ON CONTACT STRUCTURES OF TANGENT SPHERE BUNDLES

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Introduction. The present author [6] has shown that an orientable hypersurface in an almost complex manifold can be given an induced almost contact structure, and studied conditions for the induced structure to be normal.

On the other hand, K.Yano and S.Kobayashi [8] and K.Yano and S.Ishihara [9] have introduced the notions of vertical, complete and horizontal lifts of tensor fields and connections to tangent bundles. Above all, if the base manifold is almost complex, then the complete or horizontal lift of the structure defines an almost complex structure in the tangent bundle. It is also known, cf. [5], that the tangent bundle of a Riemannian manifold is given an almost Kählerian structure.[1]

Basing on these two kinds of results, we shall be able to induce various kinds of almost contact structures into tangent sphere bundles. In this paper, we shall consider a class of hypersurfaces with some property in tangent bundles; the class contains the tangent sphere bundle of a manifold.

In §§1 to 3, we shall introduce a tensor field for the study of hypersurfaces in almost complex manifolds and state some theorems on normality of induced almost contact structures in the language of the tensor field. In §§4 to 8, various kinds of almost contact structures will be induced to hypersurfaces in tangent bundles and the normality of the structures will be discussed. In §8, we shall show that, given the almost Kählerien structure in the tangent bundle of a Riemannian manifold, the induced almost contact structure of the tangent unit-sphere bundule is K-contact if and only if the base manifold is of positive constant curvature.

1. Almost complex and almost contact structures.[2] Let M be an almost complex manifold of even dimension $2n$. Denote by (x^A)[3] a local coordinate system and by $F=(F_B{}^A)$ the tensor field of the almost complex structure, which satisfies the equation

$$(1.1) \qquad\qquad F_C{}^B F_B{}^A = -\delta_C^A,$$

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1) As to differeerential geometry of tangent bundles, we refer K. Yano [10].
2) Refer [7] as to almost complex structures and [1, 2, 3, 4] as to almost contact structures.
3) In §§1 to 3, indices A, B, C, D, E run from 1 to $2n$, and $\alpha, \beta, \gamma, \delta, \varepsilon$ from 1 to $2n-1$.

$E=(\delta_B^A)$ being the unit tensor field in M. The $(1,2)$-tensor field $N=(N_{CB}{}^A)$ defined by

$$(1.2) \qquad N_{CB}{}^A = F_C{}^E(\partial_E F_B{}^A - \partial_B F_E{}^A) - F_B{}^E(\partial_E F_C{}^A - \partial_C F_E{}^A)$$

is called the *Nijenhuis tensor* of the almost complex structure F. As is well known, the structure F or the manifold M is *complex* if and only if N vanishes.

If $\Gamma=(\Gamma_{CB}^A)$ is a symmetric affine connection of M and we denote by ∇ covariant differentiation with respect to the connection, then the Nijenhuis tensor N is written as

$$(1.3) \qquad N_{CB}{}^A = F_C{}^E(\nabla_E F_B{}^A - \nabla_B F_E{}^A) - F_B{}^E(\nabla_E F_C{}^A - \nabla_C F_E{}^A).$$

A necessary and sufficient condition for F to be complex is that there exists a symmetric affine connection in M such as

$$(1.4) \qquad \nabla_C F_B{}^A = 0.$$

In an almost complex manifold M, there is a Riemannian metric $g=(g_{BA})$ satisfying the relation

$$(1.5) \qquad g_{BA} = F_B{}^D F_A{}^C g_{DC}.$$

Such a structure (F,g) or an almost complex manifold M with metric g is said to be *almost Hermitian*. The covariant tensor field (F_{BA}) given by

$$(1.6) \qquad F_{BA} = F_B{}^C g_{CA}$$

is skew symmetric. We put

$$(1.7) \qquad \Theta = (1/2)F_{BA}dx^B \wedge dx^A$$

and call it the *fundamental form* of M. If the form Θ is closed, that is,

$$(1.8) \qquad \partial_C F_{BA} + \partial_B F_{AC} + \partial_A F_{CB} = 0,$$

then the metric structure (F,g) or the manifold M is said to be *almost Kählerian*. The left hand side of (1.8) is put and rewritten as

$$F_{CBA} = \partial_C F_{BA} + \partial_B F_{AC} + \partial_A F_{CB}$$
$$= \nabla_C F_{BA} + \nabla_B F_{AC} + \nabla_A F_{CB},$$

where ∇ indicates covariant differentiation with respect to the Riemannian connection of g. If F is complex, then an almost Hermitian or almost Kählerian manifold turns to the so-called *Hermitian* or *Kählerian* manifold, respectively. A necessary and sufficient condition for an almost Hermitian manifold to be Kählerian is that the Riemannian connection satisfies $\nabla_C F_B{}^A = 0$.

Next let $\bar{M}$ be an almost contact manifold of odd dimension $2n-1$. Denote by (u^α) a local coordinate system and by (f, ξ, η) the almost contact structure, where $f = (.f_\alpha{}^\beta)$ is a $(1,1)$-tensor field of rank $2n-2$, $\xi = (\xi^\alpha)$ a contravariant vector field and $\eta = (\eta_\beta)$ a covariant vector field, and they satisfy the relations

$$(1.9) \qquad \left\{ \begin{array}{l} f_\beta{}^\varepsilon f_\varepsilon{}^\alpha - \eta_\beta \xi^\alpha = -\delta_\beta^\alpha, \\[2mm] \xi^\varepsilon f_\varepsilon{}^\alpha = 0, \quad f_\beta{}^\varepsilon \eta_\varepsilon = 0, \quad \xi^\alpha \eta_\alpha = 1. \end{array} \right.$$

