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SASAKIAN MANIFOLD WITH PSEUDO-RIEMANNIAN METRIC

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Introduction. Sasakian manifold with Riemannian metric is defined by S. Sasaki [5]. In this paper, we want to define Sasakian manifold with pseudo-Riemannian metric, and discuss the classification of Sasakian manifolds.

In section 1, we define a Sasakian manifold (with pseudo-Riemannian metric). In section 2, we define the model spaces of Sasakian manifolds which are used in section 4 for the classification of Sasakian manifolds of constant ϕ-sectional curvatures. In section 3, we discuss D-homothetic deformation which is defined by S. Tanno [9], and prove some fundamental lemmas concerning completeness of the deformed metric. In section 5, we prove that a Sasakian manifold, satisfying $R(X,Y)\cdot R=0$ for all tangent vectors X and Y, is of constant curvature. In section 6, we discuss a Sasakian manifold M^{2n+1} which is properly and isometrically immersed in E_s^{2n+2}.

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1. Preliminaries. Manifolds and tensor fields are supposed to be of class C^∞.

Let $M=M^{2n+1}$ be a connected differentiable manifold, and let ϕ, ξ and η be tensor fields of type (1, 1), (1, 0) and (0, 1), respectively, on M.

DEFINITION. (ϕ,ξ,η) is called an *almost contact structure* on M, if the followings are satisfied:

$$\eta(\xi)=1\,,$$

$$\eta(\phi(X))=0\,,\qquad X\in\mathfrak{X}(M)\,,$$

$$\phi^2(X)=-X+\eta(X)\xi,\qquad X\in\mathfrak{X}(M)\,.$$

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DEFINITION. $(\phi, \xi, \eta, g, \varepsilon)$ is called an *almost contact metric structure* on M, if (ϕ, ξ, η) is an almost contact structure on M and g is a pseudo-Riemannian metric on M such that

$$g(\xi, \xi) = \varepsilon, \quad \varepsilon = +1 \text{ or } -1,$$
$$\eta(X) = \varepsilon g(\xi, X), \quad X \in \mathfrak{X}(M),$$
$$g(\phi X, \phi Y) = g(X, Y) - \varepsilon\eta(X)\eta(Y), \quad X, Y \in \mathfrak{X}(M).$$

DEFINITION. $(\phi, \xi, \eta, g, \varepsilon)$ is a *contact metric structure* on M, if it is an almost contact metric structure on M and satisfies

$$d\eta(X, Y) = g(\phi X, Y), \quad X, Y \in \mathfrak{X}(M).$$

DEFINITION. $(\phi, \xi, \eta, g, \varepsilon)$ is a *normal contact metric structure* on M, if it is a contact metric structure and satisfies

$$(\nabla_X \phi) Y = \varepsilon\eta(Y) X - g(X, Y)\xi, \quad X, Y \in \mathfrak{X}(M),$$

where ∇ indicates the Levi-Civita connection for the pseudo-Riemannian metric g. In this case, we call $M(\phi, \xi, \eta, g, \varepsilon)$ a *Sasakian manifold.*

The following example shows that we may assume $\varepsilon = 1$ without loss of generality.

EXAMPLE. Let $(\phi, \xi, \eta, g, \varepsilon)$ be an almost contact metric structure (resp. a normal contact metric structure) on M. We put

$$\bar{g} = -g, \quad \bar{\xi} = -\xi, \quad \bar{\eta} = -\eta, \quad \bar{\phi} = \phi.$$

Then $(\bar{\phi}, \bar{\xi}, \bar{\eta}, \bar{g}, \bar{\varepsilon})$, $\bar{\varepsilon} = -\varepsilon$, is an almost contact metric structure (resp. a normal contact metric structure) on M.

PROOF. It is easy to see that $(\bar{\phi}, \bar{\xi}, \bar{\eta}, \bar{g}, \bar{\varepsilon})$ is an almost contact metric structure, and it is a contact metric structure if $(\phi, \xi, \eta, g, \varepsilon)$ is a contact metric structure. Suppose $(\phi, \xi, \eta, g, \varepsilon)$ is a normal contact metric structure. Since the parallelism with respect to g and the parallelism with respect to $\bar{g}$ are the same, we get

$$\begin{aligned}(\bar{\nabla}_X \bar{\phi}) Y &= (\nabla_X \phi) Y \\ &= \varepsilon\eta(Y) X - g(X, Y)\xi \\ &= \bar{\varepsilon}\bar{\eta}(Y) X - \bar{g}(X, Y)\bar{\xi}.\end{aligned}$$

Thus $(\bar{\phi}, \bar{\xi}, \bar{\eta}, \bar{g}, \bar{\varepsilon})$ is normal.

Hereafter, we assume $\varepsilon=1$, and drop it.

REMARK. A contact metric structure is normal if and only if the following tensor field vanishes:

$$N(X, Y) = [\phi X, \phi Y] - \phi[\phi X, Y] - \phi[X, \phi Y] + \phi^2[X, Y] + 2d\eta(X, Y)\xi .$$

(cf. S. Sasaki [7], Theorem 11.1)

By the same method as in the case of Riemannian metric, we get the following, which we use later:

PROPOSITION 1. *For an almost contact metric structure* (ϕ, ξ, η, g) *on* M,

$$(1) \qquad (\nabla_X\phi) Y = \eta(Y) X - g(X, Y)\xi$$

implies

(i) $\nabla_X\xi = \phi(X)$,
(ii) ξ *is a Killing vector field,*
(iii) $d\eta(X, Y) = g(\phi X, Y)$.

Let (M^n, g) be a pseudo-Riemannian manifold. Let X and Y be tangent vectors at a point of M^n. If X and Y satisfy

$$g(X, X) g(Y, Y) - g(X, Y)^2 \neq 0,$$

then we say that X and Y span a non-degenerate 2-plane $X\wedge Y$. This definition is independent of the choice of X and Y which span the 2-plane $X\wedge Y$. For a non-degenerate 2-plane $X\wedge Y$, we define a sectional curvature $K(X, Y)$ by

$$K(X, Y) = \frac{g(R(X, Y)Y, X)}{g(X, X)\, g(Y, Y) - g(X, Y)^2}.$$

If $K(X, Y)$ is constant for all X and Y in $T_x(M^n)$ such that $X\wedge Y$ is a non-degenerate 2-plane, we call (M^n, g) to be of constant curvature at x. If (M^n, g) is of constant curvature at every point of M^n, $K(X, Y)$ is a function of $x \in M^n$, say $k(x)$. If $k(x)$ is constant on M^n, we call (M^n, g) to be of constant curvature. It is known that if (M^n, g) is of constant curvature at every point and if $n \geqq 3$, then (M^n, g) is of constant curvature (J. A. Wolf [10], p. 57, Cor. 2.2.7). Suppose (M^n, g) is of constant curvature k, then we have

$$(2) \qquad R(X, Y)Z = k\{g(Y, Z)X - g(X, Z)Y\}$$

for all tangent vectors X, Y and Z (cf. J. A. Wolf [10], p. 56, Cor. 2.2.5).

