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AUTOMORPHISMS OF CROSSED PRODUCTS

HORST BEHNCKE*)

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Introduction. Some results concerning outer automorphisms of continuous factors have been known for some time. Notably for the hyperfinite factor Dixmier [1], Suzuki [2], Saito [3] and Blattner [4] showed that it has a large set of outer automorphisms. Dealing with the hyperfinite factor has the advantage that there are many distinct realizations for it, in each of which a particular class of groups is shown to be representable as groups of outer automorphisms. For non-hyperfinite continuous factors far fewer results are known. In fact the example by Kadison [5, 6] seems to be the only one in the published literature. A more systematic attempt to study automorphisms of W^*-algebras has been made by I. M. Singer [7]. He considered automorphisms of certain finite W^*-algebras, which are crossed products of a commutative W^*-algebra and a countable discrete group. In certain respects the present study of automorphisms of crossed products can be considered a sequel to Singer's paper. In fact in Section 2 we present generalizations of some of his results. Following this we give a complete description of the group of automorphisms of the crossed product of a factor $\mathfrak{A}$ and a countable discrete group G, which leave $\mathfrak{A}$ invariant. These results suggest to study automorphisms of crossed product $(\mathfrak{A}, G)$ which are combinations of automorphisms of $\mathfrak{A}$ and G. Criteria are given in Section 4 for such automorphisms to be outer. The remainder is devoted to the study of particular examples. Among other results we show that outer automorphisms of groups extend under very general conditions to outer automorphisms of the corresponding left rings. In Section 6 we study factors of type II and III, which have been introduced by von Neumann [8]. All these examples possess outer automorphisms. We further consider automorphisms of certain finite factors, which apparently have not been studied before. Our results lend further credibility to the hypothesis that continuous factors possess outer automorphisms in contradistinction to factors of type I.

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1. Crossed products have been introduced by Turumaru [9] and have been studied by Suzuki [10], Zeller-Meier [11], Leptin [12] and other authors. The construction of crossed products is analogous to that of semi-direct products of groups. Let $\mathfrak{A}$ be a W^*-algebra and G a countably infinite discrete group, which has a representation $G \to \widetilde{G}$ as a group of $*$-automorphisms of $\mathfrak{A}$. Elements of $\mathfrak{A}$ will be denoted by $a, b, \cdots$ and elements of G by $\alpha, \beta, \cdots$. e will denote the identity of G and $\widetilde{\alpha}a$ will stand for the image of $a \in \mathfrak{A}$ under the automorphism $\widetilde{\alpha}$, with $\alpha \in G$. Throughout this paper automorphism will always mean $*$-automorphism. The group of automorphisms of $\mathfrak{A}$ and G will be denoted by $\operatorname{Aut}\mathfrak{A}$ and $\operatorname{Aut}G$ respectively. Let φ be a normal $\widetilde{G}$-invariant state of $\mathfrak{A}$. For our construction we can assume without loss of generality that φ is faithful. The faithful normal state φ leads by the Gelfand-Segal construction to a faithful W^*-representation π of $\mathfrak{A}$ on a Hilbert space $\mathcal{K}$. $\pi(\mathfrak{A})$ has the cyclic and separating vector $\xi \in \mathcal{K}$ with

$$<\pi(a)\xi \mid \xi> = \varphi(a) \qquad \forall a \in \mathfrak{A}\,. \tag{1}$$

Since φ is $\widetilde{G}$-invariant we also have a unitary representation u of G on $\mathcal{K}$. This representation satisfies

$$u_\alpha \pi(a) u_\alpha^* = \pi(\widetilde{\alpha}a) \tag{2}$$

The representations π and u can be extended to $\mathcal{K} \otimes l^2(G)$ by

$$\begin{aligned} \Pi(a)\Sigma\zeta_\alpha \otimes \varepsilon_\alpha &= \Sigma\pi(a)\zeta_\alpha \otimes \varepsilon_\alpha \\ U_\beta \Sigma\zeta_\alpha \otimes \varepsilon_\alpha &= \Sigma u_\beta \zeta_\alpha \otimes \varepsilon_{\beta\alpha} \end{aligned} \tag{3}$$

Here $\zeta_\alpha \in \mathcal{K}$ and $\varepsilon_\alpha \in l^2(G)$ with $\varepsilon_\alpha(\beta) = \delta_{\alpha,\beta}$. The W^*-algebra on $\mathcal{K} \otimes l^2(G)$ generated by all operators $\Pi(a)$ and U_α is then called the crossed product of $\mathfrak{A}$ and G, induced by φ. If will be denoted by $(\mathfrak{A}, G)$. $(\mathfrak{A}, G)$ can also be considered the weak closure of the $*$-algerba $(\mathfrak{A}, G)_0$, which consists of all finite linear combinations of elements $\Pi(a)U_\alpha$. Multiplication and involution in $(\mathfrak{A}, G)$ are given by

$$\begin{aligned} \Pi(a)U_\alpha\Pi(b)U_\beta &= \Pi(a\widetilde{\alpha}b)U_{\alpha\beta} \\ (\Pi(a)U_\alpha)^* &= U_\alpha^{-1}\Pi(a^*) = \Pi(\widetilde{\alpha}^{-1}a^*)U_{\alpha^{-1}}\,. \end{aligned} \tag{4}$$

Since hardly any confusion is possible we shall identify the algebra $\pi(\mathfrak{A}) \otimes 1$ and $\mathfrak{A}$ in the sequel. As in [10] one shows:

LEMMA 1.1. *To each $A \in (\mathfrak{A}, G)$ there is associated a unique family $\{a_\alpha\} \subset \mathfrak{A}$ such that*

$$A\zeta \otimes \varepsilon_\beta = \Sigma \pi(a_\alpha) u_\alpha \zeta \otimes \varepsilon_{\alpha\beta} \tag{5}$$

and

$$\Sigma \varphi(a_\alpha^* a_\alpha) < \infty . \tag{6}$$

This correspondence will be denoted by $A \sim (a_\alpha)$ and we call the set of all $\alpha \in G$ with $\varphi(a_\alpha^* a_\alpha) > 0$ the G-support of A, for short G-supp A. It is easy to see that for $A \sim (a_\alpha)$ and $B \sim (b_\alpha)$ we have:

$$\begin{aligned} &A + \lambda B \sim (a_\alpha + \lambda b_\alpha), \quad A^* \sim (\widetilde{\alpha}\, a_{\alpha^{-1}}^*) \\ &AB \sim (\Sigma\, a_\beta \widetilde{\beta}\, b_{\beta^{-1}\alpha}) \end{aligned} \tag{7}$$

The sum in the last correspondence is taken in the sense of strong convergence in $\mathcal{K}$. It is easy to see that the vector $\xi \otimes \varepsilon_e$ is cyclic and separating for $(\mathfrak{A}, G)$. We further have for $A \sim (a_\alpha)$

$$|A| \geqq |a_\alpha| \qquad \forall \alpha \in G .$$

If $\mathfrak{A}$ is a W^*-algebra $\mathfrak{A}_u$ will denote the group of unitary elements of $\mathfrak{A}$ and $\mathcal{Z}(\mathfrak{A})$ will mean the center of $\mathfrak{A}$. Since we are dealing with automorphisms of crossed products it is advantageous to introduce a particular notation for automorphisms. Assume $S \in \operatorname{Aut}(\mathfrak{A}, G)$ is a spatial (inner) automorphism, then $\widetilde{S}$ will denote the unitary operator (a suitable element of $(\mathfrak{A}, G)$) which induces S, i.e.

$$\widetilde{S} A \widetilde{S}^* = S(A) \tag{8}$$

2. In [7] Singer considers automorphisms of crossed products $(\mathfrak{A}, G)$ with $\mathfrak{A} = \mathcal{L}^\infty(X, \Sigma, \mu)$, where (X, Σ, μ) is a finite separable nonatomic measure space on which G acts as a group of measure-preserving, free and ergodic automorphisms. However it is apparent that his proofs carry over also to the case when (X, Σ, μ) is σ-finite and where G leaves μ only quasi-invariant. Thus his results can be extended also to a certain class of infinite factors. Since the proofs require only minor modifications we shall not state these extended results here. Instead we shall try to apply some of his methods to the noncommutative case. Before we state the generalization of his key lemma, let us begin with a few general remarks.