There are defined the following tensor fields in $\bar{M}$:

$$(1.10) \qquad \left\{ \begin{array}{l} \bar{N}_{\gamma\beta}{}^\alpha = f_\gamma{}^\varepsilon (\partial_\varepsilon f_\beta{}^\alpha - \partial_\varepsilon f_\beta{}^\alpha) - f_\beta{}^\varepsilon (\partial_\varepsilon f_\gamma{}^\alpha - \partial_\gamma f_\varepsilon{}^\alpha) \\[2mm] \qquad\qquad + \eta_\gamma \partial_\beta \xi^\alpha - \eta_\beta \partial_\gamma \xi^\alpha, \\[3mm] \bar{N}_{\gamma\beta} = -f_\gamma{}^\varepsilon (\partial_\varepsilon \eta_\beta - \partial_\beta \eta_\varepsilon) + f_\beta{}^\varepsilon (\partial_\varepsilon \eta_\gamma - \partial_\gamma \eta_\varepsilon), \\[3mm] \bar{N}_\beta{}^\alpha = \xi^\varepsilon (\partial_\varepsilon f_\beta{}^\alpha - \partial_\beta f_\varepsilon{}^\alpha) - f_\beta{}^\varepsilon \partial_\varepsilon \xi^\alpha, \\[3mm] \bar{N}_\beta = \xi^\varepsilon (\partial_\varepsilon \eta_\beta - \partial_\beta \eta_\varepsilon). \end{array} \right.$$

If $\bar{\Gamma} = (\bar{\Gamma}_{\gamma\beta}^\alpha)$ is a symmetric affine connection in $\bar{M}$ and we denote by $\bar{\nabla}$ covariant differentiation with respect to the connection, the above tensor fields are written as

$$(1.11) \qquad \left\{ \begin{array}{l} \bar{N}_{\gamma\beta}{}^\alpha = f_\gamma{}^\varepsilon (\bar{\nabla}_\varepsilon f_\beta{}^\alpha - \bar{\nabla}_\beta f_\varepsilon{}^\alpha) - f_\beta{}^\varepsilon (\bar{\nabla}_\varepsilon f_\gamma{}^\alpha - \bar{\nabla}_\gamma f_\alpha{}^\varepsilon) \\[2mm] \qquad\qquad + \eta_\gamma \bar{\nabla}_\beta \xi^\alpha - \eta_\beta \bar{\nabla}_\gamma \xi^\alpha, \\[3mm] \bar{N}_{\gamma\beta} = -f_\gamma{}^\varepsilon (\bar{\nabla}_\varepsilon \eta_\beta - \bar{\nabla}_\beta \eta_\varepsilon) + f_\beta{}^\varepsilon (\bar{\nabla}_\varepsilon \eta_\gamma - \bar{\nabla}_\gamma \eta_\varepsilon), \\[3mm] \bar{N}_\beta{}^\alpha = \xi^\varepsilon (\bar{\nabla}_\varepsilon f_\beta{}^\alpha - \bar{\nabla}_\beta f_\varepsilon{}^\alpha) - f_\beta{}^\varepsilon \bar{\nabla}_\varepsilon \xi^\alpha, \\[3mm] \bar{N}_\beta = \xi^\varepsilon (\bar{\nabla}_\varepsilon \eta_\beta - \bar{\nabla}_\beta \eta_\varepsilon). \end{array} \right.$$

If we denote by $\mathcal{L}(\xi)$ Lie differentiation with respect to ξ in $\bar{M}$ and take account of (1.9), then the third and fourth tensor fields of (1.10) can be written as

$$(1.12) \qquad \left\{ \begin{array}{l} \bar{N}_\beta{}^\alpha = \xi^\varepsilon \partial_\varepsilon f_\beta{}^\alpha + (\partial_\beta \xi^\varepsilon) f_\varepsilon{}^\alpha - (\partial_\varepsilon \xi^\alpha) f_\beta{}^\varepsilon = \mathcal{L}(\xi) f_\beta{}^\alpha, \\[3mm] \bar{N}_\beta = \xi^\varepsilon \partial_\varepsilon \eta_\beta + (\partial_\beta \xi^\varepsilon) \eta_\varepsilon = \mathcal{L}(\xi) \eta_\beta. \end{array} \right.$$

The first tensor field $\overline{N}=(\overline{N}_{\gamma\beta}{}^\alpha)$ is called the *Nijenhuis tensor* of the almost contact structure. If $\overline{N}$ does vanish, then so do the other three and the almost contact structure is said to be *normal*.

In an almost contact manifold, there is a Riemannian metric $\overline{g}=(\overline{g}_{\beta\alpha})$ satisfying the relations

$$(1.13) \qquad \begin{cases} \overline{g}_{\beta\alpha} - \eta_\beta\eta_\alpha = f_\beta{}^\delta f_\alpha{}^\gamma \overline{g}_{\delta\gamma}, \\ \eta_\beta = \overline{g}_{\beta\alpha}\xi^\alpha. \end{cases}$$

The structure $(f,\xi,\eta,\overline{g})$ of an almost contact structure associated with metric $\overline{g}$ is called an *almost contact metric structure* of $\overline{M}$. The covariant tensor field $(f_{\beta\alpha})$ given by

$$(1.14) \qquad f_{\beta\alpha} = f_\beta{}^\gamma \overline{g}_{\gamma\alpha}$$

is skew symmetric, and we put

$$(1.15) \qquad \theta_1 = \eta_\alpha du^\alpha, \ \theta_2 = (1/2)f_{\beta\alpha}du^\beta \wedge du^\alpha.$$

If $d\theta_1 = \theta_2$, or

$$(1.16) \qquad f_{\beta\alpha} = \partial_\beta\eta_\alpha - \partial_\alpha\eta_\beta = \overline{\nabla}_\beta\eta_\alpha - \overline{\nabla}_\alpha\eta_\beta,$$

where $\overline{\nabla}$ indicates covariant differentiation with respect to the Riemannian connection of $\overline{g}$, then the structure $(f,\xi,\eta,\overline{g})$ or the manifold $\overline{M}$ is said to be *contact metric*. If, in addition, the vector field ξ is a Killing vector field with respect to the metric $\overline{g}$, that is,

$$(1.17) \qquad \overline{\mathcal{L}}(\xi)\overline{g}_{\beta\alpha} = \overline{\nabla}_\beta\eta_\alpha + \overline{\nabla}_\alpha\eta_\beta = 0,$$

then the structure or the manifold is said to be *K-contact metric*, and we have the equation

$$(1.18) \qquad f_{\beta\alpha} = 2\overline{\nabla}_\beta\eta_\alpha.$$

If a contact metric structure is normal, then the structure $(f,\xi,\eta,\overline{g})$ is said to be *Sasakian* and it is characterized by the equations

$$(1.19) \qquad \begin{cases} 2\overline{\nabla}_\beta\eta_\alpha = f_{\beta\alpha}, \\ 2\overline{\nabla}_\gamma f_{\beta\alpha} = \eta_\beta \overline{g}_{\gamma\alpha} - \eta_\alpha \overline{g}_{\gamma\beta}. \end{cases}$$

The first equation means that a Sasakian structure is a special one of K-contact structures.