Suppose we have a Sasakian manifold $M^{2n+1}(\phi, \xi, \eta, g)$. Let

$$D_x = \{X \in T_x(M^{2n+1});\ \eta(X) = 0\}\ .$$

For a non-null vector X in D_x, X and ϕX span a non-degenerate 2-plane, and hence we can consider a sectional curvature $K(X) = K(X, \phi X)$. If $K(X)$ is constant for all non-null vectors X in D_x, we call (M^{2n+1}, g) to be of constant ϕ-sectional curvature at x. If (M^{2n+1}, g) is of constant ϕ-sectional curvature at every point, $K(X)$ is a function of $x \in M^{2n+1}$, say $k(x)$. In this case, if $k(x)$ is constant on M^{2n+1}, we call (M^{2n+1}, g) to be of constant ϕ-sectional curvature. If (M^{2n+1}, g) is of constant ϕ-sectional curvature at every point and if $n \geqq 2$, (M^{2n+1}, g) is of constant ϕ-sectional curvature (cf. K. Ogiue [4]). Suppose (M^{2n+1}, g) is of constant ϕ-sectional curvature k, then we have, for any tangent vectors X, Y and Z,

$$(3) \qquad \begin{aligned} 4R(X, Y)Z = (k+3)\{g(Y, Z)X - g(X, Z)Y\} + (k-1)\{&\eta(X)\eta(Z)Y \\ - \eta(Y)\eta(Z)X + g(X, Z)\eta(Y)\xi - g(Y, Z)\eta(X)\xi \\ + g(\phi Y, Z)\phi X + g(\phi Z, X)\phi Y - 2g(\phi X, Y)\phi Z\}\ . \end{aligned}$$

(cf. K. Ogiue [4]). Thus, if (M^{2n+1}, g) is of constant ϕ-sectional curvature 1, it is of constant curvature 1.

REMARK. If we do not assume $\varepsilon = 1$, (3) should be

$$(3') \qquad \begin{aligned} 4R(X, Y)Z = (k+3\varepsilon)\{g(Y, Z)X - g(X, Z)Y\} \\ + (\varepsilon k-1)\{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X\} \\ + (k-\varepsilon)\{g(X, Z)\eta(Y)\xi - g(Y, Z)\eta(X)\xi \\ + g(\phi Y, Z)\phi X + g(\phi Z, X)\phi Y - 2g(\phi X, Y)\phi Z\}\ . \end{aligned}$$

2. Model spaces. Let b_s^{n+1} be an "inner product" on C^{n+1}, defined by

$$(1) \qquad b_s^{n+1}(u, v) = \mathrm{Re}\left(-\sum_{i=1}^{s} u_i \bar{v}_i + \sum_{j=s+1}^{n+1} u_j \bar{v}_j\right).$$

Let $\tilde{g} = g_{2s}^{2n+2}$ be a pseudo-Riemannian metric on C^{n+1} defined by the parallel translation of b_s^{n+1}. Let J be a complex structure on C^{n+1} defined by the parallel translation of the map

$$u \in C^{n+1} \longrightarrow \sqrt{-1}\, u\,.$$

For $n \geqq 0$ and $0 \leqq s \leqq n$, let $M = S_{2s}^{2n+1}$ be a hypersurface of C^{n+1} defined by

$$(2) \qquad S_{2s}^{2n+1} = \{u \in C^{n+1}\,;\ b_s^{n+1}(u, u) = 1\}\,,$$

and let $g = \tilde{g}\,|\,S_{2s}^{2n+1}$. Then (M, g) is a pseudo-Riemannian manifold of constant curvature 1, of dimension $2n+1$ and of signature $2s$ (cf. J. A. Wolf [10], pp. 62-68). If $s = 0$, M is nothing but the unit sphere S^{2n+1}; S. Sasaki and Y. Hatakeyama [6] defined a Sasakian structure on it. Similarly, we can define a Sasakian structure on $M = S_{2s}^{2n+1}$, $n \geqq 0$, $0 \leqq s \leqq n$, as follows:

For $x \in M$, the tangent space of M at x is given by

$$T_x(M) = \{X \in T_x(C^{n+1})\,;\ \tilde{g}(X, x) = 0\}\,,$$

where we consider x as its position vector. Let ξ be a vector field on M defined by

$$(3) \qquad \xi : x \in M \longrightarrow \xi_x = Jx\,,$$

where Jx is considered as a tangent vector of C^{n+1} at x by the parallel translation. Since J is skew-symmetric with respect to $\tilde{g}$, $\tilde{g}(Jx, x) = 0$; hence Jx is in $T_x(M)$, and

$$g(\xi_x, \xi_x) = \tilde{g}(x, x) = 1\,.$$

Let η be a 1-form on M defined by

$$(4) \qquad \eta(X) = g(\xi, X)\,, \quad X \in \mathfrak{X}(M)\,.$$

Since $x \in M$ is a non-null vector in C^{n+1}, we have an orthogonal projection

$$\pi : T_x(C^{n+1}) \longrightarrow T_x(M)$$

with respect to $\tilde{g}$, that is,

$$(5) \qquad \pi(X) = X - \tilde{g}(x, X)\,x\,, \quad X \in T_x(C^{n+1})\,, \quad x \in M\,.$$