Let $T \in (\mathfrak{A}, G)_u$, the unitary group of $(\mathfrak{A}, G)$, such that $T\mathfrak{A}T^* = \mathfrak{A}$ and

$T^*\mathfrak{A}T=\mathfrak{A}$, i.e. T induces an automorphism τ on $\mathfrak{A}$, then

$$T\Pi(a)T^* = \Pi(\tau a), T^*\Pi(a)T = \Pi(\tau^{-1}a) \qquad \forall a \in \mathfrak{A}\,. \tag{9}$$

We assume that $T\sim(b_\beta)$. After rewriting (9) as

$$T\Pi(a) = \Pi(\tau a)\,T, T^*\Pi(a) = \Pi(\tau^{-1}a)T^*$$

and applying Lemma 1.1 we obtain

$$b_\beta\widetilde{\beta}a = \tau(a)b_\beta,\ \ \widetilde{\beta}^{-1}(b_\beta^*\,a) = \tau^{-1}(a)\widetilde{\beta}^{-1}(b_\beta^*) \tag{10}$$

A simple computation finally shows that (10) is only possible if $b_\beta b_\beta^*$ and $b_\beta^* b_\beta$ belong to the center $\mathcal{Z}(\mathfrak{A})$ of $\mathfrak{A}$. At this point it is advantageous to introduce the polar decomposition of b_β.

$$b_\beta = v_\beta|b_\beta|\,. \tag{11}$$

Since $b_\beta b_\beta^*, b_\beta^* b_\beta \in \mathcal{Z}(\mathfrak{A})$, v_β satisfies

$$v_\beta v_\beta^* = v_\beta^*\,v_\beta = E_\beta \in \mathcal{Z}(\mathfrak{A})\,. \tag{12}$$

Thus the v_β are even partial unitary operators. This allows us to rewrite (10) as

$$v_\beta\widetilde{\beta}(a)v_\beta^* = \tau(a)E_\beta, \widetilde{\beta}^{-1}(v_\beta^*)\widetilde{\beta}^{-1}(a)\widetilde{\beta}^{-1}(v_\beta) = \tau^{-1}(a)\widetilde{\beta}^{-1}(E_\beta) \tag{13}$$

So far we have been very general and we have not yet used any particular hypotheses on G or $\mathfrak{A}$.

LEMMA 2.1. *Let $\mathfrak{A}, G, \varphi$ be as in Section 1 and assume $\widetilde{G}$ acts freely on the center $\mathcal{Z}(\mathfrak{A})$ of $\mathfrak{A}$. Then any $T \in (\mathfrak{A}, G)_u$, $T\sim(b_\beta)$, which satisfies (9) has a unique decomposition*

$$T = \Pi(b)T', \qquad b \in \mathfrak{A}_u \tag{14}$$

and

$$T' \sim (E_\beta),\ b_\beta b_\beta^* = b_\beta^*\,b_\beta = E_\beta \in \mathcal{Z}(\mathfrak{A}) \tag{15}$$

$$E_\beta E_\gamma = \delta_{\beta,\gamma}E_\beta,\ \widetilde{\beta}^{-1}(E_\beta)\,\widetilde{\gamma}^{-1}(E_\gamma) = \delta_{\beta,\gamma}\widetilde{\beta}^{-1}(E_\beta) \tag{16}$$

$$\Sigma E_\beta = \Sigma\widetilde{\beta}^{-1}E_\beta = 1\,. \tag{17}$$

Conversely any such $T' \in (\mathfrak{A}, G)$ is a unitary operator which leaves $\mathfrak{A}$ invariant

$$T'\Pi(a)T'^{*} = \Pi(\Sigma E_{\beta}\widetilde{\beta}(a)). \tag{18}$$

PROOF. As before let $b_{\beta} = v_{\beta}|b_{\beta}|$ be the polar decomposition of b_{β}. (13) shows that for any $a \in \mathcal{Z}(\mathfrak{A})$ with support contained in $\widetilde{\beta}^{-1}(E_{\beta})\widetilde{\gamma}^{-1}(E_{\gamma})$ we have $\widetilde{\beta}a = \widetilde{\gamma}a$. Since $\widetilde{G}$ operates freely on $\mathcal{Z}(\mathfrak{A})$ this is only possible if the $\widetilde{\beta}^{-1}E_{\beta}$ are orthogonal central projections. The other half of (16) is shown similarly. Since T is unitary we have $1 = \Sigma b_{\beta}b_{\beta}^{*} = \Sigma E_{\beta}|b_{\beta}|^{2}$. Since all E_{β} are orthogonal this implies $|b_{\beta}| = E_{\beta}, b_{\beta} = v_{\beta}$ and (17). Now set $b = \Sigma b_{\beta}$, then b is obviously unitary and $T' = \Pi(b^{*})T \sim (b^{*}b_{\beta}) = (E_{\beta})$. The converse can be shown easily by considering

$$T'\Pi(a)T'^{*}\zeta \otimes \varepsilon_{e} = \Pi(\Sigma E_{\beta}\widetilde{\beta}a)\zeta \otimes \varepsilon_{e}.$$

COROLLARY. *$\mathfrak{A}' \cap (\mathfrak{A}, G) = \mathcal{Z}(\mathfrak{A})$ and $(\mathfrak{A}, G)$ is a factor if $\widetilde{G}$ is a group of ergodic automorphisms of $\mathcal{Z}(\mathfrak{A})$.*

PROOF. Let $T \in \mathfrak{A}' \cap (\mathfrak{A}, G)_{u}$ and assume $T \sim (b_{\beta})$. Then (13) shows that for $a \in \mathcal{Z}(\mathfrak{A})$ we have $E_{\beta}\widetilde{\beta}a = aE_{\beta}$. Since $\widetilde{G}$ operates freely on $\mathcal{Z}(\mathfrak{A})$, this is only possible if $E_{\beta} = 0$ for $\beta \neq e$. Thus $T = \Pi(b)$ and obviously $b \in \mathcal{Z}(\mathfrak{A})$. If furthermore $\Pi(b) \in \mathcal{Z}(\mathfrak{A}, G)$ then $\Pi(b) = U_{\alpha}\Pi(b)U_{\alpha}^{*} = \Pi(\widetilde{\alpha}b)$, and this shows the remainder.

As in Singer's paper let $\mathcal{S}$ be the group of all automorphisms of $(\mathfrak{A}, G)$ which leave $\mathfrak{A}$ invariant. $\mathfrak{K}$ will denote the group of all automorphisms which leave $\mathfrak{A}$ elementwise invariant.

THEOREM 2.1. *Let $\mathfrak{A}, G, \varphi$ be as in Section 1 and assume that $\widetilde{G}$ operates freely and ergodically on $\mathfrak{A}$, then any $S \in \mathfrak{K}$ satisfies:*

$$S\Pi(a) = \Pi(a), SU_{\alpha} = \Pi(a_{\alpha})U_{\alpha} \tag{19}$$

with

$$a_{\alpha} \in \mathcal{Z}(\mathfrak{A})_{u} \quad \textit{and} \quad a_{\alpha\beta} = a_{\alpha}\widetilde{\alpha}a_{\beta}. \tag{20}$$

PROOF. The equation

$$(SU_{\alpha})\Pi(a)(SU_{\alpha})^{*} = S(U_{\alpha}\Pi(a)U_{\alpha}^{*}) = S\Pi(\widetilde{\alpha}a) = U_{\alpha}\Pi(a)U_{\alpha}^{*}$$

implies that $U_{\alpha}^{-1}S(U_{\alpha})$ commutes elementwise with $\mathfrak{A}$. The above corollary shows then that $U_\alpha^{-1}S(U_\alpha) = \Pi(\tilde{\alpha}^{-1}a_\alpha)$. The remainder is a consequence of the multiplicativity of S.

COROLLARY. *$S \in \mathfrak{K}$ is an inner automorphism if $a_\alpha = b\,\tilde{\alpha}\,b^*$ for some $b \in \mathcal{Z}(\mathfrak{A})_u$.*

PROOF. Apply the corollary of Lemma 2.1.

THEOREM 2.2. *Any $S \in \mathcal{S}$ satisfies*

$$S\Pi(a) = \Pi(\sigma a) \qquad \sigma \in \operatorname{Aut}\mathfrak{A} \tag{21}$$

$$SU_\alpha = \Pi(b^\alpha)B^\alpha \qquad b^\alpha \in \mathfrak{A}_u \tag{22}$$

$$B^\alpha B^\beta = B^{\alpha\beta}, \Pi(b^{\alpha\beta}) = \Pi(b^\alpha)B^\alpha\Pi(b^\beta)B^{\alpha*} \tag{23}$$

$$B^\alpha \sim (E_\beta^\alpha) \quad \textit{and } B^\alpha \textit{ satisfies } (15)-(17) \tag{24}$$

$$\sigma\tilde{\alpha}\sigma^{-1}a = \Sigma E_\beta^\alpha b^\alpha \tilde{\beta}(a) b^{\alpha*} \tag{25}$$

$$E_\beta^\alpha E_\beta^\gamma = \delta_{\alpha,\gamma}E_\beta^\alpha \tag{26}$$

Further S is spatial, whenever σ is spatial, in particular $\mathfrak{K}$ is a group of spatial automorphisms.

PROOF. It is easy to realize that SU_α is a unitary operator in $(\mathfrak{A}, G)$ which leaves $\mathfrak{A}$ invariant. The decomposition (22) as well as (24) are therefore consequences of Lemma 2.1. The multiplicativity of S implies (23), whereas (21) holds by definition of $\mathcal{S}$. By definition (21) and by (4) we have

$$\begin{aligned}\Pi(\sigma\tilde{\alpha}\sigma^{-1}a) &= S[U_\alpha\Pi(\sigma^{-1}a)U_\alpha^*] = S(U_\alpha)\Pi(a)S(U_\alpha)^* \\ &= \Pi(b^\alpha)\Pi(\Sigma_\beta E_\beta^\alpha\tilde{\beta}[a])\Pi(b^\alpha)^*\,.\end{aligned}$$

This shows (25).