2. Almost contact structure of a hypersurface in an almost complex manifold.

Now we consider an almost complex manifold M with structure tensor F and an orientable hypersurface $\bar{M}$ in M. Assume that $\bar{M}$ is locally represented by parametric equations

$$(2.1) \qquad x^A = x^A(u^\alpha).$$

We put

$$(2.2) \qquad B_\alpha{}^A = \partial_\alpha x^A,$$

which span the tangent hyperplane of $\bar{M}$ at each point. Further we choose a vector field $C=(C^A)$ complementary to the tangent hyperplane of $\bar{M}$ at each point, and call it a *pseudo-normal* to $\bar{M}$. The matrix $\begin{pmatrix} B_\alpha^A \\ C^A \end{pmatrix}$ is regular and the inverse matrix will be denoted by $(B^\alpha{}_B, C_B)$. Then we have the equations

$$(2.3) \qquad \begin{cases} B_\beta{}^A B^\alpha{}_A = \delta_\beta^\alpha, \; C^A B^\alpha{}_A = 0, \\ B_\beta{}^A C_A = 0, \quad C^A C_A = 1 \end{cases}$$

and

$$(2.4) \qquad B_\alpha{}^A B^\alpha{}_B + C^A C_B = \delta_B^A.$$

For an arbitrary $(1,1)$-tensor field $F=(F_B{}^A)$ in M, we put

$$(2.5) \qquad \begin{aligned} f_\beta{}^\alpha &= B_\beta{}^B F_B{}^A B_A{}^\alpha, \;\; f_\infty{}^\alpha = C^B F_B{}^A B^\alpha{}_A, \\ f_\beta{}^\infty &= B_\beta{}^B F_B{}^A C_A, \;\; f_\infty{}^\infty = C^B F_B{}^A C_A. \end{aligned}$$

If, in particular, F is an almost complex structure in M, then the pseudo-normal vector field $C=(C^A)$ can be chosen such as

$$(2.6) \qquad f_\infty{}^\infty = C^B F_B{}^A C_A = 0.$$

Indeed, since there is an almost Hermitian metric g associated with F, we may take C as the unit normal vector field to the hypersurface $\bar{M}$ with respect to g. If we once choose such a pseudo-normal vector C and put

$$(2.7) \quad \begin{cases} \xi^\alpha = f_\infty{}^\alpha = C^B F_B{}^A B_A{}^\alpha, \\ \eta_\beta = -f_\beta{}^\infty = -B_\beta{}^B F_B{}^A C_A, \end{cases}$$

then we see that (f, ξ, η) defines an almost contact structure in the hypersurface $\bar{M}$, which will be said to be *induced* in $\bar{M}$ from an almost complex structure F in M, [6].

Given a symmetric affine connection $\Gamma = (\Gamma_{CB}^A)$ in M, the induced connection $\bar{\Gamma} = (\bar{\Gamma}_{\gamma\beta}^\alpha)$ in the hypersurface $\bar{M}$ is defined by

$$(2.8) \quad \bar{\Gamma}_{\gamma\beta}^\alpha = (\partial_\gamma B_\beta{}^A + \Gamma_{CB}^A B_\gamma{}^C B_\beta{}^B) B^\alpha{}_A,$$

and the tensor fields $h = (h_{\gamma\beta})$, $l = (l_\gamma{}^\alpha)$ and $m = (m_\gamma)$ in $\bar{M}$ by

$$(2.9) \quad \begin{cases} h_{\gamma\beta} = (\partial_\gamma B_\beta{}^A + \Gamma_{CB}^A B_\gamma{}^C B_\beta{}^B) C_A, \\ l_\gamma{}^\alpha = (\partial_\gamma C^A + \Gamma_{CB}^A B_\gamma{}^C C^B) B^\alpha{}_A, \\ m_\gamma = (\partial_\gamma C^A + \Gamma_{CB}^A B_\gamma{}^C C^B) C_A, \end{cases}$$

respectively. Then the so-called van der Waerden-Bortolloti covariant derivatives of $B_\beta{}^A$ and C^A are expressed by

$$(2.10) \quad \begin{cases} \nabla_\gamma B_\beta{}^A = \partial_\gamma B_\beta{}^A + B_\gamma{}^C B_\beta{}^B \Gamma_{CB}^A - \bar{\Gamma}_{\gamma\beta}^\alpha B_\alpha{}^A = h_{\gamma\beta} C^A, \\ \nabla_\gamma C^A = \partial_\gamma C^A + B_\gamma{}^C C^B \Gamma_{CB}^A = l_\gamma{}^\alpha B_\alpha{}^A + m_\gamma C^A, \end{cases}$$

and those of $B^\alpha{}_B$ and C_B by

$$(2.11) \quad \begin{cases} \nabla_\gamma B^\alpha{}_B = -l_\gamma{}^\alpha C_B, \\ \nabla_\gamma C_B = -h_{\gamma\beta} B^\beta{}_B - m_\gamma C_B. \end{cases}$$

