda, \mu \in I$ $\{h_{\lambda\mu}^{(\alpha)}\}_{\alpha \in A}$ is a normal family in $\Gamma(U_{\lambda\mu}, \mathcal{O})$. By Th. 4 there exists $k^{(\alpha)} = \{k_\lambda^{(\alpha)}\}'_{\lambda \in I} \in C^0(V, \mathcal{O})$, $\alpha \in A$, such that $\delta k^{(\alpha)} = h^{(\alpha)}$ on V, $\alpha \in A$, and for fixed $\lambda \in I$ $\{k_\lambda^{(\alpha)}\}_{\alpha \in A}$ is a normal family in $\Gamma(V_\lambda, \mathcal{O})$. Define $g^{(\alpha)} \in \Gamma(M, \mathcal{M})$, $\alpha \in A$, by setting $g^{(\alpha)} = f_\lambda^{(\alpha)} - k_\lambda^{(\alpha)}$ on V_λ, $\alpha \in A$. $\{g^{(\alpha)}\}_{\alpha \in A}$ is a normal family in $\Gamma(M, \mathcal{M})$ and $g^{(\alpha)}$ is a solution of $f^{(\alpha)}$, $\alpha \in A$. q. e. d.

PROOF OF THEOREM 2. Suppose $F = \{f^{(\alpha)}\}_{\alpha \in A}$ is a normal family of Cousin II data on M. There exist an open covering $U = \{U_\lambda\}_{\lambda \in I}$ of M and $\{f_\lambda^{(\alpha)}\}'_{\lambda \in I} \in C^0(U, \mathcal{M}^*)$, $\alpha \in A$, such that (1) $f_\lambda^{(\alpha)}$ is a solution of $f^{(\alpha)} | U_\lambda, \alpha \in A$, $\lambda \in I$, and (2) for fixed $\lambda \in I$ $\{f_\lambda^{(\alpha)}\}_{\alpha \in A}$ is a normal family in $\Gamma(U_\lambda, \mathcal{M}^*)$. We can suppose w. l. o. g. that (1) intersections of finite subcollections of U are empty or contractible open subsets of M, (2) U is locally finite, and $I = N$ and (3) there exists an open covering $V = \{V_\lambda\}_{\lambda \in I}$ of M such that V_λ^- is a compact subset of U_λ, $\lambda \in I$.

Let $h_{\lambda\mu}^{(\alpha)} = f_\mu^{(\alpha)}(f_\lambda^{(\alpha)})^{-1} | U_{\lambda\mu}$, $\alpha \in A, \lambda, \mu \in I$. By Lemma 1 for fixed $\lambda, \mu \in I$ $\{h_{\lambda\mu}^{(\alpha)}\}_{\alpha \in A}$ is a relatively compact subset of $\Gamma(U_{\lambda\mu}, \mathcal{O}^*)$ (where $\Gamma(U_{\lambda\mu}, \mathcal{O}^*)$ is given the topology induced from $\Gamma(U_{\lambda\mu}, \mathcal{O})$). Fix $x_{\lambda\mu} \in U_{\lambda\mu}$ for $U_{\lambda\mu} \neq \emptyset$. Since $U_{\lambda\mu}$ is empty or contractible, for $U_{\lambda\mu} \neq \emptyset$ and $\lambda < \mu$ there exists a unique

$k_{\lambda\mu}^{(\alpha)} \in \Gamma(U_{\lambda\mu}, \mathcal{O})$ such that $\exp(2\pi i k_{\lambda\mu}^{(\alpha)}) = h_{\lambda\mu}^{(\alpha)}$ and the real part of $k_{\lambda\mu}^{(\alpha)}(x_{\lambda\mu})$ is in $[0\ 1)$. Set $k_{\mu\lambda}^{(\alpha)} = -k_{\lambda\mu}^{(\alpha)}$ and $k_{\lambda\lambda}^{(\alpha)} = 0$. $k^{(\alpha)} = \{k_{\lambda\mu}^{(\alpha)}\}'_{\lambda,\mu\in I} \in C^1(\boldsymbol{U}, \mathcal{O})$. Then $\delta k^{(\alpha)} \in Z^2(\boldsymbol{U}, \boldsymbol{Z})$ and for fixed $\lambda, \mu \in I$ $\{k_{\lambda\mu}^{(\alpha)}\}_{\alpha\in A}$ is a normal family in $\Gamma(U_{\lambda\mu}, \mathcal{O})$. By Lemmas 2 and 7 there exists $u^{(\alpha)} \in C^1(U, \boldsymbol{Z})$ $\alpha \in A$, such that $\delta u^{(\alpha)} = \delta k^{(\alpha)}$ and $\{u^{(\alpha)}\}_{\alpha\in A}$ is a relatively compact subset of $C^1(\boldsymbol{U}, \boldsymbol{Z})$. Let $v^{(\alpha)} = k^{(\alpha)} - u^{(\alpha)}$. Then $\{v^{(\alpha)}\}_{\alpha\in A}$ is a relatively compact subset of $Z^1(\boldsymbol{U}, \mathcal{O})$. By Th. 4 there exists $w^{(\alpha)} = \{w_\lambda^{(\alpha)}\}'_{\lambda\in I} \in C^0(\boldsymbol{V}, \mathcal{O})$, $\alpha \in A$, such that $\delta w^{(\alpha)} = v^{(\alpha)}$ and $\{w^{(\alpha)}\}_{\alpha\in A}$ is a relatively compact subset of $C^0(\boldsymbol{V}, \mathcal{O})$. Define $g^{(\alpha)} \in \Gamma(M, \mathcal{M}^*)$ by setting $g^{(\alpha)} = f_\lambda^{(\alpha)} \exp(-2\pi i w_\lambda^{(\alpha)})$ on V_λ. Then $\{g^{(\alpha)}\}_{\alpha\in A}$ is a normal family in $\Gamma(M, \mathcal{M}^*)$ and $g^{(\alpha)}$ is a solution of $f^{(\alpha)}, \alpha \in A$. q. e. d.