This equation means that restricted to $\tilde{\beta}^{-1}(E_\beta^\alpha)$ the automorphism $\sigma\tilde{\alpha}\sigma^{-1}$ looks like $b^\alpha\tilde{\beta}[\cdots]b^{\alpha*}$. This shows that $\sigma\tilde{\alpha}\sigma^{-1}$ and $\sigma\tilde{\gamma}\sigma^{-1}$ act alike on all central elements with support in $\tilde{\beta}^{-1}[E_\beta^\alpha E_\beta^\gamma]$. Since $\tilde{G}$ operates freely on $\mathcal{Z}(\mathfrak{A})$ also $\sigma\tilde{G}\sigma^{-1}$ has this property. This shows (26). Assume $\sigma \in \operatorname{Aut}\mathfrak{A}$ is induced by u_σ on $\mathcal{K}$, then define $\tilde{S}\xi\otimes\varepsilon_\alpha = \Pi(b^\alpha)B^\alpha u_\sigma u_\alpha^{-1}\xi\otimes\varepsilon_e$. Using (21)–(26) it is easy to see that $\tilde{S}$ is a unitary operator on $\mathcal{K}\otimes l^2(G)$, which satisfies (8).

The inner automorphisms of $\mathcal{S}$ have already been described in the lemma. Again we can single out a subgroup $\mathcal{S}_1$ of $\mathcal{S}$, which consists of all $S \in \mathcal{S}$ with

$S(U_\alpha) = B^\alpha \sim (E^\alpha_\beta)$. However only if $\mathfrak{A}$ is commutative can we assert that $\mathcal{S}$ is the semi-direct product of $\mathfrak{R}$ and $\mathcal{S}_1$. This can already be seen for inner automorphisms.

3. In the last section the relevant conditions were given on the center of $\mathfrak{A}$. Now we shall study the opposite case. In particular we shall assume $\mathfrak{A}$ to be a factor and G to be a group of outer automorphisms. The latter condition seems to be the most suitable replacement for the free action of G on $\mathcal{Z}(\mathfrak{A})$, which we had assumed previously.

LEMMA 3.1. *Let $\mathfrak{A}$ be a factor and $\widetilde{G}$ a countable discrete group of outer automorphisms of $\mathfrak{A}$. Then any $T \in (\mathfrak{A}, G)_u$ which satisfies* (9) *is of the form $T = \Pi(b) U_\beta$ with $b \in \mathfrak{A}_u$.*

PROOF. Assume $T \sim (b_\beta)$; since $\mathfrak{A}$ is a factor we either have in (13) $E_\beta = 0$ or $E_\beta = 1$. Assume $E_\beta = 1 = E_\gamma$ for $\gamma \neq \beta$, then (13) implies

$$\widetilde{\beta}\widetilde{\gamma}^{-1}(a) = v_\beta^* v_\gamma a v_\gamma^* v_\beta \qquad \text{for all } a \in \mathfrak{A}\,.$$

This however is impossible, because $\widetilde{G}$ is a group of outer automorphisms. Thus there exist only one $\beta \in G$ with $E_\beta = 1$ and $T = \Pi(b) U_\beta$.

COROLLARY. *$\mathfrak{A}' \cap (\mathfrak{A}, G) = \{\lambda 1\}$ and $(\mathfrak{A}, G)$ is a factor.*

PROOF. $\Pi(b) U_\beta \Pi(a) U_\beta^* \Pi(b^*) = \Pi(a)$ implies $\beta = e$ and $b \in \mathcal{Z}(\mathfrak{A}) = \{\lambda 1\}$.

This corollary generalizes a result by Suzuki [10], who proved it for the case where $\mathfrak{A}$ is a finite factor and φ its trace.

Again let $\mathcal{S}$ be the group of all automorphisms of $(\mathfrak{A}, G)$, leaving $\mathfrak{A}$ invariant, and let again $\mathfrak{R}$ denote the fixgroup of $\mathfrak{A}$.

THEOREM 3.1. *Any $S \in \mathfrak{R}$ satisfies*

$$S\Pi(a) = \Pi(a),\ SU_\alpha = \lambda_\alpha U_\alpha\,, \qquad \lambda_\alpha \textit{ complex numbers} \tag{27}$$

$$|\lambda_\alpha| = 1,\ \lambda_\alpha \lambda_\beta = \lambda_{\alpha\beta} \tag{28}$$

$$\mathfrak{R} \cong (G/CG)^*\,. \tag{29}$$

Here CG denotes the commutator subgroup of G, and $(G/CG)^*$ stands for the dual group of the abelian group G/CG. Moreover $\mathfrak{R}$ is a group of spatial outer automorphisms.

PROOF. As before we use the fact that $U_\alpha^{-1}S(U_\alpha)$ belongs to the relative commutant of $\mathfrak{A}$. Lemma 3.1 shows then (27), and the multiplicativity of S yields (28). This formula also shows that any $S \in \mathfrak{K}$ thus induces a homomorphism Λ_s of G into the circle group. Since the circle group is abelian CG will be in the kernel of any such homomorphism. With these remarks (29) is now nearly obvious. We only have to remember that for S_1, $S_2 \in \mathfrak{K}$ the induced homomorphism is $\alpha \to \lambda_{1\alpha}\lambda_{2\alpha}$, when S_1 corresponds to $\lambda_{1\alpha}$ and S_2 to $\lambda_{2\alpha}$. To see that these automorphisms are spatial we define the unitary operator $\widetilde{S}$ on $\mathcal{K}\otimes l^2(G)$ by $\widetilde{S}\Sigma\xi_\alpha\otimes\varepsilon_\alpha = \Sigma\lambda_\alpha\xi_\alpha\otimes\varepsilon_\alpha$. Then $\widetilde{S}\Pi(a)U_\alpha\widetilde{S}^* = \Pi(a)\lambda_\alpha U_\alpha$ is easy to check. The corollary to Lemma 3.1 shows that $\mathfrak{K}$ is a group of outer automorphisms.

THEOREM 3.2. *Any $S \in \mathcal{S}$ satisfies*

$$S(\Pi(a)) = \Pi(\sigma[a]),\ S(U_\alpha) = \Pi(a_\alpha)U_{s(\alpha)} \tag{30}$$

with $s \in \operatorname{Aut} G$, $\sigma \in \operatorname{Aut}\mathfrak{A}$, $a_\alpha \in \mathfrak{A}_u$ *and*

$$a_{\alpha\beta} = a_\alpha s(\alpha)[a_\beta],\ a_\alpha s(\alpha)\cdot\sigma[a]a_\alpha^* = \sigma\cdot\alpha[a] \tag{31}$$
$$\forall a \in \mathfrak{A}\,.$$

Conversely any such S defines an automorphism of $(\mathfrak{A}, G)_0$. If φ is moreover σ-invariant, this S extends to a spatial automorphism of $(\mathfrak{A}, G)$.

PROOF. We proceed as in Theorem 2.2. Since S leaves $\mathfrak{A}$ invariant and because the $S(U_\alpha)$ leave $\mathfrak{A}$ invariant (30) follows easily. Equation (31) is a consequence of the multiplicativity of S. Conversely let $s \in \operatorname{Aut} G$, $\sigma \in \operatorname{Aut}\mathfrak{A}$ and $\{a_\alpha\} \subset \mathfrak{A}_u$ be given such that (31) holds. Then it is easy to see that $S(\Pi(a)U_\alpha) = \Pi(\sigma(a))\Pi(a_\alpha)U_{s(\alpha)}$ defines an automorphism of $(\mathfrak{A}, G)_0$. If φ is even σ-invariant, then there exists a unitary operator u_σ on $\mathcal{K}$ with $u_\sigma\pi(a)\xi = \pi(\sigma[a])\xi$ and $u_\sigma\pi(a)u_\sigma^* = \pi(\sigma(a))$. In this case we define an operator $\widetilde{S}$ on $\mathcal{K}\otimes l^2(G)$ by $\widetilde{S}\Sigma\zeta_\alpha\otimes\varepsilon_\alpha = \Sigma\pi(a_\alpha)u_{s(\alpha)}u_\sigma u_\alpha^{-1}\zeta_\alpha\otimes\varepsilon_{s(\alpha)}$. A tedious but simple computation shows then that $\widetilde{S}$ is unitary and that $\widetilde{S}\Pi(a)U_\alpha\widetilde{S}^* = S(\Pi(a)U_\alpha)$.