For any $(1, 1)$-tensor field F in M, the covariant derivatives of the four tensor fields defined by (2.5) with respect to the induced connection $\bar{\Gamma}$ are written in the forms

$$(2.12) \quad \begin{cases} \bar{\nabla}_\gamma f_\beta{}^\alpha = B_\gamma{}^C B_\beta{}^B B^\alpha{}_A \nabla_C F_B{}^A - l_\gamma{}^\alpha f_\beta{}^\infty + h_{\gamma\beta} f_\infty{}^\alpha, \\ \bar{\nabla}_\gamma f_\beta{}^\infty = B_\gamma{}^C B_\beta{}^B C_A \nabla_C F_B{}^A - h_{\gamma\alpha} f_\beta{}^\alpha - m_\gamma f_\beta{}^\infty + h_{\gamma\beta} f_\infty{}^\infty, \\ \bar{\nabla}_\gamma f_\infty{}^\alpha = B_\gamma{}^C C^B B^\alpha{}_A \nabla_C F_B{}^A + l_\gamma{}^\alpha f_\beta{}^\alpha + m_\gamma f_\infty{}^\alpha - l_\gamma{}^\alpha f_\infty{}^\infty, \\ \bar{\nabla}_\gamma f_\infty{}^\infty = B_\gamma{}^C C^B C_A \nabla_C F_B{}^A + l_\gamma{}^\beta f_\beta{}^\infty - h_{\gamma\alpha} f_\infty{}^\alpha. \end{cases}$$

If F is an almost complex structure in M, then the covariant derivatives of the induced almost contact structure (f, ξ, η) in $\overline{M}$ are given by

$$(2.13) \quad \left\{ \begin{array}{l} \overline{\nabla}_\gamma f_\beta{}^\alpha = B_\gamma{}^C B_\beta{}^B B^\alpha{}_A \nabla_C F_B{}^A + l_\gamma{}^\alpha \eta_\beta + h_{\gamma\beta} \xi^\alpha, \\[2mm] \overline{\nabla}_\gamma \eta_\beta = - B_\gamma{}^C B_\beta{}^B C_A \nabla_C F_B{}^A + h_{\gamma\alpha} f_\beta{}^\alpha - m_\gamma \eta_\beta, \\[2mm] \overline{\nabla}_\gamma \xi^\alpha = B_\gamma{}^C C^B B^\alpha{}_A \nabla_C F_B{}^A + l_\gamma{}^\beta f_\beta{}^\alpha + m_\gamma \xi^\alpha, \end{array} \right.$$

and we have

$$(2.14) \quad B_\gamma{}^C C^B C_A \nabla_C F_B{}^A = l_\gamma{}^\beta \eta_\beta + h_{\gamma\alpha} \xi^\alpha.$$

Substituting these into (1.11), we obtain the equations

$$(2.15) \quad \left\{ \begin{array}{l} \overline{N}_{\gamma\beta}{}^\alpha = B_\gamma{}^C B_\beta{}^B B^\alpha{}_A N_{CB}{}^A + \eta_\gamma C^E B_\beta{}^B B^\alpha{}_A \nabla_E F_B{}^A - \eta_\beta C^E B_\gamma{}^C B^\alpha{}_A \nabla_E F_C{}^A \\[2mm] \qquad + (f_\gamma{}^\varepsilon l_\varepsilon{}^\alpha - l_\gamma{}^\varepsilon f_\varepsilon{}^\alpha - m_\gamma \xi^\alpha) \eta_\beta - (f_\beta{}^\varepsilon l_\varepsilon{}^\alpha - l_\beta{}^\varepsilon f_\varepsilon{}^\alpha - m_\beta \xi^\alpha) \eta_\gamma, \\[2mm] \overline{N}_{\gamma\beta} = B_\gamma{}^C B_\beta{}^B C_A N_{CB}{}^A + \eta_\gamma C^E B_\beta{}^B C_A \nabla_E F_B{}^A - \eta_\beta C^E B_\gamma{}^C C_A \nabla_E F_C{}^A \\[2mm] \qquad + (l_\gamma{}^\alpha \eta_\alpha + f_\gamma{}^\alpha m_\alpha) \eta_\beta - (l_\beta{}^\alpha \eta_\alpha + f_\beta{}^\alpha m_\alpha) \eta_\gamma, \\[2mm] \overline{N}_\beta{}^\alpha = C^C B_\beta{}^B B^\alpha{}_A N_{CB}{}^A - f_\beta{}^\varepsilon C^C B_\varepsilon{}^B B^\alpha{}_A \nabla_C F_B{}^A \\[2mm] \qquad - (l_\beta{}^\alpha + f_\beta{}^\varepsilon l_\varepsilon{}^\delta f_\delta{}^\alpha + f_\beta{}^\varepsilon m_\varepsilon \xi^\alpha - \eta_\beta \xi^\varepsilon l_\varepsilon{}^\alpha), \\[2mm] \overline{N}_\beta = - C^C B_\beta{}^B C_A N_{CB}{}^{\cdot A} + f_\beta{}^\varepsilon C^C B_\varepsilon{}^B C_A \nabla_C F_B{}^A - (f_\beta{}^\varepsilon l_\varepsilon{}^\alpha \eta_\alpha + \eta_\beta \xi^\alpha m_\alpha - m_\beta). \end{array} \right.$$

If we put

$$(2.16) \quad L_\beta{}^\alpha = l_\beta{}^\alpha - \eta_\beta \xi^\varepsilon l_\varepsilon{}^\alpha + f_\beta{}^\varepsilon (l_\varepsilon{}^\delta f_\delta{}^\alpha + m_\varepsilon \xi^\alpha + C^C B_\varepsilon{}^B B^\alpha{}_A \nabla_C F_B{}^A),$$