Let ϕ be a tensor field of type (1, 1) on M defined by

$$\phi = \pi \circ J \,. \tag{6}$$

It is easy to see that (ϕ, ξ, η, g) is an almost contact metric structure on M. We want to show that this structure is a Sasakian structure. According to Proposition 1, it is sufficient to show

$$(\nabla_X \phi) Y = \eta(Y) X - g(X, Y) \xi \,. \tag{7}$$

Consider M to be a hypersurface of C^{n+1}. Then the vector field

$$\zeta : x \in M \longrightarrow \zeta_x = x$$

is a field of unit normal vectors to M in C^{n+1}. For any vector fields X and Y tangent to M, we have the formulas of Gauss and Weingarten:

$$D_X Y = \nabla_X Y + h(X, Y) \zeta \,, \tag{8}$$

$$D_X \zeta = -AX \,, \tag{9}$$

where D_X and ∇_X denote covariant differentiations for $\tilde{g}$ and g, respectively. A is a field of symmetric endomorphisms (with respect to g) satisfying

$$h(X, Y) = g(AX, Y) \tag{10}$$

for tangent vectors X and Y (cf. L. P. Eisenhart [1]). Since the pseudo-Riemannian metric g is defined by the parallel translation,

$$D_X \zeta = X \tag{11}$$

for any tangent vector X to M. Thus, (8), (9), (10) and (11) imply

$$D_X Y = \nabla_X Y - g(X, Y) \zeta \,. \tag{12}$$

Now, we have

$$(\nabla_X \phi) Y = \nabla_X(\phi Y) - \phi \nabla_X Y \,, \tag{13}$$

for any vector fields X and Y tangent to M. We want to show that the right hand side of the above equation is nothing but the right hand side of (7). Using (12), we get

(14) $$\nabla_X(\phi Y) = D_X(\phi Y) + g(X, \phi Y)\zeta$$
$$= D_X(\pi JY) + \tilde{g}(X, \phi Y)\zeta\,.$$

On the other hand, we have

$$D_X(\pi JY) = D_X(JY - \tilde{g}(\zeta, JY)\zeta)$$
$$= JD_XY - \tilde{g}(X, JY)\zeta - \tilde{g}(\zeta, JD_XY)\zeta - \tilde{g}(\zeta, JY)X\,,$$

$$\tilde{g}(X, \phi Y) - \tilde{g}(X, JY) = \tilde{g}(X, \pi JY - JY)$$
$$= \tilde{g}(X, -\tilde{g}(\zeta, JY)\zeta)$$
$$= 0\,.$$

Thus (14) becomes

(15) $$\nabla_X(\phi Y) = JD_XY - \tilde{g}(\zeta, JD_XY)\zeta - \tilde{g}(\zeta, JY)X\,.$$

The second term of the right hand side of (13) is

(16) $$\phi\nabla_XY = \pi J(D_XY + g(X, Y)\zeta)$$
$$= \pi JD_XY + g(X, Y)\xi\,.$$

Hence, (13), (15) and (16) imply

$$(\nabla_X\phi)Y = JD_XY - \tilde{g}(\zeta, JD_XY)\zeta - \tilde{g}(\zeta, JY)X - \pi JD_XY - g(X, Y)\xi$$
$$= \tilde{g}(\zeta, JD_XY)\zeta - \tilde{g}(\zeta, JD_XY)\zeta + g(\xi, Y)X - g(X, Y)\xi$$
$$= \eta(Y)X - g(X, Y)\xi\,.$$

REMARK. If we replace (2),(3),(4) and (6) by

(2′) $$H_{2s-1}^{2n+1} = \{u \in C^{n+1}\,;\ b_s^{n+1}(u, u) = -1\}, \quad 1 \leqq s \leqq n+1\,,$$

(3′) $$\bar{\xi} : x \in H_{2s-1}^{2n+1} \longrightarrow \bar{\xi}_x = -Jx\,,$$

(4′) $$\bar{\eta}(X) = -\bar{g}(\bar{\xi}, X), \quad \bar{g} = \tilde{g}\,|\,H_{2s-1}^{2n+1}\,,$$

(6′) $$\bar{\phi} = \bar{\pi} \circ J, \quad \bar{\pi}X = X + \tilde{g}(x, X)x, \quad X \in T_x(H_{2s-1}^{2n+1})\,,$$

then $H_{2s-1}^{2n+1}(\bar{\phi}, \bar{\xi}, \bar{\eta}, \bar{g}, -1)$ is a Sasakian manifold and $H_{2s-1}^{2n+1}(\bar{\phi}, -\bar{\xi}, -\bar{\eta}, -\bar{g}, +1)$

is nothing but $S^{2n+1}_{2(n-s+1)}(\phi, \xi, \eta, g)$ (cf. Example of §1).

It is known that S^{2n+1}_{2s} is diffeomorphic to $R^{2s}\times S^{2n+1-2s}$. Thus S^{2n+1}_{2s} is simply connected for $s\neq n$; S^{2n+1}_{2n} is connected with infinite cyclic fundamental group. We define

$$\text{(17)}\qquad \begin{aligned} &\tilde{S}^{2n+1}_{2s} = S^{2n+1}_{2s} \quad \text{for} \quad s \neq n\,; \\ &\tilde{S}^{2n+1}_{2n} = \text{universal pseudo-Riemannian covering manifold of } S^{2n+1}_{2n}\,. \end{aligned}$$

The Sasakian structure on S^{2n+1}_{2n}, which we defined above, induces a Sasakian structure on $\tilde{S}^{2n+1}_{2n}$. We call $\tilde{S}^{2n+1}_{2s}$ with the Sasakian structure to be the model spaces of Sasakian manifolds, and denote by $\tilde{S}^{2n+1}_{2s}(\tilde{\phi}, \tilde{\xi}, \tilde{\eta}, \tilde{g})$.

LEMMA 1. *Let* (M^{2n+1}, h) *be a pseudo-Riemannian manifold. Suppose* (M^{2n+1}, h) *is complete and of constant curvature* 1, M^{2n+1} *is simply connected and* h *is of signature* $2s$, $0\leqq s\leqq n$, $n\geqq 1$. *Then,* (M^{2n+1}, h) *is isometric to the model space* $\tilde{S}^{2n+1}_{2s}$. (cf. J. A. Wolf [10], p. 68, Theorem 2.4.9).

LEMMA 2. *Suppose we have two Sasakian manifolds* $M^{2n+1}(\phi, \xi, \eta, g)$ *and* $\bar{M}^{2n+1}(\bar{\phi}, \bar{\xi}, \bar{\eta}, \bar{g})$ *such that* M *and* $\bar{M}$ *are simply connected,* g *and* $\bar{g}$ *have the same signature. If* (M, g) *and* $(\bar{M}, \bar{g})$ *are complete and of constant curvature* 1, *then there is an isometry*

$$f\colon M \longrightarrow \bar{M}$$

such that $f_*\xi=\bar{\xi}$, $f^*\bar{\eta}=\eta$, $f_*\circ\phi=\bar{\phi}\circ f_*$; *that is,* $M(\phi, \xi, \eta, g)$ *and* $\bar{M}(\bar{\phi}, \bar{\xi}, \bar{\eta}, \bar{g})$ *are equivalent.*