We should add here that as a consequence of Lemma 3.1 all inner automorphisms of $\mathcal{S}$ are induced by elements of the form $\Pi(b)U_\beta$. This shows, in particular, that an inner automorphism of $\mathcal{S}$ will lead to an inner automorphism $s \in \operatorname{Aut} G$. Theorem 3.1 shows that $\mathfrak{K}$ does not only leave $\mathfrak{A}$ elementwise invariant, but also the much larger subalgebra $(\mathfrak{A}, CG)$, which consists of all elements $A \in (\mathfrak{A}, G)$ with G-supp $A \subset CG$. This indicates that it will be impossible in general to prove Galois-like theorems in this connection.

4. Our results in the previous section suggest to consider automorphisms of $(\mathfrak{A}, G)$, which are combinations of automorphisms of $\mathfrak{A}$ and of automorphisms of G. Of course we can no longer hope to describe completely the group $\mathcal{S}$ of arbitrary crossed products $(\mathfrak{A}, G)$, but a number of useful results can still be derived for certain subgroups of $\mathcal{S}$.

THEOREM 4.1. *Let $\mathfrak{A}, G$ and φ be as in Section I and assume we are given $\sigma \in \operatorname{Aut} \mathfrak{A}$, $s \in \operatorname{Aut} G$ and $\{a_\alpha\} \subset \mathfrak{A}_u$ such that*

$$a_{\alpha\beta} = a_\alpha s(\alpha)[a_\beta], \quad a_\alpha s(\alpha) \cdot \sigma[a] a_\alpha^* = \sigma \cdot \alpha[a] \qquad \forall a \in \mathfrak{A} \tag{32}$$

then $S(\Pi(a)U_\alpha) = \Pi(\sigma(a))\Pi(a_\alpha)U_{s(\alpha)}$ defines an automorphism of $(\mathfrak{A}, G)_0$. This automorphism extends to a spatial automorphism of $(\mathfrak{A}, G)$ if φ is even σ-invariant.

The proof follows along the lines of the proof of Theorem 3.2 and is therefore omitted.

It is obvious that all automorphisms $S \in \mathcal{S}$ described by (32) form a group, which we call $\mathcal{S}_2$. A very simple method of satisfying (32) is given by the following corollary.

COROLLARY. *Let $\{\lambda_\alpha\}_{\alpha \in G}$ be a family of complex numbers of absolute value 1, which satisfy $\lambda_{\alpha\beta} = \lambda_\alpha \lambda_\beta$, then $\tilde{S} \sum \zeta_\alpha \otimes \varepsilon_\alpha = \sum \lambda_\alpha \zeta_\alpha \otimes \varepsilon_\alpha$ defines a spatial automorphism of $(\mathfrak{A}, G)$ with $\tilde{S}\Pi(a)U_\alpha \tilde{S}^{-1} = \lambda_\alpha \Pi(a)U_\alpha$. The group of all such automorphisms is isomorphic to $(G/CG)^*$.*

This corollary is shown as Theorem 3.1.

It is now of interest to determine, which automorphisms of $\mathcal{S}_2$ are actually inner automorphisms of $(\mathfrak{A}, G)$.

THEOREM 4.2. *Let $\mathfrak{A}, G$ and φ be as in Section 1 and let $S \in \mathcal{S}_2$ be an automorphism, which is described by (32). If S is an inner automorphism induced by $\tilde{S} \in (\mathfrak{A}, G)$, then $\beta \in G\text{-}\operatorname{supp} \tilde{S}$ implies*

$$E_\beta = \{s(\alpha)\beta\alpha^{-1} \mid \alpha \in G\} \text{ is a finite set.} \tag{33}$$

PROOF. We have by assumption $\tilde{S}U_\alpha\tilde{S}^* = \Pi(a_\alpha)U_{s(\alpha)}$ or $\tilde{S}U_\alpha = \Pi(a_\alpha)U_{s(\alpha)}\tilde{S}$. Assume $S \sim (b_\beta)$ and use Lemma 1.1, then $b_{s(\alpha)\beta\alpha^{-1}} = a_\alpha \widetilde{s(\alpha)}[b_\beta]$. However, since $\tilde{S}$ is unitary, we must have

$$1 = |\widetilde{S}\zeta \otimes \varepsilon_e|^2 = \Sigma\,\varphi(b_\beta^* b_\beta) \geqq \Sigma_{s(\alpha)\beta\alpha^{-1} \in E_\beta}\, \varphi(\widetilde{s(\alpha)}(b_\beta)^* a_\alpha^* a_\alpha \widetilde{s(\alpha)}(b_\beta))$$
$$= \text{card}\, E_\beta \cdot \varphi(b_\beta^* b_\beta)\,.$$

This proves the theorem.

We should remark here that only the condition $S(U_\alpha) = \Pi(a_\alpha)U_{s(\alpha)}$ was used. Otherwise the theorem is quite general.

COROLLARY 1. *Let G be an R-group and $S \in \mathcal{S}$ as described by* (32), *then S is an outer automorphism of* $(\mathfrak{A}, G)$, *whenever s is an outer automorphism of G.*

PROOF. We assume that S is an inner automorphism of $(\mathfrak{A}, G)$. Let $\beta \in G$-supp $\widetilde{S}$, then by Theorem 4.2 E_β must be finite. This implies in particular that for all $\alpha \in G$ the set $\{s(\alpha)^n \beta\alpha^{-n} \mid n=0, \pm 1, \cdots\}$ is finite. Thus there exists a positive integer k with $s(\alpha)^{+k}\beta\alpha^{-k} = \beta$ or $s(\alpha)^k = \beta\alpha^k\beta^{-1} = (\beta\alpha\beta^{-1})^k$. Since G is an R-group $s(\alpha) = \beta\alpha\beta^{-1}$, and s is an inner automorphism of G.

COROLLARY 2. *Let G be a group with no normal subgroups of finite index and assume s is an outer automorphism of G, then S as described* (32) *is an outer automorphism of* $(\mathfrak{A}, G)$.

PROOF. We assume that S is an inner automorphism of $(\mathfrak{A}, G)$. For $\beta \in G$-supp $\widetilde{S}$ we know that E_β is finite (33), $E_\beta = \{\beta, s(\alpha_2)\beta\alpha_2^{-1}, \cdots, s(\alpha_n)\beta\alpha_n^{-1}\}$. This shows that for any $\alpha \in G$ there exists an α_i $i=1, \cdots, n$ such that $s(\alpha)\beta\alpha^{-1} = s(\alpha_i)\beta\alpha_i^{-1}$ or $\alpha_i^{-1}\alpha \in N_\beta = \{\gamma \mid s(\gamma)\beta\gamma^{-1} = \beta\}$. Thus the subgroup N_β is of finite index. Therefore also the subgroup $N = \bigcap_{i=1}^{n} N_{s(\alpha_i)\beta\alpha_i^{-1}} = \bigcap_{\alpha \in G} N_{s(\alpha)\beta\alpha^{-1}}$ is of finite index. However it is easy to see that N is a normal subgroup of G. By assumption $N = G = N_\beta$, and this implies $s(\gamma) = \beta\gamma\beta^{-1}$ $\forall \gamma \in G$.

COROLLARY 3. *Let G be an R-group or a group with no normal subgroups of finite index and assume the center of G is trivial, then any inner automorphism S as described by* (32) *is of the form* $\widetilde{S} = \Pi(b)U_\beta$.

PROOF. In Corollary 1 and 2 we had seen that S can only be inner on $(\mathfrak{A}, G)$, if it extends the inner automorphism s of G. By modifying S we can assume that $s(\alpha) = \alpha$ $\forall \alpha \in G$. Then for $\beta \in G$-supp $\widetilde{S}$ we must have that $E_\beta = \{\alpha\beta\alpha^{-1} \mid \alpha \in G\}$ is finite. Since G is an R-group or has no normal subgroups of finite index β must lie in the center of G, which by assumption is e.

COROLLARY 4. *Let S be an inner automorphism of* $(\mathfrak{A}, G)$ *as described by* (32), *then* G-supp $\widetilde{S}$ *lies in a coset of* G_0. G_0 *denotes the normal subgroup of* G *consisting of all elements in* G *with finite conjugacy classes.*

PROOF. Let $\alpha \in G$-supp $\widetilde{S}$ and modify S by U_α^{-1}, i.e. instead of the automorphism S we consider the automorphism induced by $U_\alpha^{-1}\widetilde{S} = \widetilde{S}'$. Thus we can assume without loss of generality that $e \in G$-supp $\widetilde{S}$. Let also $\beta \in G$-supp $\widetilde{S}$, then we have for the conjugacy class C_β of β $C_\beta = \{\alpha\beta\alpha^{-1} = [\alpha s(\alpha)^{-1}][s(\alpha)\beta\alpha^{-1}] \mid \alpha \in G\} \subset E_e^{-1} E_\beta$. Since both E_e and E_β are finite by (33), C_β must be finite.

This corollary shows in particular that if all nontrivial conjugacy classes in G are infinite any inner automorphism of $\mathcal{S}_2$ is induced by some $\Pi(b)U_\beta$. It is now easy to prove related results along similar lines as above. However we want to study now means of satisfying the conditions in (32).