then we see the tensor field $L = (L_\beta{}^\alpha)$ satisfies the equations

$$(2.17) \quad \left\{ \begin{array}{l} f_\gamma{}^\beta L_\beta{}^\alpha = f_\gamma{}^\beta l_\beta{}^\alpha - l_\gamma{}^\beta f_\beta{}^\alpha - m_\gamma \xi^\alpha - C^C B_\beta{}^B B^\alpha{}_A \nabla_C F_B{}^A \\[2mm] \qquad + \eta_\gamma \xi^\beta (l_\beta{}^\delta f_\delta{}^\alpha + m_\beta \xi^\alpha + C^C B_\beta{}^B B^\alpha{}_A \nabla_C F_B{}^A), \\[2mm] L_\beta{}^\alpha \eta_\alpha = l_\beta{}^\alpha \eta_\alpha + f_\beta{}^\alpha m_\alpha - C^C B_\beta{}^B C_A \nabla_C F_B{}^A \\[2mm] \qquad - \eta_\beta (\xi^\varepsilon l_\varepsilon{}^\alpha \eta_\alpha - C^C C^D F_D{}^B C_A \nabla_C F_B{}^A), \\[2mm] \xi^\beta L_\beta{}^\alpha = 0, \\[2mm] f_\gamma{}^\beta L_\beta{}^\alpha \eta_\alpha = f_\gamma{}^\beta l_\beta{}~\eta_\alpha - m_\gamma + \eta_\gamma \xi^\alpha m_\alpha - f_\gamma{}^\beta C^C B_\beta{}^B C_A \nabla_C F_B{}^A, \end{array} \right.$$

and the expressions (2.15) are written as

$$(2.18) \quad \begin{cases} \overline{N}_{\gamma\beta}{}^\alpha = B_\gamma{}^C B_\beta{}^B B^\alpha{}_A N_{CB}{}^A - \eta_\gamma f_\beta{}^\varepsilon L_\varepsilon{}^\alpha + \eta_\beta f_\gamma{}^\varepsilon L_\varepsilon{}^\alpha, \\[2mm] \overline{N}_{\gamma\beta} = B_\gamma{}^C B_\beta{}^B C_A N_{CB}{}^A - \eta_\gamma L_\beta{}^\alpha \eta_\alpha + \eta_\beta L_\gamma{}^\alpha \eta_\alpha, \\[2mm] \overline{N}_\beta{}^\alpha = C^C B_\beta{}^B B^\alpha{}_A N_{CB}{}^A - L_\beta{}^\alpha, \\[2mm] \overline{N}_\beta = - C^C B_\beta{}^B C_A N_{CB}{}^A - f_\beta{}^\varepsilon L_\varepsilon{}^\alpha \eta_\alpha. \end{cases}$$

Now we can obtain the following

THEOREM 1.[4] *Let M be a complex manifold and $\overline{M}$ an orientable hypersurface in M. Then the conditions in each of the following triples are equivalent to one another and the first implies the second*:

(1) *The induced almost contact structure in $\overline{M}$ is normal, i.e.,*

$$\overline{N}_{\gamma\beta}{}^\alpha = 0 \Longleftrightarrow \overline{N}_\beta{}^\alpha = 0 \Longleftrightarrow L_\beta{}^\alpha = 0.$$

(2)
$$\overline{N}_{\gamma\beta} = 0 \Longleftrightarrow \overline{N}_\beta = 0 \Longleftrightarrow L_\beta{}^\alpha \eta_\alpha = 0.$$

PROOF. It is known [2] that $\overline{N}_{\gamma\beta}{}^\alpha = 0$ implies $\overline{N}_\beta{}^\alpha = 0$, $\overline{N}_{\gamma\beta} = 0$ and $\overline{N}_\beta = 0$ and that $\overline{N}_{\gamma\beta} = 0$ implies $\overline{N}_\beta = 0$. By the assumption, the Nijenhuis tensor N of M vanishes. Hence $\overline{N}_\beta{}^\alpha = 0$ implies $L_\beta{}^\alpha = 0$, then $\overline{N}_{\gamma\beta}{}^\alpha = 0$. On the other hand, $\overline{N}_\beta = 0$ implies $f_\beta{}^\varepsilon L_\varepsilon{}^\alpha \eta_\alpha = 0$, from which and the third equation of (2.17) follows $L_\varepsilon{}^\alpha \eta_\alpha = 0$, then $\overline{N}_{\gamma\beta} = 0$.   Q.E.D.

3. Almost contact metric structure of a hypersurface in an almost Hermitian manifold.

Suppose now that M is an almost Hermitian manifold and $\overline{M}$ an orientable hypersurface in M. We take $C = (C^A)$ as the unit normal vector field to $\overline{M}$. Then the induced metric $\overline{g} = (\overline{g}_{\beta\alpha})$ is defined by

$$(3.1) \qquad\qquad \overline{g}_{\beta\alpha} = g_{BA} B_\beta{}^B B_\alpha{}^A,$$

and it is associated with the induced almost contact structure (f, ξ, η) in $\overline{M}$,

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4) Cf. Y. Tashiro [6, Theorem 2].

see [6]. If the contravariant components of the metric tensor $\overline{g}$ are denoted by $\overline{g}^{\beta\alpha}$, then we have

$$(3.2) \qquad B^{\alpha}{}_{B} = \overline{g}^{\alpha\beta} g_{BA} B_{\beta}{}^{A}, \quad C_{B} = g_{BA} C^{A},$$

and the tensor fields h, l and m have the properties

$$(3.3) \qquad l_{\gamma}{}^{\alpha} = -\overline{g}^{\beta\alpha} h_{\gamma\beta}, \quad m_{\gamma} = 0.$$

The covariant components of the tensor field L defined by (2.16) are equal to

$$(3.4) \qquad L_{\beta\alpha} = -h_{\beta\alpha} + \eta_{\beta} \xi^{\varepsilon} h_{\varepsilon\alpha} + f_{\beta}{}^{\varepsilon} (h_{\varepsilon\delta} f_{\alpha}{}^{\delta} + C^{C} B_{\varepsilon}{}^{B} B_{\alpha}{}^{A} \nabla_{C} F_{BA})$$

and satisfy the equation

$$(3.5) \qquad \begin{aligned} f_{\gamma}{}^{\beta} L_{\beta\alpha} &= -f_{\gamma}{}^{\beta} h_{\beta\alpha} - f_{\alpha}{}^{\beta} h_{\beta\gamma} - C^{C} B_{\gamma}{}^{B} B_{\alpha}{}^{A} \nabla_{C} F_{BA} \\ &\quad + \eta_{\gamma} \xi^{\varepsilon} (f_{\alpha}{}^{\delta} h_{\varepsilon\delta} + C^{C} B_{\varepsilon}{}^{B} B_{\alpha}{}^{A} \nabla_{C} F_{BA}). \end{aligned}$$