PROOF. Let $x\in M$ and $\bar{x}\in\bar{M}$ be arbitrary points. Since g and $\bar{g}$ have the same signature, we can find an isometry

$$F\colon T_x(M) \longrightarrow T_{\bar{x}}(\bar{M})$$

such that $F(\xi_x) = \bar{\xi}_{\bar{x}}$, $\bar{\eta}(F(X)) = \eta(X)$ for $X\in T_x(M)$ and $F\circ\phi=\bar{\phi}\circ F$. Since M and $\bar{M}$ are simply connected, and since (M, g) and $(\bar{M}, \bar{g})$ are complete, we have a unique isometry

$$f\colon M \longrightarrow \bar{M}$$

such that $f(x)=\bar{x}$ and $f_*|T_x(M)=F$ (cf. J. A. Wolf [10], p. 61, Corollary 2.3.12). Since f is an isometry and since ξ is a Killing vector field by Proposition 1,

$f_*\xi$ is a Killing vector field on $\bar{M}$. For any tangent vector $\bar{X}$ to $\bar{M}$, we have

$$\bar{\nabla}_{\bar{X}}(f_*\xi) = f_*(\nabla_{f_*^{-1}\bar{X}}\xi) = f_*(\phi f_*^{-1}\bar{X}) .$$

Hence, for $\bar{X} \in T_{\bar{x}}(\bar{M})$, we get

$$\bar{\nabla}_{\bar{X}}(f_*\xi) = \bar{\phi}\bar{X} . \tag{18}$$

Thus, since $\bar{\xi}$ is a Killing vector field, (18), $\bar{\nabla}_{\bar{X}}\bar{\xi} = \bar{\phi}\bar{X}$ and $(f_*\xi)_{\bar{x}} = \bar{\xi}_{\bar{x}}$ imply $f_*\xi = \bar{\xi}$, and hence $f^*\bar{\eta} = \eta$. Finally, for any $X \in \mathfrak{X}(M)$ and $\bar{Y} \in \mathfrak{X}(\bar{M})$, we have

$$\begin{aligned} \bar{g}(f_* \cdot \phi X, \bar{Y}) \cdot f &= (f^*\bar{g})(\phi X, f_*^{-1}\bar{Y}) = g(\phi X, f_*^{-1}\bar{Y}) \\ &= d\eta(X, f_*^{-1}\bar{Y}) = (f^*d\bar{\eta})(X, f_*^{-1}\bar{Y}) \\ &= d\bar{\eta}(f_*X, \bar{Y}) \cdot f = \bar{g}(\bar{\phi} \cdot f_*X, \bar{Y}) \cdot f , \end{aligned}$$

showing $f_* \circ \phi = \bar{\phi} \circ f_*$.

3. *D*-homothetic deformations. Suppose we have a Sasakian manifold $M^{2n+1}(\phi, \xi, \eta, g)$. Let

$$\bar{g} = \alpha g + (\alpha^2 - \alpha)\, \eta \otimes \eta , \tag{1}$$

where α is a non-zero constant, and let

$$\bar{\xi} = (1/\alpha)\xi , \quad \bar{\eta} = \alpha\eta , \quad \bar{\phi} = \phi .$$

Then $(\bar{\phi}, \bar{\xi}, \bar{\eta}, \bar{g})$ is a Sasakian structure on $M = M^{2n+1}$, and we say that $M(\phi, \xi, \eta, g)$ is D-homothetic to $M(\bar{\phi}, \bar{\xi}, \bar{\eta}, \bar{g})$. If (M, g) is of constant ϕ-sectional curvature k, we have

$$\begin{aligned} \bar{K}(X) &= \bar{K}(X, \bar{\phi}X) \\ &= (1/\alpha)\{k - 3(\alpha - 1)\} \end{aligned} \tag{2}$$

for any non-null vector $X \in D_x$, and hence $(M, \bar{g})$ is of constant ϕ-sectional curvature $(1/\alpha)\{k-3(\alpha-1)\}$. Thus if $k \neq -3$, and if we take $\alpha = (k+3)/4$, $(M, \bar{g})$ is of constant ϕ-sectional curvature 1, and hence of constant curvature 1. (cf. S. Tanno [8], [9]). We summarize as follows:

PROROSITION 2. *A Sasakian manifold of constant ϕ-sectional curvature $k \neq -3$ is D-homothetic to a Sasakian manifold of constant curvature* 1.

Let $M=M^{2n+1}(\phi, \xi, \eta, g)$ be a Sasakian manifold.

DEFINITION. We call a geodesic $x(t)$, $\alpha < t < \beta$, to be *ξ-geodesic* (resp. *D-geodesic*) if $\phi(\dot{x}(t)) = 0$ (resp. $\eta(\dot{x}(t)) = 0$) for $\alpha < t < \beta$.

DEFINITION. We call M to be *ξ-complete* (resp. *D-complete*) if every ξ-geodesic (resp. D-geodesic) is complete.

LEMMA 1. *Let $M^{2n+1}(\phi, \xi, \eta, g)$ be a Sasakian manifold. If (M^{2n+1}, g) is complete, then $(M^{2n+1}, \bar{g})$ is ξ-* and *D-complete, where*

$$\bar{g} = \alpha g + (\alpha^2 - \alpha)\, \eta \otimes \eta\,, \quad \alpha \neq 0\,.$$

PROOF. Let $\bar{\nabla}_X$ and ∇_X denote covariant differentiations for $\bar{g}$ and g, respectively. For any vector fields X, Y and Z, we have

$$\begin{aligned} 2\bar{g}(\bar{\nabla}_X Y, Z) &= X\bar{g}(Y, Z) + Y\bar{g}(X, Z) - Z\bar{g}(X, Y) \\ &\quad + \bar{g}([X, Y], Z) + \bar{g}([Z, X], Y) + \bar{g}([Z, Y], X) \\ &= 2\alpha g(\nabla_X Y, Z) + (\alpha^2 - \alpha)\{X(\eta(Y)\,\eta(Z)) + Y(\eta(X)\,\eta(Z)) \\ &\quad - Z(\eta(X)\,\eta(Y)) + \eta([X, Y])\,\eta(Z) + \eta([Z, X])\,\eta(Y) \\ &\quad + \eta([Z, Y])\,\eta(X)\}\,. \end{aligned}$$

On the other hand, by the definition of contact metric structure,

$$\begin{aligned} 2g(\phi X, Y) &= 2d\eta(X, Y) \\ &= X\eta(Y) - Y\eta(X) - \eta([X, Y])\,. \end{aligned}$$