LEMMA 4.1. *Let* $\mathfrak{A}$ *be a* W^*-*algebra and let* $\widetilde{H}$ *be a group of automorphisms of* $\mathfrak{A}$, *such that the normal faithful state* φ *is* $\widetilde{H}$-*invariant. Let* $\widetilde{G}$ *be a normal subgroup of* $\widetilde{H}$. *Then any* $\widetilde{\sigma} \in \widetilde{H}$ *extends to a spatial automorphism* S *of* $(\mathfrak{A}, G)$.

PROOF. For $\widetilde{\sigma} \in \widetilde{H}$ define for all $\widetilde{\alpha} \in \widetilde{G}$ $\widetilde{s(\alpha)} = \widetilde{\sigma}\,\widetilde{\alpha}\,\widetilde{\sigma}^{-1}$.

Then $\widetilde{\sigma} \in \operatorname{Aut}\mathfrak{A}$, $s \in \operatorname{Aut} G$ and for all $a \in \mathfrak{A}$ we have $\widetilde{\sigma} \cdot \widetilde{\alpha}[a] = \widetilde{\sigma}\,\widetilde{\alpha}\,\widetilde{\sigma}^{-1}\widetilde{\sigma}[a] = \widetilde{s(\alpha)}\,\widetilde{\sigma}[a]$, which shows (32). Thus $S[\Pi(a)U_\alpha] = \Pi(\widetilde{\sigma}[a])U_{\sigma\alpha\sigma^{-1}}$ defines a spatial automorphism of $(\mathfrak{A}, G)$, which we call the extension of $\widetilde{\sigma} \in \operatorname{Aut}\mathfrak{A}$.

If $\widetilde{H}$ is a semidirect product, we can obtain an even stronger result.

THEOREM 4.3. *Let* $\mathfrak{A}$ *be a* W^*-*algebra and assume the semi-direct product* $\widetilde{G} \circledS \widetilde{H}$ *of the countable discrete groups* $\widetilde{G}$ *and* $\widetilde{H}$ *acts on* $\mathfrak{A}$ *as a group of automorphisms such that the normal faithful state* φ *is* $\widetilde{G} \circledS \widetilde{H}$ *invariant. Then* $\widetilde{H}$ *extends to a group* $S(H)$ *of spatial automorphisms of* $(\mathfrak{A}, G)$ *and* $(\mathfrak{A}, G \circledS H)$ *and* $((\mathfrak{A}, G), S(H))$ *are spatially isomorphic.*

PROOF. The semi-direct product $\widetilde{G} \circledS \widetilde{H}$ is the system of all pairs $(\widetilde{\sigma}, \widetilde{\alpha})$ with $\widetilde{\sigma} \in \widetilde{H}$ and $\widetilde{\alpha} \in \widetilde{G}$. Multiplication is given by $(\widetilde{\sigma}, \widetilde{\alpha})(\widetilde{\rho}, \widetilde{\beta}) = (\widetilde{\sigma}\widetilde{\rho}, \widetilde{\alpha}\,\widetilde{s(\beta)})$, where $s \in \operatorname{Aut}\widetilde{G}$ is associated to $\widetilde{\sigma} \in \widetilde{H}$. For $\widetilde{\sigma} \in \widetilde{H}$ we therefore define its extension as

$$S_\sigma(\Pi(a)U_\alpha) = \Pi(\widetilde{\sigma}[a])U_{s(\alpha)}$$

Since φ is $\widetilde{H}$-invariant, S_σ is even spatial on $\mathcal{K}\otimes l^2(G)$, given by

$$\widetilde{S}_\sigma \Sigma \zeta_\alpha \otimes \varepsilon_\alpha = \Sigma u_{s(\alpha)} u_\sigma u_\alpha^{-1} \zeta_\alpha \otimes \varepsilon_{s(\alpha)} .$$

To see the remainder we define an operator U from $\mathcal{K}\otimes l^2(G)\otimes l^2(S(H))$ onto $\mathcal{K}\otimes l^2(G \circledS H)$ by $U\Sigma[\zeta_{\alpha,\sigma}\otimes \varepsilon_\alpha]\otimes \varepsilon_S = \Sigma \zeta_{\alpha,\sigma}\otimes \varepsilon_{(\sigma,\alpha)}$. U is obviously unitary, and a simple computation shows

$$U U_{S_\sigma} = U_{(\sigma,e)} U, \quad U U_\alpha = U_{(e,\alpha)} U$$

and

$$U \Pi(a) = \Pi(a) U$$

where

$$U_{S_\sigma} \Sigma[\zeta_{\alpha,\sigma'}\otimes \varepsilon_\alpha]\otimes \varepsilon_{S_{\sigma'}} = \Sigma \widetilde{S}_\sigma[\zeta_{\alpha,\sigma'}\otimes \varepsilon_\alpha]\otimes \varepsilon_{S_\sigma S_{\sigma'}} .$$

This proves the theorem.

DEFINITION. A group of automorphisms $\widetilde{G}$ of a W^*-algebra $\mathfrak{A}$ acts ergodically, if it leaves only multiples of the identity invariant.

THEOREM 4.4. *Let $\mathfrak{A}, G, H$ and φ be as in Lemma* 4.1. *A subgroup $S(H')$ of $S(H)$ acts ergodically on $(\mathfrak{A}, G)$, if $\widetilde{H}'$ acts ergodically on $\mathfrak{A}$ and if all nontrivial H' orbits in G are infinite. These two conditions are also necessary. Here $S(H)$ denotes the extension of H to a group of automorphisms of $(\mathfrak{A}, G)$ as given in Lemma* 4.1.

PROOF. Assume $A \in (\mathfrak{A}, G)$ is $S(H')$ invariant and let $A \sim (a_\alpha)$. Then the vector $A\xi \otimes \varepsilon_e = \Sigma \Pi(a_\alpha)\xi \otimes \varepsilon_\alpha$ is invariant under $S(H')$ too, i.e.

$$\Sigma \Pi(a_\alpha)\xi \otimes \varepsilon_\alpha = \Sigma \Pi(\sigma[a_\alpha])\xi \otimes \varepsilon_{\sigma\alpha\sigma^{-1}} \quad \forall \sigma \in \widetilde{H}'$$

Since all nontrivial H' orbits in G are infinite, we must have $A\xi\otimes\varepsilon_e = \Pi(a)\xi\otimes\varepsilon_e$ or $A = \Pi(a) = \Pi(\sigma[a])$. This finally implies $A = \lambda 1$, because $\widetilde{H}'$ acts ergodically on $\mathfrak{A}$. If conversely $\widetilde{H}'$ does not act ergodically on $\mathfrak{A}$, then there exists an $a \in \mathfrak{A}$ with $a = \widetilde{\sigma}[a]$ $\forall \widetilde{\sigma} \in \widetilde{H}'$. Then $\Pi(a)$ is $\widetilde{H}'$ invariant. If there exists a finite H' orbit $\mathcal{O}$ in G, $\mathcal{O} = \{\sigma\alpha\sigma^{-1} | \sigma \in H'\}$, then $\sum_{\gamma\in\mathcal{O}} U_\gamma$ is $S(H')$ invariant. In particular we shall say that an automorphism $\widetilde{\alpha}$ acts ergodically on $\mathfrak{A}$ if the group $\{\widetilde{\alpha}^n\}$ acts ergodically on $\mathfrak{A}$.

5. This section is devoted to application of our results in Section 4 to the left regular representation of countable discrete groups. The left regular representation, or the left ring, of a countable discrete group G, can be considered a crossed product of the complex numbers C with G. (C, G) is traditionally denoted by $\mathcal{U}(G)$. Our results in Section 4 yield immediately

THEOREM 5.1. *Any outer automorphism of G extends to a spatial automorphism of $\mathcal{U}(G)$. This extension is an outer automorphism of $\mathcal{U}(G)$, if one of the following conditions holds*

a) *G is an R-group*

b) *G has no normal subgroup of finite index*

c) *All nontrivial conjugacy classes in G are infinite*

d) *G_0 agrees with the center of G.*

COROLLARY 1. *Any countably infinite group H has an ergodic and a nonergodic outer automorphic representation on the hyperfinite factor of type II. The nonergodic automorphic representation can be chosen in such a way that all automorphisms leave a given subfactor M of type I_n of the hyperfinite factor elementwise invariant.*

PROOF. Let N be a countably infinite set. By $\mathcal{S}_\infty = \mathcal{S}_\infty(N)$ we denote the group of all permutations of N, whereas $\Pi_\infty = \Pi_\infty(N)$ stands for the group of all finite permutations of N. Since Π_∞ is locally finite, $\mathcal{U}(\Pi_\infty)$ is the hyperfinite factor of type II_1.