In this case, the second equation of (2.13) is written in the form

$$(3.6) \qquad \overline{\nabla}_{\gamma} \eta_{\beta} = -B_{\gamma}{}^{C} B_{\beta}{}^{B} C^{A} \nabla_{C} F_{BA} + h_{\gamma\alpha} f_{\beta}{}^{\alpha}.$$

Substituting (3.6) into the equation (1.16), which characterizes for $\overline{M}$ to be contact metric, and using (3.5), we have the equation

$$(3.7) \qquad \begin{aligned} f_{\gamma\beta} &= -B_{\gamma}{}^{C} B_{\beta}{}^{B} C^{A} (\nabla_{C} F_{BA} - \nabla_{B} F_{CA}) + h_{\gamma\alpha} f_{\beta}{}^{\alpha} - h_{\beta\alpha} f_{\gamma}{}^{\alpha} \\ &= -C^{C} B_{\gamma}{}^{B} B_{\beta}{}^{A} (F_{CBA} - \nabla_{C} F_{BA}) + h_{\gamma\alpha} f_{\beta}{}^{\alpha} - h_{\beta\alpha} f_{\gamma}{}^{\alpha} \\ &= -C^{C} B_{\gamma}{}^{B} B_{\beta}{}^{A} F_{CBA} - f_{\gamma}{}^{\alpha} L_{\alpha\beta} - 2 f_{\gamma}{}^{\alpha} h_{\alpha\beta} \\ &\quad + \eta_{\gamma} \xi^{\varepsilon} (f_{\beta}{}^{\delta} h_{\varepsilon\delta} + C^{C} B_{\varepsilon}{}^{B} B_{\beta}{}^{A} \nabla_{C} F_{BA}). \end{aligned}$$

If, in particular, the manifold M is almost Kählerian, $F_{CBA} = 0$, then we contract (3.7) with ξ^{γ} and see that

$$(3.8) \qquad \xi^{\varepsilon} (f_{\beta}{}^{\delta} h_{\varepsilon\delta} + C^{C} B_{\varepsilon}{}^{B} B_{\alpha}{}^{A} \nabla_{C} F_{BA}) = 0$$

and consequently, from (3.5),

$$f_{\gamma}{}^{\alpha} (\overline{g}_{\alpha\beta} + L_{\alpha\beta} + 2 h_{\alpha\beta}) = 0.$$

Hence we can put

$$\overline{g}_{\alpha\beta} + L_{\alpha\beta} + 2h_{\alpha\beta} = \eta_\alpha \nu_\beta$$

or

(3.9) $$L_{\alpha\beta} = -2h_{\alpha\beta} - \overline{g}_{\alpha\beta} + \eta_\alpha \nu_\beta.$$

Contracting this expression with ξ^β, and using the third equation of (2.17), we obtain

$$\nu_\alpha = \eta_\alpha + 2\xi^\varepsilon h_{\varepsilon\alpha}.$$

Thus we have the following

THEOREM 2. *Let M be an almost Kählerian manifold and $\overline{M}$ an orientable hypersurface in M. If the induced almost contact metric structure in $\overline{M}$ is contact metric, then the tensor field $L_{\beta\alpha}$ has the form*

(3.10) $$L_{\beta\alpha} = -2h_{\beta\alpha} - \overline{g}_{\beta\alpha} + \eta_\beta(\eta_\alpha + 2\xi^\varepsilon h_{\varepsilon\alpha}).$$

Now we can state the following

THEOREM 3. *Let M be a Kählerian manifold and $\overline{M}$ an orientable hypersurface in M. If the induced almost contact metric structure in $\overline{M}$ is K-contact, then the structure is normal, that is, Sasakian. A necessary and sufficient condition for the case is that the second fundamental tensor $h_{\beta\alpha}$ has the form*

(3.11) $$2h_{\beta\alpha} = -\overline{g}_{\beta\alpha} + \alpha\eta_\beta\eta_\alpha,$$

α being a scalar field in $\overline{M}$.[5]

PROOF. Since we have $\nabla F = 0$ in a Kählerian manifold M, the equation (3.6) reduces to

$$\overline{\nabla}_\gamma \eta_\beta = h_{\gamma\alpha} f_\beta{}^\alpha,$$

---

5) Cf. Y. Tashiro [6, Theorem 8]. The difference of the coefficients of (3.11) from those of the theorem is not essential.

hence it follows from the Killing equation (1.17) that

$$h_{\gamma\alpha}f_\beta{}^\alpha + h_{\beta\alpha}f_\gamma{}^\alpha = 0,$$

and from (3.5)

$$f_\gamma{}^\beta L_{\beta\alpha} = \eta_\gamma \xi^\varepsilon f_\alpha{}^\delta h_{\varepsilon\delta}$$

which is equal to zero as is seen by contraction with ξ^γ. By means of this equation and the third of (2.17), we see that the tensor field L vanishes. By virtue of Theorem 1, the induced structure is normal.

Then it follows from (3.10) that

$$2h_{\beta\alpha} = -\bar{g}_{\beta\alpha} + \eta_\beta(\eta_\alpha + 2\xi^\varepsilon h_{\varepsilon\alpha}).$$

Since both $h_{\beta\alpha}$ and $g_{\beta\alpha}$ are symmetric, we may put

$$\eta_\alpha + 2\xi^\varepsilon h_{\varepsilon\alpha} = \alpha\eta_\alpha,$$

where α is a scalar field in $\bar{M}$ given by

$$\alpha = 2h_{\beta\alpha}\bar{g}^{\beta\alpha} + n - 1.$$

Conversely, if $h_{\beta\alpha}$ has the form (3.11), it follows from (3.4) that the tensor field L vanishes.

Q.E.D.