PROOF OF THEOREM 3. Suppose $F = \{f^{(\alpha)}\}_{\alpha\in A}$ is a small normal family of non-negative divisors on M. We can assume w. l. o. g. that F is compact in the topological space of all non-negative divisors. There exist an open covering $\boldsymbol{U} = \{U_\lambda\}_{\lambda\in I}$ of M and a subset K of M such that (1) to (5) of Def. 4 are satisfied. We can assume w. l. o. g. that $\boldsymbol{U}$ is countable. Fix $x_\lambda \in K \cap U_\lambda$ for $U_\lambda \neq \emptyset$. Fix $x_{\lambda\mu} = x_{\mu\lambda} \in K \cap U_{\lambda\mu}$ for $U_\lambda \cap U_\mu \neq \emptyset$. By Ths. 1.9 (p. 168) and 2.25 (p. 188) of [8] there exists $h_\lambda^{(\alpha)} \in \Gamma(U_\lambda, \mathcal{O})$, $\alpha \in A$, $\lambda \in I$, such that (1) $h_\lambda^{(\alpha)}$ is a solution of $f^{(\alpha)}|U_\lambda$, (2) $h_\lambda^{(\alpha)}(x_\lambda) = 1$, and (3) for fixed $\lambda \in I$ the map from F to $\Gamma(U_\lambda, \mathcal{O})$ defined by $f^{(\alpha)} \to h_\lambda^{(\alpha)}$ is continuous.

For $U_\lambda \cap U_\mu \neq \emptyset$ choose a continuous map $\xi_{\lambda\mu}\colon [0,1] \to K \cap U_\lambda$ such that $\xi_{\lambda\mu}(0) = x_\lambda$ and $\xi_{\lambda\mu}(1) = x_{\lambda\mu}$. For $\alpha \in A$ and $U_\lambda \cap U_\mu \neq \emptyset$ there is a unique continuous map $\eta_{\lambda\mu}^{(\alpha)}\colon [0,1] \to \boldsymbol{R}$ such that $\eta_{\lambda\mu}^{(\alpha)}(0) = 0$ and $\exp(2\pi i \eta_{\lambda\mu}^{(\alpha)}(\theta)) = h_\lambda^{(\alpha)}(\xi_{\lambda\mu}(\theta)) | h_\lambda^{(\alpha)}(\xi_{\lambda\mu}(\theta))|^{-1}$ for $\theta \in [0,1]$. Let $k_{\lambda\mu}^{(\alpha)} = h_\mu^{(\alpha)}(h_\lambda^{(\alpha)})^{-1}|U_{\lambda\mu}$, $\alpha \in A, \lambda, \mu \in I$. There is a unique holomorphic function $u_{\lambda\mu}^{(\alpha)}$ on $U_{\lambda\mu}$ such that $\exp(2\pi i u_{\lambda\mu}^{(\alpha)}) = k_{\lambda\mu}^{(\alpha)}$ and the real part of $u_{\lambda\mu}^{(\alpha)}(x_{\lambda\mu})$ is $\eta_{\mu\lambda}^{(\alpha)}(1) - \eta_{\lambda\mu}^{(\alpha)}(1)$. Let $u^{(\alpha)} = \{u_{\lambda\mu}^{(\alpha)}\}'_{\lambda,\mu\in I}$. $\delta u^{(\alpha)} \in Z^2(\boldsymbol{U}, \boldsymbol{Z})$. The map from F to $C^1(\boldsymbol{U}, \mathcal{O})$ defined by $f^{(\alpha)} \to u^{(\alpha)}$ is continuous. $B = \{\delta u^{(\alpha)}\}_{\alpha\in A}$ is a compact subset of $Z^2(\boldsymbol{U}, \boldsymbol{Z})$. By Lemma 7 there exists $a^{(\alpha)} = \{a_{\lambda\mu}^{(\alpha)}\}'_{\lambda,\mu\in I} \in C^1(\boldsymbol{U}, \boldsymbol{R})$, $\alpha \in A$, such that $\delta a^{(\alpha)} = \delta u^{(\alpha)}$ and the map from B to $C^1(\boldsymbol{U}, \boldsymbol{R})$ defined by $\delta u^{(\alpha)} \to a^{(\alpha)}$ is continuous. Let $v^{(\alpha)} = u^{(\alpha)} - a^{(\alpha)}$. Then $C = \{v^{(\alpha)}\}_{\alpha\in A}$ is a compact subset of $Z^1(\boldsymbol{U}, \mathcal{O})$.