1) Let us assume the group H has a representation $\sigma \to P_\sigma$ $\forall \sigma \in H$ as a group of infinite permutations of N. Then $\sigma \to s_\sigma$ with $s_\sigma(\alpha) = P_\sigma \alpha P_\sigma^{-1}$ $\forall \alpha \in \Pi_\infty$ determines uniquely a representation of H as a group of outer automorphisms of Π_∞. By Theorem 5.1. c the s_σ extend uniquely to spatial outer automorphisms S_σ of $\mathcal{U}(\Pi_\infty)$.

2) In order to find a representation of H as a group of permutations of N, let H' be any countable group which contains H as a subgroup. Set $N = H'$ and for any $\sigma \in H$ let P_σ be given by $P_\sigma(\tau) = \sigma\tau$ for all $\tau \in H'$. It is easy to see that the group $\{S_\sigma | \sigma \in H\}$ acts ergodically on $\mathcal{U}(\Pi_\infty)$ if H is infinite. Since there are continuously many nonisomorphic countable discrete groups H', we have determined in this way continuously many outer automorphic representations of H. However it is not clear whether some of these representations are not equivalent, i.e. whether there exists an automorphism T of $\mathcal{U}(\Pi_\infty)$ such that $TS_\sigma T^{-1} = S'_\sigma$ for two automorphic representations S and S' of H. So far I have been unable to determine when two automorphic representations S and S' are equivalent in the above sense.

3) To find nonergodic outer automorphic representations of H we split N into two disjoint parts $N = N_1 \cup N_2$ such that N_2 is infinite. Now we

represent H as a group of permutations of N, which operate on N_2 only. The extensions of these permutations the leave invariant all elements in $\mathcal{U}(\Pi_\infty)$, whose $\Pi_\infty(N)$ support consists of permutations, which operate on N_1 only. These elements constitute a regular representation of the group of permutations of N_1. Thus by appropriately choosing the cardinality of N_1 we can achieve that a certain subfactor $\widetilde{M}$ of type I_n is elementwise invariant under all S_σ, $\sigma \in H$. By [13, Lemma 3.3.] there exists an inner automorphism T connecting M and $\widetilde{M}$, $M = T(\widetilde{M})$. Then $\sigma \rightarrow TS_\sigma T^{-1}$ is the desired nonergodic outer automorphic representation of H on $\mathcal{U}(\Pi_\infty)$, which leaves M elementwise invariant.

This corollary gives a new and more general proof of a result by Suzuki [2]. In the next chapter we shall see a still more general result than this.

COROLLARY 2. *Corollary* 1 *is also valid for at least one finite non-hyperfinite factor of type* II_1.

PROOF. Let F^n be a free product of countably (infinite) many cyclic groups of order $n \geqq 2$ and let $\alpha_1, \alpha_2, \cdots$ be the generators of F^n. Any permutation of the generators extends to an outer automorphism of F^n and then by Theorem 5.1. c to an outer automorphism of $\mathcal{U}(F^n)$. Now proceed as in the previous corollary.

The results in Corollary 1 show that $\mathcal{U}(\Pi_\infty)$ is an ideal candidate for constructing crossed products. To those constructions we can then apply our results in Section 3 and Lemma 4.1 easily.

The corollary of Theorem 4.1 can be applied as follows.

THEOREM 5.2. *Let* λ_α, $\alpha \in G$ *be a set of complex numbers of absolute value* 1 *with* $\lambda_{\alpha\beta} = \lambda_\alpha \lambda_\beta$ *then* $S(U_\alpha) = \lambda_\alpha U_\alpha$ *defines a spatial automorphism of* $\mathcal{U}(G)$. *The group of these automorphisms is isomorphic to* $(G/CG)^*$. *Any inner automorphism of this type is of finite order. This group is a group of outer automorphisms if* G_0 *is the center of* G.

PROOF. The first part is already clear from the corollary. Assume now this automorphism is inner, induced by $\widetilde{S} \in \mathcal{U}(G)_u$. Then $\widetilde{S} U_\alpha \widetilde{S}^* = \lambda_\alpha U_\alpha$, and for $\widetilde{S} \sim (b_\beta)$ we have $b_{\alpha\beta\alpha^{-1}} = \lambda_\alpha b_\beta$. This implies immediately that G-supp $\widetilde{S} \subset G_0$. Let now $\beta \in G$-supp $\widetilde{S}$ and assume α belongs to the centralizer $\mathcal{Z}_\beta$ of β, then $b_{\alpha\beta\alpha^{-1}} = b_\beta = \lambda_\alpha b_\beta$ or $\lambda_\alpha = 1$. Since $\beta \in G$-supp $\widetilde{S} \subset G_0$ the centralizer $\mathcal{Z}_\beta$ of β is of finite index. Thus the kernel of the homomorphism $\alpha \rightarrow \lambda_\alpha$ of G into the circle group is of finite index. This shows that S is of finite order.

In particular our results show that factors $\mathcal{U}(G)$ possess outer automorphisms if G is not complete or if G is not perfect.

6. Now we want to apply the results of Section 4 to certain factors which have been introduced by von Neumann [8] and which have been studied by Pukanszky [14] and Schwartz [15]. The construction is based on the following: Let N be a countably infinite set. For any $n \in N$ let $(X_n, \Sigma_n, \mu_{pn})$ be the finite measure space with $X_n = \{0, 1\}$, Σ_n the discrete Borel structure of X_n and $\mu_{pn}(\{0\}) = p$, $\mu_{pn}(\{1\}) = 1-p = q$ with $0 < p \leqq 1/2$. Then let $(X, \Sigma, \mu_p) = (\otimes X_n, \otimes \Sigma_n, \otimes \mu_{pn})$ be the product Borel structure of the $(X_n, \Sigma_n, \mu_{pn})$. The measure space (X, Σ, μ_p) is familiar from probability theory, where it describes infinitely repeated coin tossing. X can also be considered the space of functions on N with values 0 and 1. The set of functions on N with values 0 and 1 and finite support will be denoted by $\Delta = \Delta(N)$. Δ becomes an abelian group if we define

$$(\delta+\delta')(n) = \delta(n) + \delta'(n) \pmod 2$$

Δ admits a representation as a group of free, ergodic Borel automorphisms, of (X, Σ, μ_p) which leave μ_p quasi-invariant [14]

$$[\delta x](n) = x(n) + \delta(n) \pmod 2 \tag{34}$$

$$(d\mu_{p\delta}/d\mu_p)(x) = (p/q)^{(2x-1)\cdot\delta} \tag{35}$$

where $x \cdot \delta = \Sigma x(i)\delta(i)$. The action of Δ can be extended to $\mathcal{L}^\infty(X, \Sigma, \mu) = \mathcal{L}^\infty$ and $\mathcal{L}^2(X, \Sigma, \mu) = \mathcal{L}^2$ by

$$\begin{aligned} (\widetilde{\delta}a)(x) &= a(\delta x) && \forall a \in \mathcal{L}^\infty \\ (u_\delta \xi)(x) &= (d\mu_{p\delta}/d\mu_p)^{1/2}(x)\,\xi(\delta x) && \forall \xi \in \mathcal{L}^2 \end{aligned} \tag{36}$$

As in section 1 (3) we now define the operators $\Pi(a)$ and U_δ on $\mathcal{L}^2 \otimes l^2(\Delta)$; the only thing that has been altered are the Radon-Nikodym factors. The operators $\Pi(a)$ and U_δ, $a \in \mathcal{L}^\infty$, $\delta \in \Delta$ span the W^*-algebra $\mathcal{M}_p$, which again is called the crossed product of $\mathcal{L}^\infty$ and Δ. $\mathcal{M}_p$ is a factor for any $0 < p \leqq 1/2$, because Δ acts freely and ergodically on (X, Σ, μ_p), [8]. $\mathcal{M}_{\frac{1}{2}}$ is the hyperfinite factor of type II, whereas the $\mathcal{M}_p$ $0 < p < 1/2$ are nonisomorphic factors of type III [13, 14]. Let us now turn to automorphisms of $\mathcal{M}_p$. We shall extend in steps any permutation P_σ of N to an automorphism of $\mathcal{M}_p$. First we extend P_σ to X and Δ.

$$(\sigma x)(n) = x(P_\sigma^{-1} n), \qquad (\sigma\delta)(n) = \delta(P_\sigma^{-1} n) \tag{37}$$

It is easy to see that this extension σ of P_σ defines a Borel automorphism of (X, Σ, μ_p), which leaves μ_p invariant, and an automorphism σ of Δ respectively. The extension of σ to $\mathcal{L}^\infty$ and $\mathcal{L}^2$ is as easy as before.

$$(\tilde{\sigma}a)(x) = a(\sigma^{-1}x) \qquad \forall a \in \mathcal{L}^\infty \tag{38}$$

$$(u_\sigma\zeta)(x) = \zeta(\sigma^{-1}x) \qquad \forall \zeta \in \mathcal{L}^2$$

The spatial automorphisms S_σ of $\mathcal{M}_p$ can now be defined

$$S_\sigma(\Pi(a)U_\delta) = \Pi(\tilde{\sigma}\,a)\,U_{\sigma(\delta)} \tag{39}$$

$$\tilde{S}_\sigma \Sigma \zeta_\delta \otimes \varepsilon_\delta = \Sigma u_\sigma \zeta_\delta \otimes \varepsilon_{\sigma(\delta)} \tag{40}$$

The necessary computations to check (39) and (40) are easy to perform and are therefore omitted.