4. Hypersurfaces in a tangent bundle. Let M be an n-dimensional manifold and $T_x(M)$ the tangent space of M at a point x and $T(M)$ the tangent bundle of M. If (x^h) is a local coordinate system in M and (y^h) the cartesian coordinate in the tangent space $T_x(M)$ at each point x with respect to the natural base $\partial_h = \partial/\partial x^h$, then (x^h, y^h) form a local coordinate system in the tangent bundle $T(M)$, called the *induced coordinate system* from (x^h). We write often $(x^{\bar{h}})$ for (y^h) and (x^A) for (x^h, y^h).[6]

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6) We refer [10] as to differential geometry of tangent bundles. From now on, various kinds of indices run respectively on the following ranges:

$A, B, C, \cdots = 1, 2, \cdots, n, n+1, \cdots, 2n$;

$h, i, j, \cdots = 1, 2, \cdots, n$;

$\bar{h}, \bar{i}, \bar{j}, \cdots = \qquad\qquad n+1, \cdots, 2n$;

$\alpha, \beta, \gamma, \cdots = 1, 2, \cdots, n, n+1, \cdots, 2n-1$;

$\kappa, \lambda, \mu, \cdots = \qquad\qquad n+1, \cdots, 2n-1$.

A regular hypersurface in the tangent space $T_x(M)$ at a point x is represented by parametric equations $y^h = y^h(u^\lambda)$ with matrix $(\partial_\lambda y^h)$ of rank $n-1$. Differentiable functions $y^h = y^h(x^i, u^\lambda)$ of local coordinates (x^i) and $n-1$ parameters (u^λ) give locally a hypersurface in the tangent space $T_x(M)$ at each point x with coordinates (x^i), and consequently a field of hypersurfaces in the tangent bundle $T(M)$. Such a field will be denoted by $S(M)$. If the local coordinates (x^h) themselves are regarded as parameters, the field $S(M)$ is a hypersurface represented by

$$(4.1) \qquad x^A = x^A(x^i, u^\lambda) \qquad \text{or} \qquad \begin{cases} x^h = x^h \\ y^h = y^h(x^i, u^\lambda) \end{cases}$$

in the tangent bundle $T(M)$. We write (u^α) for parameters (x^i, u^λ).

Now we consider $S(M)$ as a hypersurface $\bar{M}$ treated in the preceeding paragraphs. The tangent vectors $B_\alpha{}^A = \partial x^A / \partial u^\alpha$ of $S(M)$ in $T(M)$ are given by

$$(4.2) \qquad B_\alpha{}^A \begin{cases} B_i{}^h = \delta_i^h, & B_\lambda{}^h = 0, \\ B_i{}^{\bar{h}} = \partial_i y^h, & B_\lambda{}^{\bar{h}} = \partial_\lambda y^h. \end{cases}$$

We suppose that the square matrix $(B_\lambda{}^{\bar{h}}, y^h)$ is regular at each point of $S(M)$. This means that $S(M)$ is a regular hypersurface in $T(M)$ and the tangent hyperplane to $S(M)$ at each point does not pass through the origin of $T_x(M)$. Then we can take the vector field C with components

$$(4.3) \qquad (C^A) = \begin{pmatrix} 0 \\ y^h \end{pmatrix}$$

as a pseudo-normal vector field of $S(M)$ in $T(M)$. The vectors $B^\alpha{}_B$ and C_B have the components

$$(4.4) \qquad B^\alpha{}_B \begin{cases} B^h{}_i = \delta_i^h, & B^h{}_{\bar{\iota}} = 0, \\ B^\kappa{}_i = - B_i{}^{\bar{h}} B^\kappa{}_{\bar{h}}, & B^\kappa{}_{\bar{\iota}}, \end{cases}$$

and

$$(4.5) \qquad C_A : C_i = - B_i{}^{\bar{h}} C_{\bar{h}}, \quad C_{\bar{h}},$$

respectively, among which $B^\kappa{}_{\bar{\iota}}$ and $C_{\bar{h}}$ are determined by the non-trivial relations

The tensor fields h, l, and m in $S(M)$ defined dy (2.9) have components

$$(5.3) \qquad h_{\gamma\beta} \begin{cases} h_{ji} = (\partial_j B_i^{\bar{h}} + \partial \Gamma_{ji}^h + \Gamma_{j\bar{l}}^h B_i^{\bar{l}} + \Gamma_{l\bar{i}}^h B_j^{\bar{l}}) C_{\bar{h}} \\ \qquad = (\mathcal{L}(y) \Gamma_{ji}^h) C_{\bar{h}}, \\ h_{\mu i} = (\partial_\mu \nabla_i y^h) C_{\bar{h}}, \\ h_{\mu\lambda} = (\partial_\mu \partial_\lambda y^h) C_{\bar{h}}, \end{cases}$$

$$(5.4) \qquad l_\gamma{}^\alpha \begin{cases} l_j{}^h = 0, \qquad\quad l_\lambda{}^h = 0, \\ l_j{}^\kappa = (\nabla_i y^h) B_{\bar{h}}^\kappa, \quad l_\lambda{}^\kappa = \delta_\lambda{}^\kappa, \end{cases}$$

$$(5.5) \qquad m_\gamma : \quad m_j = (\nabla_j y^h) C_{\bar{h}}, \quad m_\mu = 0,$$

respectively.