Hence, we have

$$\begin{aligned} \eta([X, Y]) &= X\eta(Y) - Y\eta(X) - 2g(\phi X, Y)\,, \\ \eta([Z, X]) &= Z\eta(X) - X\eta(Z) - 2g(\phi Z, X)\,, \\ \eta([Z, Y]) &= Z\eta(Y) - Y\eta(Z) - 2g(\phi Z, Y)\,. \end{aligned}$$

Thus we get

$$(3) \quad \begin{aligned} \bar{g}(\bar{\nabla}_X Y, Z) &= \alpha g(\nabla_X Y, Z) + (\alpha^2 - \alpha)\{(X\eta(Y))\,\eta(Z) - g(\phi X, Y)\,\eta(Z) \\ &\quad - g(\phi Z, X)\,\eta(Y) - g(\phi Z, Y)\,\eta(X)\}\,. \end{aligned}$$

Now, suppose $x(t)$, $\beta < t < \gamma$, be a geodesic in M^{2n+1} with respect to $\bar{g}$. Since $\bar{\xi}$ is a Killing vector field,

$$\bar{g}(\bar{\nabla}_{\dot{x}(t)}\bar{\xi}, \dot{x}(t)) = (1/2)(L(\bar{\xi})\bar{g})(\dot{x}(t), \dot{x}(t))$$
$$= 0 .$$

Hence we get

$$\dot{x}(t)\,\bar{\eta}(\dot{x}(t)) = \dot{x}(t)\,\bar{g}(\bar{\xi}, \dot{x}(t)) = 2\bar{g}(\bar{\nabla}_{\dot{x}(t)}\bar{\xi}, \dot{x}(t)) = 0 . \tag{4}$$

Since ϕ is skew symmetric with respect to g,

$$g(\phi\dot{x}(t), \dot{x}(t)) = 0 . \tag{5}$$

If we put $X = Y = \dot{x}(t)$ in (3), then (4) and (5) imply

$$\alpha g(\nabla_{\dot{x}(t)}\dot{x}(t), Z) - (\alpha^2 - \alpha)\, g(\phi Z, \dot{x}(t))\,\eta(\dot{x}(t)) = 0 . \tag{6}$$

This formula says that $x(t), \beta < t < \gamma$, is a geodesic with respect to g if $x(t)$ is either ξ-geodesic or D-geodesic with respect to $\bar{g}$. Thus, since (M^{2n+1}, g) is complete, $(M^{2n+1}, \bar{g})$ is ξ- and D-complete.

The following lemma is due to S. Tanno:

LEMMA 2. *If a simply connected Sasnkian manifold $M = M^{2n+1}(\phi, \xi, \eta, g)$ is ξ- and D-complete, and of constant curvature* 1, *then it is complete.*

PROOF. Let $\widetilde{S}$ be one of the model spaces such that the signature of $\widetilde{S}$ is the same as that of M. Let $\bar{x}(t)$, $\alpha < t < \beta$, be a geodesic in M. We want to show that the geodesic can be extended for $\alpha < t < \beta + \varepsilon$ for some $\varepsilon > 0$. We may suppose $0 \in (\alpha, \beta)$. Let us take any point $x_0 \in \widetilde{S}$. Since $\widetilde{S}$ and M are of constant curvature, we can find a local isomorphism f_0 such that $f_0(x_0) = {}^-(0)$. Let X be a tangent vector to $\widetilde{S}$ at x_0 such that $f_{0*}(X) = \dot{\bar{x}}(0)$, and let $x(t)$ be a geodesic in $\widetilde{S}$ such that $x(0) = x_0$ and $\dot{x}(0) = X$. Since $\widetilde{S}$ is complete, we can extend $x(t)$ for $-\infty < t < +\infty$. Thus we can extend the local isomorphism f_0 along $x(t)$ for $\alpha < t < \beta$, say f_1. To show that $\bar{x}(t)$ can be extended for $\alpha < t < \beta + \varepsilon$ for some $\varepsilon > 0$, it is sufficient to show that f_0 can be extended along $x(t)$ for $\alpha < t \leqq \beta$. If $x(t)$ is either ξ-geodesic or

D-geodesic it can be done, because M is ξ- and D-complete. So, we may suppose that $x(t)$ is neither ξ-geodesic nor D-geodesic. By considering a normal coordinate neighborhood of $\widetilde{S}$ at $x(\beta)$, we can find $t_1 \in (0, \beta)$ such that, there exists $Y \in T_{x(t_1)}(\widetilde{S})$ such that $\tilde{\eta}(Y)=0$ and the D-geodesic $y(t), y(0) = x(t_1)$ and $\dot{y}(0) = Y$, intersects the trajectory L of ξ passing through $x(\beta)$ at $z \in \widetilde{S}$. Since M is D-complete, we can extend f_1 along the D-geodesic $y(t)$, say f_2; especially, the domain of f_2 contains a neighborhood of z. Since M is ξ-complete, we can extend f_2 along L, say f_3; in particular, the domain of f_3 contains a neighborhood of $x(\beta)$. Since $\widetilde{S}$ and M are simply connected, these extensions are unique. Thus f_0 is extended along $x(t)$ for $\alpha < t \leqq \beta$.

4. Main theorems.

THEOREM 1. *If a Sasakian manifold* $M^{2n+1}(\phi, \xi, \eta, g)$, $n \geqq 1$, *is complete, simply connected and of constant* ϕ*-sectional curvature* $k \neq -3$, *then it is* D*-homothetic to the model space* $\widetilde{S}^{2n+1}_{2s}$ *of Sasakian manifolds, where*

$$2s = \text{the signature of } g \text{ if } k > -3\,,$$
$$2s = 2n - \text{the signature of } g \text{ if } k < -3\,.$$

PROOF. Let

$$\bar{g} = \alpha g + (\alpha^2 - \alpha)\,\eta \otimes \eta\,,$$
$$\bar{\xi} = (1/\alpha)\,\xi\,, \quad \bar{\eta} = \alpha\eta\,, \quad \bar{\phi} = \phi\,,$$
$$\alpha = (k+3)/4\,.$$

Then Proposition 2 says that $M^{2n+1}(\bar{\phi}, \bar{\xi}, \bar{\eta}, \bar{g})$ is a Sasakian manifold of constant curvature 1. According to Lemma 1 of §3, $(M^{2n+1}, \bar{g})$ is $\bar{\xi}$- and D-complete, and hence it is complete by Lemma 2 of §3. Since $(M^{2n+1}, \bar{g})$ is complete, Lemma 1 of §2 says that it is isometric to $\widetilde{S}^{2n+1}_{2s}$, where

$$2s = \text{the signature of } g \text{ if } \alpha > 0\,,$$
$$2s = 2n - \text{the signature of } g \text{ if } \alpha < 0\,.$$

It is clear that $\alpha > 0$ (resp. $\alpha < 0$) is equivalent to $k > -3$ (resp. $k < -3$). Then, Lemma 2 of §2 says that $M^{2n+1}(\bar{\phi}, \bar{\xi}, \bar{\eta}, \bar{g})$ is equivalent to the model space $\widetilde{S}^{2n+1}_{2s}$ of Sasakian manifold; that is, the Sasakian manifold $M^{2n+1}(\phi, \xi, \eta, g)$ is D-homothetic to $\widetilde{S}^{2n+1}_{2s}$.