THEOREM 6.1. *$\mathcal{S}_\infty(N)$ has a representation as a group of spatial automorphisms of the hyperfinite factors $\mathcal{M}_p$ given by* (37)–(40). *S_σ* (39) *defines an outer automorphism of the $\mathcal{M}_p$ iff $P_\sigma \in \mathcal{S}_\infty - \Pi_\infty$.*

PROOF. The first part is already clear from above. Now let $P_\sigma \in \mathcal{S}_\infty - \Pi_\infty$ and assume S_σ is an inner automorphism of $\mathcal{M}_p$ with $\tilde{S}_\sigma \sim (b_\beta)$. Then $\tilde{S}_\sigma U_\delta \tilde{S}_\sigma = U_{\sigma(\delta)}$ implies $b_{\delta' + \sigma(\delta') + \delta} = \tilde{\sigma}(\delta')(b_\delta)$. Arguments analogous to those in Theorem 4.2 show that $\{\delta' + \tilde{\sigma}\delta' \mid \delta' \in \Delta\}$ must be a finite set. This however is impossible since $P_\sigma \in \mathcal{S}_\infty - \Pi_\infty$.

Now let $P_\sigma \in \Pi_\infty$, then we can split N into a finite set N_1 and a set N_2 such that P_σ leaves N_2 elementwise fixed and operates only on N_1. Then consider $X_i = \bigotimes_{n \in N_i} X_n$, $\Sigma_i = \bigotimes_{n \in N_i} \Sigma_n$ and $\mu_{pi} = \bigotimes_{n \in N_i} \mu_{pn}$ $i = 1, 2$. We have obviously $X = X_1 \otimes X_2$, $\Sigma = \Sigma_1 \otimes \Sigma_2$ and $\mu_p = \mu_{p1} \otimes \mu_{p2}$. This also implies a factorization of $\mathcal{M}_p$ as $\mathcal{M}_p = \mathcal{M}_{p1} \otimes \mathcal{M}_{p2}$ with $\mathcal{M}_{pi} = (\mathcal{L}^\infty(X_i, \Sigma_i, \mu_{pi}), \Delta_i)$ $i = 1, 2$ [13]. Since P_σ operates only on N_1, S_σ will leave $\mathcal{M}_{p2}$ elementwise invariant. It is known [16] that $\mathcal{M}_{p1}$ is a factor of type I_{2^n}, where n is the cardinality of N_1. Therefore $S_\sigma | \mathcal{M}_{p1}$ is an inner automorphism, induced by $\tilde{S}_{\sigma 1}$. Then $\tilde{S}_{\sigma 1} \otimes 1$ induces S_σ.

As in Section 5 one shows now:

COROLLARY. *Any countable discrete group admits an ergodic and nonergodic outer automorphic representation of the hyperfinite factor $\mathcal{M}_p$, $0 < p < 1/2$. The nonergodic representation can be chosen in such a way that a given subfactor of type I_n is elementwise invariant.*

This corollary generalizes a result by T. Saitô [3], however the proof is different and more straightforward. For $p=1/2$ this is just the corollary of Theorem 5.1. The corollary shows that the $\mathcal{M}_p$ are ideal candidates for constructing crossed products.

Let G be a countable discrete group and let $P: \alpha \to P_\alpha$ be a representation of G as a group of infinite permutations of N. Then $P(G)$ extends by (37)-(40) to a group $S(G)$ of outer automorphisms of $\mathcal{M}_p$. All $\widetilde{S}_\alpha$, $\alpha \in G$, leave the vector $1 \otimes \varepsilon_0 \in \mathcal{L}^2 \otimes l^2(\Delta)$ invariant. Thus we can form the crossed product of $\mathcal{M}_p$ and G, which we denote by $(\mathcal{M}_p, G, P)$ since it also depends on the permutation representation P of G. The corollary of Lemma 3.1 shows that all $(\mathcal{M}_p, G, P)$ are factors, regardless of P. Since $\mathcal{M}_p$ $0<p<1/2$ is a factor of type III, $(\mathcal{M}_p, G, P)$ are factors of type III [17]. However the $(\mathcal{M}_{\frac{1}{2}}, G, P)$ are finite factors. This gives us a large source of continuous factors, because not only is p variable, but also P can be changed. On the basis of the recent results by Powers [13] one would conjecture that all $(\mathcal{M}_p, G, P)$ are non-isomorphic for different p. It is also probable that distinct P will lead to nonisomorphic factors for certain groups.

In (37) we had seen that a permutation representation P of G on N leads to a representation of G as a group of automorphisms of Δ, $(\alpha\delta)(n) = \delta(P_\alpha^{-1} n)$ $\forall \delta \in \Delta$, $\alpha \in G$. This allows us to construct the semidirect product of Δ and G. $\Delta Ⓢ_P G = \{(\alpha, \delta) \mid \alpha \in G, \delta \in \Delta\}$ and the multiplication is defined by

$$(\alpha, \delta)(\alpha', \delta') = (\alpha\alpha', \delta + \alpha(\delta'))$$

Theorem 4.3 now suggests:

THEOREM 6.2. *$(\mathcal{M}_p, G, P)$ is spatially isomorphic to $(\mathcal{L}^\infty, \Delta Ⓢ_P G)$. The action of $\Delta Ⓢ_P G$ on (X, Σ, μ_p) is given by*

$$[(\alpha, \delta)x](n) = x(P_\alpha^{-1} n) + \delta(n) \pmod 2 \tag{41}$$

and extended to $\mathcal{L}^\infty$ and $\mathcal{L}^2$ as

$$\begin{aligned} [\widetilde{(\alpha, \delta)}\, a](x) &= a((\alpha, \delta)^{-1} x) \qquad \forall a \in \mathcal{L}^\infty \\ [u_{(\alpha,\delta)}\, \zeta](x) &= (d\mu_{p(\alpha,\delta)}/d\mu_p)^{\frac{1}{2}}(x)\, \zeta((\alpha, \delta)^{-1} x) \quad \forall \zeta \in \mathcal{L}^2 . \end{aligned} \tag{42}$$

The proof proceeds as in Theorem 4.3. A particular permutation representation P_0, the Cayley representation can be obtained by identifying N and G and by defining

$$P_{0\alpha}\beta = \alpha\beta \qquad \alpha, \beta \in G = N. \tag{43}$$

Then Theorem 6.2 shows that $(\mathcal{M}_p, G, P_0)$ are just the factors which have been constructed by von Neumann [8], Pukanszky [14] and Schwartz [15]. Since $\mathcal{M}_p$ is already a crossed product in its own right, $(\mathcal{M}_p, G, P_0)$ can be considered a "doubly crossed" product.

THEOREM 6.3. *Any nontrivial automorphism μ of G extends to an outer spatial automorphism S_μ of $(\mathcal{L}^\infty, \Delta \circledS_{P_0} G)$. The extension of μ $\tilde{\mu} \in \mathrm{Aut}\, \mathcal{L}^\infty$ and u_μ are given by* (37) *and* (38). *The corresponding automorphism m of $\Delta \circledS_{P_0} G$ is given as*

$$m(\alpha, \delta) = (\mu\alpha, \mu\delta) \tag{44}$$

and S_μ is defined as

$$S_\mu(\Pi(a) U_{(\alpha,\delta)}) = \Pi(\tilde{\mu}(a)) U_{m(\alpha,\delta)} \tag{45}$$

and

$$\widetilde{S}_\mu \Sigma \zeta \otimes \varepsilon_{(\alpha,\delta)} = \Sigma u_\mu \zeta \otimes \varepsilon_{m(\alpha,\delta)} \tag{46}$$

PROOF. The equations (32) are easy to check and thus (45) defines an automorphism of $(\mathcal{L}^\infty, \Delta \circledS_{P_0} G)$. $\widetilde{S}_\mu$ is obviously a unitary operator and a simple computation shows that $\widetilde{S}_\mu \Pi(a) U_{(\alpha,\delta)} \widetilde{S}_\mu = \Pi(\tilde{\mu} a) U_{m(\alpha,\delta)}$. If μ is an outer automorphism of G, S_μ is an outer automorphism of $(\mathcal{M}_p, G, P_0)$ by Lemma 3.1. Thus we can assume that μ is inner on G, $\mu(\alpha) = \beta\alpha\beta^{-1}$. Instead of S_μ consider now the automorphism $T: A \to U_{(\beta^{-1},0)} S_\mu(A) U_{(\beta,0)}$. In particular $T(U_{(\alpha,\delta)}) = U_{(\alpha,\delta^\beta)}$ with $\delta^\beta(\alpha) = \delta(\alpha\beta)$. Standard arguments (33) now show that T is an outer automorphism, unless $\beta = e$.