Let $c_{\lambda\mu}^{(\alpha)} = \exp(-2\pi i a_{\lambda\mu}^{(\alpha)})$. Then

$$c_{\lambda\mu}^{(\alpha)} c_{\mu\nu}^{(\alpha)} c_{\nu\lambda}^{(\alpha)} = 1 \text{ for } U_{\lambda\mu\nu} \neq \emptyset. \tag{4}$$

Fix $U_{\lambda_0} \neq \emptyset$. Since M is connected, for every $U_\lambda \neq \emptyset$ we can find $\lambda_1, \cdots, \lambda_m \in I$ such that $\lambda_m = \lambda$ and $U_{\lambda_j} \cap U_{\lambda_{j+1}} \neq \emptyset$ for $0 \leqq j < m$. Let $d_\lambda^{(\alpha)} = \prod_{j=0}^{m-1} c_{\lambda_j\lambda_{j+1}}^{(\alpha)}$. Since $\pi_1(M) = 0$, (4) implies that $d_\lambda^{(\alpha)}$ is independent of the choice of $\lambda_1, \cdots, \lambda_{m-1}$. $d_\mu^{(\alpha)}(d_\lambda^{(\alpha)})^{-1} = c_{\lambda\mu}^{(\alpha)}$ for $U_{\lambda\mu} \neq \emptyset$.

We can choose a locally finite open covering $\boldsymbol{V} = \{V_\lambda\}_{\lambda\in I}$ of M such that

V_λ^- is a compact subset of U_λ, $\lambda \in I$. By Th.4 there exists $w_\lambda^{(\alpha)} = \{w_\lambda^{(\alpha)}\}'_{\lambda \in I}$ $\in C^0(\boldsymbol{V}, \mathcal{O})$ such that $\delta w'^{(\alpha)} = v^{(\alpha)}$ on $\boldsymbol{V}$ and the map from C to $C^0(\boldsymbol{V}, \mathcal{O})$ defined by $v^{(\alpha)} \rightarrow w^{(\alpha)}$ is continuous.

Define $g^{(\alpha)} \in \Gamma(M, \mathcal{O})$ by setting $g^{(\alpha)} = d_\lambda^{(\alpha)} h_\lambda^{(\alpha)} \exp(-2\pi i w_\lambda^{(\alpha)})$ on V_λ. $g^{(\alpha)}$ is well-defined, because $\exp(-2\pi i w_\lambda^{(\alpha)}) \exp(2\pi i w_\mu^{(\alpha)}) = \exp(2\pi i v_{\lambda\mu}^{(\alpha)}) = \exp(2\pi i u_{\lambda\mu}^{(\alpha)})$ $\exp(-2\pi i a_{\lambda\mu}^{(\alpha)}) = k_{\lambda\mu}^{(\alpha)} c_{\lambda\mu}^{(\alpha)} = h_\mu^{(\alpha)} (h_\lambda^{(\alpha)})^{-1} d_\mu^{(\alpha)} (d_\lambda^{(\alpha)})^{-1}$ on $V_{\lambda\mu}$. The map $\gamma: F \rightarrow \Gamma(M, \mathcal{O})$ defined by $\gamma(f^{(\alpha)}) = g^{(\alpha)}$ satisfies the requirement. q. e. d.

COROLLARY. (1) *holds for $M =$ the product of $\boldsymbol{C}^q$ and the unit ball of $\boldsymbol{C}^p$.*

PROOF. We can assume w. l. o. g. that $x_0 = 0$, because any point of M can be mapped to the origin by some biholomorphic map of M onto M. Let $I = \boldsymbol{N}$. For $\lambda \in I$ let $U_\lambda = \{(z, w) \in \boldsymbol{C}^q \times \boldsymbol{C}^p \,|\, \lambda^{-2}|z|^2 + (\lambda+1)^2\lambda^{-2}|w|^2 < 1\}$. U_λ is biholomorphic to a ball in $\boldsymbol{C}^{q+p}$ and is relatively compact in M. $\boldsymbol{U} = \{U_\lambda\}_{\lambda \in I}$ covers M. Let $K = \{0\}$. Then (1) to (5) of Def. 4 are satisfied with $F = N$. N is a small normal family of non-negative divisors on M. q. e. d.

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MATHEMATICS DEPARTMENT
UNIVERSITY OF NOTRE DAME
NOTRE DAME, INDIANA, U. S. A.