COROLLARY. *If a Sasakian manifold* $M^{2n+1}(\phi, \xi, \eta, g)$, $n \geqq 1$, *with a Riemannian metric* g *is complete, simply connected and of constant*

ϕ-sectional curvature $k \neq -3$, then it is D-homothetic to either the unit sphere S^{2n+1} if $k > -3$ or $\tilde{S}^{2n+1}_{2n}$ if $k < -3$.

REMARK. The above Corollary was proved by S. Tanno [9] in the case of $k > -3$.

EXAMPLE. Let us consider the model space $(\tilde{S}^{2n+1}_{2n}, \tilde{g})$. $\tilde{S}^{2n+1}_{2n}$ is the universal pseudo-Riemannian covering manifold of S^{2n+1}_{2n}, which is diffeomorphic to $R^{2n} \times S^1$. Let us consider a D-homothetic deformation

$$\bar{\tilde{g}} = -\tilde{g} + 2\tilde{\eta} \otimes \tilde{\eta} ,$$

i.e., $\alpha = -1$ in (1) of §3. It is clear that $\bar{\tilde{g}}$ is a Riemannian metric of $\tilde{S}^{2n+1}_{2n}$, and (2) of §3 says that $(\tilde{S}^{2n+1}_{2n}, \bar{\tilde{g}})$ is of constant ϕ-sectional curvature -7.

THEOREM 2. *Let $M_i = M^{2n+1}_i(\phi_i, \xi_i, \eta_i, g_i)$, $i = 1, 2$, $n \geqq 1$, be complete, simply connected Sasakian manifolds. Suppose they are of the same signature $2s$ and of the same constant ϕ-sectional curvature $k \neq -3$, then they are equivalent; that is, there is an isometry*

$$f\colon M_1 \longrightarrow M_2$$

such that $f_*\xi_1 = \xi_2$, $f^*\eta_2 = \eta_1$ and $f^* \circ \phi_1 = \phi_2 \circ f_*$.

PROOF. Theorem 1 says that $\bar{M}_i = M^{2n+1}_i(\bar{\phi}_i, \bar{\xi}_i, \bar{\eta}_i, \bar{g}_i)$, $i = 1, 2$, are equivalent to $\tilde{S}^{2n+1}_{2s}$, where

$$\bar{g}_i = \alpha g_i + (\alpha^2 - \alpha)\, \eta_i \otimes \eta_i ,$$
$$\bar{\xi}_i = (1/\alpha)\, \xi_i , \quad \bar{\eta}_i = \alpha \eta_i, \quad \bar{\phi}_i = \phi_i, \qquad i = 1, 2,$$
$$\alpha = (k+3)/4 .$$

Hence, Lemma 2 of §2 implies that $\bar{M}_1$ and $\bar{M}_2$ are equivalent; that is, there is an isometry

$$f\colon \bar{M}_1 \longrightarrow \bar{M}_2$$

such that $f_*\bar{\xi}_1 = \bar{\xi}_2$, $f^*\bar{\eta}_2 = \bar{\eta}_1$ and $f_* \circ \bar{\phi}_1 = \bar{\phi}_2 \circ f_*$. Since

$$g_i = (1/\alpha)\, \bar{g}_i + ((1/\alpha^2) - (1/\alpha))\, \bar{\eta}_i \otimes \bar{\eta}_i , \qquad i = 1, 2 ,$$

f is an isometry

$$f: M_1 \longrightarrow M_2.$$

Moreover, we have

$$f_*\xi_1 = f_*(\alpha\overline{\xi_1}) = \alpha\overline{\xi_2} = \xi_2,$$
$$f^*\eta_2 = f^*((1/\alpha)\overline{\eta_2}) = (1/\alpha)\overline{\eta_1} = \eta_1,$$
$$f_*\circ\phi_1 = f_*\circ\overline{\phi}_1 = \overline{\phi}_2\circ f_* = \phi_2\circ f_*.$$

Thus f gives the equivalence of M_1 and M_2.

5. Sasakian manifold with $\boldsymbol{R(X, Y)\cdot R = 0}$. Let $M^{2n+1}(\phi, \xi, \eta, g)$ be a Sasakian manifold. Then, by the definition of Sasakian manifold, we get

(1)
$$\begin{aligned} R(X, \xi)Y &= \nabla_X\nabla_Y\xi - \nabla_{\nabla_X Y}\xi \quad (\because \xi \text{ is a Killing vector field}) \\ &= \nabla_X(\phi Y) - \phi(\nabla_X Y) \\ &= (\nabla_X\phi)Y + \phi(\nabla_X Y) - \phi(\nabla_X Y) \\ &= \eta(Y)X - g(X, Y)\xi, \end{aligned}$$

(2)
$$\begin{aligned} R(X, Y)\xi &= \nabla_X\nabla_Y\xi - \nabla_Y\nabla_X\xi - \nabla_{[X,Y]}\xi \\ &= \nabla_X(\phi Y) - \nabla_Y(\phi X) - \phi([X, Y]) \\ &= (\nabla_X\phi)Y + \phi(\nabla_X Y) - (\nabla_Y\phi)X - \phi(\nabla_Y X) - \phi([X,Y]) \\ &= \eta(Y)X - g(X, Y)\xi - (\eta(X)Y - g(Y, X)\xi) \\ &= \eta(Y)X - \eta(X)Y \end{aligned}$$

for any vector fields X and Y. Suppose $R(X, Y)\cdot R=0$ for all tangent vectors X and Y, where $R(X, Y)$ operates on R as a derivation of the tensor algebra at each point. Now, let X and Y be tangent vectors such that $\eta(X)=\eta(Y)=0$ and $g(X, Y) = 0$. Then, using (1) and (2) above,