Similar methods can also be applied to other factors $(\mathcal{M}_p, G, P)$, where $P \neq P_0$. If $G \neq CG$ the group $\mathfrak{K}$ (Theorem 3.1) is nonempty and a group of outer automorphisms of $(\mathcal{M}_p, G, P)$ regardless of p or P.

We shall now turn to the construction of certain finite factors which apparently have not been considered before. Again this construction is based on the crossed product.

Let G' be a countably infinite discrete group and G a countably infinite subgroup of G'. For $G' = N$ we construct again (X, Σ, μ_p) and Δ as we did earlier in this section.

LEMMA 6.1. *G acts as a group of free, ergodic and measure preserving Borel automorphisms on (X, Σ, μ_p) by*

$$[\beta x](\alpha) = x(\beta^{-1}\alpha) \qquad \forall \beta \in G \subset G',\ \alpha \in G \tag{37}$$

The μ_p are inequivalent for different $0 < p \leqq 1/2$.

PROOF. It is easy to see that the point transformations $\beta \in G$ as defined above define Borel automorphisms of (X, Σ, μ_p) which leave the μ_p invariant. G acts freely on (X, Σ, μ_p) if for $\beta \neq e$ the set $E_\beta = \{x \mid \beta x = x\}$ has μ_p-measure zero. We have to consider two cases.

a) $\beta \in G$ has infinite order. Then we write

$$E_\beta = E_\beta^{(1)} \cup E_\beta^{(0)} = \{x \in E_\beta \mid x(e)=1\} \cup \{x \in E_\beta \mid x(e)=0\}$$
$$\subset \{x \mid x(e)=x(\beta^{-1}) = \cdots = x(\beta^{-n})=1\} \cup \{x \mid x(e) = \cdots = x(\beta^{-n})=0\} .$$

This shows that $\mu^p(E_\beta) \leqq p^n + q^n$ for any $n < \infty$.

b) $\beta \in G$ has finite order $m \geqq 2$. Then find n elements $\alpha_1, \cdots, \alpha_n$ such that $\alpha_k \neq \beta^j \alpha_l$ for all $j < m$ and $k \neq l$. We write

$$E_\beta = \bigcup_{\delta \in \Delta^n} \{x \in E_\beta \mid x(\alpha_i) = \delta(i)\} \subset \bigcup_{\delta \in \Delta^n} \{x \mid x(\alpha_i) = x(\beta^{-1}\alpha_i) = \cdots = \delta(i)\}$$

where Δ^n is the set of all n-tuples with 0 and 1 as entries. This shows that

$$\mu_p(E_\beta) \leqq \sum_{\delta \in \Delta^n} p^{(n-|\delta|)m} q^{|\delta|m} = (p^m + q^m)^n$$

with $|\delta| = \Sigma \delta(i)$. Since $p^m + q^m < 1$ we see again that $\mu_p(E_\beta) = 0$. To show the ergodicity of G let $a \in \mathcal{L}^\infty(X, \Sigma, \mu)$ such that

$$a = \tilde{\beta}(a) \quad \forall \beta \in G, \text{ with } \beta \text{ given by (38)}.$$

Since (X, Σ, μ) is a finite measure space $a \in \mathcal{L}^2$ and a has an expansion with respect to the orthonormal basis $\{\omega_\delta \mid \delta \in \Delta\}$ [14] with

$$\omega_\delta(x) = (-1)^{\delta \cdot x}(p/q)^{(x-1/2)\cdot\delta} \tag{44}$$
$$a = \Sigma \lambda_\delta \omega_\delta .$$

It is easy to see that $\tilde{\beta}\omega_\delta = \omega_{\beta\delta}$ where $(\beta\delta)(\alpha) = \delta(\beta^{-1}\alpha)$. We therefore have $\lambda_\delta = \lambda_{\beta(\delta)}$. Since the set $\{\beta\delta \mid \beta \in G\}$ is infinite if $\delta \neq 0$, a must be of the form $a = \lambda\omega_0 =$constant.

Assume now that for $p \neq p'$ $\mu_p \gg \mu_{p'}$. Then the Radon-Nikodym theorem tells us that $\mu_{p'} = f \cdot \mu_p$. Since $\mu_{p'}$ and μ_p are G-invariant, f is G-invariant. But G acts ergodically on (X, Σ, μ_p). Hence f is constant and $\mu_p = \mu_{p'}$, which is easily seen to be impossible.

COROLLARY. *The crossed products* $(\mathcal{L}^\infty(X, \Sigma, \mu_p), G)$ *are factors of type* II_1.

In view of the recent results by Powers [13] it would be interesting to find out if for certain groups G these factors are nonisomorphic for distinct p. For $G=G'$ denote $(\mathcal{L}^\infty(X, \Sigma, \mu_p), G)$ by $\mathfrak{N}_p(G)$ then one can easily show

THEOREM 6.5. *Every automorphism* μ *of* G *extends to a spatial outer automorphism* S_μ *of* $\mathfrak{N}_p(G)$.

$$(\mu x)(\alpha) = x(\mu^{-1}\alpha),\ (\tilde{\mu}a)(x) = a(\mu^{-1}x) \quad \forall a \in \mathcal{L}^\infty \tag{45}$$

$$(u_\mu \zeta)(x) = \zeta(\mu^{-1}x) \tag{46}$$

$$S_\mu(\Pi(a)U_\alpha) = \Pi(\tilde{\mu}a)U_{\mu(\alpha)} \tag{47}$$

$$\widetilde{S}_\mu \Sigma \zeta_\alpha \otimes \varepsilon_\alpha = \Sigma u_\mu \zeta_\alpha \otimes \varepsilon_{\mu(\alpha)} \tag{48}$$

PROOF. It is easy to show that S_μ defines an automorphism of $\mathfrak{N}_p(G)$, which is induced by $\widetilde{S}_\mu$. If S_μ is an inner automorphism with $\widetilde{S}_\mu \sim (b_\beta)$ then $\widetilde{S}_\mu \Pi(a) = \Pi(\tilde{\mu}a)\widetilde{S}_\mu$ or $b_\beta \widetilde{\beta a} = b_\beta \tilde{\mu} a$, $\forall a \in \mathcal{L}^\infty$. Thus the set $\{x \mid b_\beta(x) \neq 0\}$ is contained, modulo a μ_p zero set, in the set $E_{\mu,\beta} = \{x \mid \beta^{-1}x = \mu^{-1}x\}$. As in Lemma 6.1 one shows that $\mu_p(E_{\mu,\beta}) = 0$ unless $\beta = e$ and μ is the identity automorphism of G. Along similar lines one can also construct outer automorphisms of $(\mathcal{L}^\infty, G)$ if $G' \neq G$. For certain groups G also $\mathfrak{R}$ will lead to outer automorphisms of $\mathfrak{N}_p(G)$.

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REFERENCES

[1] J. DIXMIER, Sous anneau abéliens maximau dans les facteurs de type fini, Ann. of Math., 59(1954), 279-286.

[2] N. SUZUKI, A linear representation of a countably infinite group, Proc. Japan Acad., 34(1958), 575-579.

[3] T. SAITÔ, On a representation of a countably infinite group, Tôhoku Math. J., 13(1961), 268-273.

[4] R. BLATTNER, Automorphic group representations, Pacific J. Math., 8(1958), 665-677.

[5] J. DIXMIER, Les algébres d'operateurs dans l'espace hilbertien, Gauthier-Villars, Paris, 1957.

[6] R. V. KADISON AND J. RINGROSE, Derivations and automorphisms of operator algebras, Comm. Math. Phys., 4(1967), 32-63.

[7] I. M. SINGER, Automorphisms of finite factors, Amer. J. Math., 77(1955), 117-133.

[8] J. VON NEUMANN, On rings of operators III, Ann. of Math., 41(1940), 94-161.

[9] T. TURUMARU, Crossed products of operator algebras, Tôhoku Math. J., 10(1958), 355-365.

[10] N. SUZUKI, Crossed products of rings of operators, Tôhoku Math. J., 11(1959), 113–124.
[11] G. ZELLER-MEIER, Produits Croisés d'une C^*algébre par un groupe d'automorphismes, C. R. Acad. Sc., 263(1966), 20–27.
[12] H. LEPTIN, Darstellungen verallgemeinerter $\mathcal{L}^1$ algebren, Preprint, Univ. of Heidelberg.
[13] R. T. POWERS, Representations of uniformly hyperfinite algebras, Ann. of Math., 85(1967), 138–137.
[14] L. PUKANSZKY, Some examples of factors, Publ. Math. Debrecen, 4(1956), 135–156.
[15] J. SCHWARTZ, Non-isomorphism of a pair of factors of type III, Comm. Pure Appl. Math., 16(1963), 111–120.
[16] J. GLIMM, Type I C^*-algebras, Ann. of Math., 73(1961), 572–612.
[17] S. SAKAI, Lecture Notes, Yale Univ., 1964.

INSTITUT FÜR ANGEWANDTE MATHEMATIK
UNIVERSITÄT HEIDELBERG
HEIDELBERG, GERMANY