$$\begin{aligned} &(R(X, \xi)\cdot R)(X, Y)Y \\ &= R(X, \xi)R(X, Y)Y - R(R(X, \xi)X, Y)Y - R(X, R(X, \xi)Y)Y - R(X, Y)R(X, \xi)Y \\ &= \eta(R(X, Y)Y)X - g(X, R(X, Y)Y)\xi - R(\eta(X)X - g(X, X)\xi, Y)Y \\ &\quad - R(X, \eta(Y)X - g(X, Y)\xi)Y - R(X, Y)(\eta(Y)X - g(X, Y)\xi) \\ &= \eta(R(X, Y)Y)X - g(X, R(X, Y)Y)\xi + g(X, X)R(\xi, Y)Y \end{aligned}$$

$$= \eta(R(X,Y)Y)X - g(X,R(X,Y)Y)\xi - g(X,X)\eta(Y)Y + g(X,X)g(Y,Y)\xi.$$

Hence,

$$\eta(R(X,Y)Y)X - g(X,R(X,Y)Y)\xi + g(X,X)g(Y,Y)\xi = 0. \tag{3}$$

Thus, considering ξ-component of (3), we get

$$g(X,R(X,Y)Y) = g(X,X)g(Y,Y),$$

showing that (M^{2n+1},g) is of constant ϕ-sectional curvature 1, and hence it is of constant curvature 1.

THEOREM 3. *A Sasakian manifold satisfying* $R(X,Y)\cdot R=0$ *for all tangent vectors* X *and* Y *is of constant curvature* 1.

6. Sasakian manifold M^{2n+1} which is isometrically immersed in E_s^{2n+2}. Let E_s^n be a Euclidean space R^n with a pseudo-Riemannian metric $\widetilde{g}_s$ which is defined by the parallel displacement of the "inner product"

$$<x,y> = -\sum_{i=1}^{s} x^i y^i + \sum_{j=s+1}^{n} x^j y^j.$$

Then the signature of $\widetilde{g}_s$ is s, and E_s^n is complete and of constant curvature 0 (cf. J. A. Wolf [10], §2. 4).

Let $M^{2n+1}(\phi, \xi, \eta, g)$ be a Sasakian manifold. Suppose we have an isometric immersion

$$f: M^{2n+1} \longrightarrow E_s^{2n+2}.$$

For each $x \in M^{2n+1}$, we can choose a unit vector field ζ normal to M^{2n+1} on some neighborhood U of x:

$$\widetilde{g}_s(\zeta,\zeta) = \varepsilon, \quad \varepsilon = 1 \quad \text{or} \quad -1 \quad \text{on } U.$$

For any vector fields X and Y on U tangent to M^{2n+1}, we have the formulas of Gauss and Weingarten:

$$D_X Y = \nabla_X Y + \varepsilon h(X,Y)\zeta,$$

$$D_X \zeta = -AX,$$

where D_X and ∇_X denote covariant differentiations for $\tilde{g}_s$ and g, respectively. A is a field of symmetric endomorphisms which corresponds to the second fundamental form h, that is, $h(X, Y) = g(AX, Y)$ for all tangent vectors X and Y. The equation of Gauss expresses the curvature tensor R of M^{2n+1} by means of A:

$$(1) \qquad R(X, Y)Z = \varepsilon\{g(Z, AY)AX - g(Z, AX)AY\}.$$

This equation implies

$$(2) \qquad R(X, \xi)Y = \varepsilon\{\eta(AY)AX - g(AX, Y)A\xi\}.$$

On the other hand, we have (1) of §5:

$$(3) \qquad R(X, \xi)Y = \eta(Y)X - g(X, Y)\xi.$$

Suppose the isometric immersion $f: M^{2n+1} \longrightarrow E_s^{2n+2}$ is proper, that is, A can be expressed by a real diagonal matrix with respect to a certain orthonormal frame at each point of M^{2n+1} (cf. A. Fialkow [2], p.764). Let $\{e_1, e_2, \cdots, e_{2n+1}\}$ be an orthonormal basis of $T_{x_0}(M^{2n+1})$ such that A is expressed by a diagonal matrix with respect to $\{e_1, e_2, \cdots e_{2n+1}\}$, i.e.,

$$(4) \qquad Ae_i = \rho_i e_i, \quad 1 \leqq i \leqq 2n+1, \quad \rho_i \in R.$$

(2), (3) with $X = e_i$, $Y = e_j$ and (4) imply

$$(5) \qquad \eta(e_j)e_i - g(e_i, e_j)\,\xi = \varepsilon\{\rho_i\rho_j\eta(e_j)e_i - \rho_i g(e_i, e_j)A\xi\}.$$

If $i \neq j$, (5) implies

$$\eta(e_j)e_i = \varepsilon\rho_i\rho_j\eta(e_j)e_i.$$

Hence $\varepsilon\rho_i\rho_j = 1$ for all $i \neq j$, or $\eta(e_j) = 0$ for some j.

(a) Suppose $\varepsilon\rho_i\rho_j = 1$ for all $i \neq j$. Then $\rho_i \neq 0$ for all i, and $\rho_1 = \rho_2 = \cdots = \rho_{2n+1} = \rho$. Thus $\varepsilon\rho^2 = 1$. This implies $\varepsilon = 1$ and $\rho^2 = 1$.

(b) Suppose $\eta(e_{j_0}) = 0$ for some j_0. Then (5) implies

$$\xi = \varepsilon\rho_{j_0}A\xi.$$

Hence $\rho_{j_0} \neq 0$ and $A\xi = (1/\varepsilon\rho_{j_0})\xi$, i. e, ξ is an eigenvector of A with eigenvalue

$1/\varepsilon\rho_{j_0}$. We may suppose $e_1 = \xi$, and hence $\eta(e_i) = 0$ for $2 \leqq i \leqq 2n+1$. (2) implies

$$K(e_i, \xi) = \varepsilon\rho_1\rho_i\,,$$

(3) implies

$$K(e_i, \xi) = 1$$

for $2 \leqq i \leqq 2n+1$. Hence we get $\rho_1\rho_i = \varepsilon$ for $2 \leqq i \leqq 2n+1$, and hence $\rho_2=\rho_3 = \cdots = \rho_{2n+1} = \rho$. Consequently, $AX = \rho X$ for any tangent vector X such that $\eta(X)=0$. Thus (1) implies (M^{2n+1}, g) is of constant ϕ-sectional curvature $\varepsilon\rho^2$, hence we have (3) of §1 with $k = \varepsilon\rho^2$. Now, if we assume $n \geqq 2$, we can find non-null tangent vectors X and Y such that $\eta(X) = \eta(Y) = 0$, $g(X, Y) = 0$ and $g(\phi X, Y) = 0$. Then (3) of §1 and (1) of this section give

$$4R(X, Y)X = -(k+3)\, g(X, X)Y$$

and

$$R(X, Y)X = -\varepsilon\rho^2 g(X, X)Y\,,$$

respectively. Hence we get

$$\frac{k+3}{4} = \varepsilon\rho^2.$$

Since $k = \varepsilon\rho^2$, this equation implies $\varepsilon\rho^2 = 1$, that is, $\rho^2 = \varepsilon$. Hence $\varepsilon = 1$ and $\rho^2=1$. Since $\rho_1\rho = \varepsilon$, we get $\rho_1 = \rho$.

Summarizing (a) and (b), if $n \geqq 2$, we have $\varepsilon = 1$, $A = \rho$ and $\rho^2 = 1$. We may suppose $\rho=1$, since the change $\zeta \longrightarrow -\zeta$ implies $A \longrightarrow -A$, $\rho=(1/(2n+1))\cdot \operatorname{Tr} A$ is a differentiable function on U.

Now, let us suppose $n \geqq 2$. Consider the R^{2n+2}-valued function

$$x \in U \subset M^{2n+1} \longrightarrow \zeta_x + f(x) \in R^{2n+2}\,.$$

For any tangent vector X to M^{2n+1}, we have

$$\begin{aligned} D_{f_*X}(\zeta+f) &= f_*(-AX+X) \\ &= 0\,. \end{aligned}$$

This implies that $\zeta+f$ is a constant map $M^{2n+1} \longrightarrow \alpha \in R^{2n+2}$, and hence

$$<f(x)-\alpha, f(x)-\alpha> = <\zeta_x, \zeta_x> \\ = 1$$

for $x \in U$. Thus $f(U)$ lies on the hypersurface $S_s^{2n+1}(\alpha)$, which is the hypersurface S_s^{2n+1} translated by the parallel translation $\beta \longrightarrow \alpha+\beta$, $\beta \in R^{2n+2}$. Let $M' = \{x \in M^{2n+1} : f(x) \in S_s^{2n+1}(\alpha)\}$. Then the above argument says that M' is open. Similarly, $M^{2n+1}-M'$ is open, showing M' to be closed. Thus, since M^{2n+1} is connected, $M'=M^{2n+1}$, i.e., $f(M^{2n+1})$ lies on $S_s^{2n+1}(\alpha)$. In particular, (M^{2n+1}, g) is of constant curvature 1.

THEOREM 4. *Suppose we have a complete Sasakian manifold* $M^{2n+1}(\phi, \xi, \eta, g)$, $n \geqq 2$, *which is properly and isometrically immersed in* E_s^{2n+2}. *Then*

(i) *if* $0 \leqq s \leqq 2n-1$, *then* s *is even, the immersion is an isometric imbedding and* $M^{2n+1}(\phi, \xi, \eta, g)$ *is equivalent to* $\tilde{S}_s^{2n+1}$,

(ii) *if* $2n \leqq s \leqq 2n+2$, *then* $s=2n$ *and* $M^{2n+1}(\phi, \xi, \eta, g)$ *is a pseudo-Riemannian covering manifold of* S_{2n}^{2n+1} *and the immersion induces the covering projection, naturally.*

We need the following Lemma:

LEMMA. *Let* $M_1=(M_1^n, h_1)$ *and* $M_2=(M_2^n, h_2)$ *be pseudo-Riemannian manifolds with the same dimension and signature. Suppose* M_1 *and* M_2 *are of the same constant curvature* k, *and suppose we have an isometric immersion*

$$f: M_1 \longrightarrow M_2 .$$

Then, if M_1 *is complete,* M_2 *is also complete and the isometric immersion* f *is a covering projection* (cf. S. Kobayashi-K. Nomizu [3], Theorem 4.6).

PROOF. Let y_2 be an arbitrary point of M_2. Let us take $x_1 \in M_1$ and let $x_2 = f(x_1)$. Then we can join x_2 and y_2 by a broken geodesic L_2. Since M_1 is complete, there is a broken geodesic L_1 in M_1 such that $f(L_1)=L_2$, showing that f is an onto mapping.

Let $x_2(t)$, $\alpha<t<\beta$, be a geodesic in M_2. Then, since M_1 is complete, we have a geodesic $x_1(t)$, $-\infty<t<+\infty$, such that $f(x_1(t))=x_2(t)$ for $\alpha<t<\beta$. Since f is an isometric immersion, there is a neighborhood U of $x_1(\alpha)$ (resp. $x_1(\beta)$) such that $f|U$ is an isometry of U onto $f(U)$ which is a neighborhood of $f(x_1(\alpha))$ (resp. $f(x_1(\beta))$). Thus the geodesic $x_2(t)$, $\alpha < t < \beta$, can be extended for $\alpha - \varepsilon' < t < \beta + \varepsilon''$ for some positive constants ε' and ε'', showing M_2 to be complete.

Let us consider the universal pseudo-Riemannian covering manifolds $\widetilde{M}_1$ and $\widetilde{M}_2$ of M_1 and M_2 with projections p_1 and p_2, respectively. Let x_1 be an arbitrary point of M_1, choose $y_1 \in p_1^{-1}(x_1)$ and $y_2 \in p_2^{-1}(f(x_1))$. Let V_{y_1}, U_{x_1}, $U_{f(x_1)}$ and V_{y_2} be neighborhoods of y_1, x_1, $f(x_1)$ and y_2, respectively, such that p_1, f and p_2 are isometries of V_{y_1}, U_{x_1} and V_{y_2} onto U_{x_1}, $U_{f(x_1)}$ and $U_{f(x_1)}$, respectively. Then we have an isometry

$$F = p_2^{-1} f p_1 : V_{y_1} \longrightarrow V_{y_2} .$$

Since $\widetilde{M}_1$ and $\widetilde{M}_2$ are complete, simply connected and of constant curvature k, the local isometry F has a unique extension, say F; that is, an isometry F: $\widetilde{M}_1 \longrightarrow \widetilde{M}_2$. Since this extension can be done along all (broken) geodesics passing through y_1, we have

$$p_2 \circ F = f \circ p_1 ,$$

which shows that f is a covering projection, since f is a continuous and open mapping.

PROOF OF THEOREM 4. The above Lemma says that the isometric immersion is a covering projection $M^{2n+1} \longrightarrow S_s^{2n+1}(\alpha)$. If $0 \leqq s \leqq 2n-1$, s is even, then $S_s^{2n+1}(\alpha)$ is simply connected, hence the covering projection is an isometry. Thus the Theorem follows from Lemma 2 of §2.

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