Given a $(1,s)$-tensor field $P = (P_{i_s \cdots i_1}{}^h)$ in M, the complete lift $P^C = (\widetilde{P}_{B_s \cdots B_1}{}^A)$ of P to the tangent bundle $T(M)$ has components

$$(5.6) \qquad \begin{cases} \widetilde{P}_{i_s \cdots i_1}{}^h = P_{i_s \cdots i_1}{}^h, \\ \widetilde{P}_{i_s \cdots i_1}{}^{\bar{h}} = \partial P_{i_s \cdots i_1}{}^h, \\ \widetilde{P}_{i_s \cdots \bar{i}_t \cdots i_1}{}^{\bar{h}} = P_{i_s \cdots i_t \cdots i_1}{}^h \quad (t = 1, \cdots, s), \end{cases}$$

all the others being zero, with respect to the induced coordinate system. The complete lift Γ^C of an affine connection Γ has the property

$$(5.7) \qquad \nabla^C P^C = (\nabla P)^C$$

for any tensor field P in M. In particular, the complete lift F^C to $T(M)$ of a $(1,1)$-tensor field $F = (F_i{}^h)$ in M has components

$$(5.8) \qquad \widetilde{F}_B{}^A \begin{cases} \widetilde{F}_i{}^h = F_i{}^h, \qquad \widetilde{F}_{\bar{i}}{}^h = 0, \\ \widetilde{F}_i{}^{\bar{h}} = \partial F_i{}^h, \qquad \widetilde{F}_{\bar{i}}{}^{\bar{h}} = F_i{}^h, \end{cases}$$

and the covariant derivative $\nabla^C F^C$ of the lift F^C with respect to Γ^C has components

$$(5.9) \qquad \begin{cases} \widetilde{\nabla}_j \widetilde{F}_i{}^h = \nabla_j F_i{}^h, \quad \widetilde{\nabla}_j \widetilde{F}_{\bar{i}}{}^h = \widetilde{\nabla}_{\bar{j}} \widetilde{F}_i{}^h = \widetilde{\nabla}_{\bar{j}} \widetilde{F}_{\bar{i}}{}^h = 0, \\ \widetilde{\nabla}_j \widetilde{F}_i{}^{\bar{h}} = \partial \nabla_j F_i{}^h, \quad \widetilde{\nabla}_j \widetilde{F}_{\bar{i}}{}^{\bar{h}} = \widetilde{\nabla}_{\bar{j}} \widetilde{F}_i{}^{\bar{h}} = \nabla_j F_i{}^h, \quad \widetilde{\nabla}_{\bar{j}} \widetilde{F}_{\bar{i}}{}^{\bar{h}} = 0. \end{cases}$$

Suppose now that M is an almost complex manifold and F is the structure. Then it is known that the complete lift F^c of F defines an almost complex structure in $T(M)$ and the Nijenhuis tensor $\widetilde{N}$ of F^c coincides with the complete lift N^c to $T(M)$ of the Nijenhuis tensor N of F. Therefore the complete lift F^c is complex if and only if F is complex.

For the pseudo-normal vector field $C=(0, y^h)$, the equation (2.6) becomes

$$(5.10) \qquad C^B \widetilde{F}_B{}^A C_A = y^i F_i{}^h C_{\bar{h}} = 0$$

with respect to the complete lift F^c. If an almost Hermitian metric $g=(g_{ji})$ in M is associated with the almost complex structure F and we consider the unit spheres defined by

$$g_{ih} y^i y^h = 1$$

in the tangent spaces $T_x(M)$, then the vectors (y^h) and $(C_{\bar{i}})=(g_{ih}y^h)$ satisfy (5.10). Hence the tangent unit-sphere bundle possesses the property. We shall confine ourselves with fields of hypersurfaces in $T(M)$ such that the vector field $C=(0, y^h)$ satisfies the equation (5.10), and call them *S-hypersurfaces*.

The tensor fields $f=(f_\beta{}^\alpha)$, $\xi=(\xi^\alpha)$ and $\eta=(\eta_\beta)$ of the induced almost contact structure in an S-hypersurface $S(M)$ from the complete lift F^c to $T(M)$ have components

$$(5.11) \qquad f_\beta{}^\alpha \begin{cases} f_i{}^h = F_i{}^h, \\ f_\lambda{}^h = 0, \\ f_i{}^\kappa = [\partial F_i{}^h + B_i{}^{\bar{j}} F_j{}^h - B_j{}^{\bar{h}} F_i{}^j] B^\kappa{}_{\bar{h}} = [\mathcal{L}(y) F_i{}^h] B^\kappa{}_{\bar{h}}, \\ f_\lambda{}^\kappa = B_\lambda{}^{\bar{i}} F_i{}^h B^\kappa{}_{\bar{h}}, \end{cases}$$

$$(5.12) \qquad \xi^\alpha \begin{cases} \xi^h = 0, \\ \xi^\kappa = y^i F_i{}^h B^\kappa{}_{\bar{h}}, \end{cases}$$

and

$$(5.13) \qquad \eta_\beta \begin{cases} \eta_i = -[\partial F_i{}^h + B_i{}^{\bar{j}} F_j{}^h - B_j{}^{\bar{h}} F_i{}^j] C_{\bar{h}} = -[\mathcal{L}(y) F_i{}^h] C_{\bar{h}}, \\ \\ \eta_\lambda{}^{\cdot} = -B_\lambda{}^{\bar{i}} F_i{}^h C_{\bar{h}}, \end{cases}$$

respectively. Substituting these components and (5.3), (5.4), (5.5) and (5.9) into the four types of the components, $L_i{}^h$, $L_\lambda{}^h$, $L_i{}^\kappa$, $L_\lambda{}^\kappa$ of the tensor field L defined by (2.16), we see that all the components vanish.

THEOREM 4. *For the induced almost contact structure of an S-hyper-surface $S(M)$ from the complete lift F^c in $T(M)$ of an almost complex structure F of M, the tensor field L defined by (2.16) vanishes identically.*

Hence the Nijenhuis tensor $\overline{N}$ and the others given by (2.17) of the induced almost contact structure in $S(M)$ have the components

$$\overline{N}_{\gamma\beta}{}^\alpha \begin{cases} \overline{N}_{ji}{}^h = N_{ji}{}^h, \quad \overline{N}_{\mu i}{}^h = \overline{N}_{j\lambda}{}^h = \overline{N}_{\mu\lambda}{}^h = 0, \\[2mm] \overline{N}_{ji}{}^\kappa = [-N_{ji}{}^l B_l{}^{\bar h} + \partial N_{ji}{}^h + B_j{}^{\bar l} N_{li}{}^h + B_i{}^{\bar l} N_{jl}{}^h]B^\kappa{}_{\bar h} \\[2mm] \qquad = [\mathcal{L}(y)N_{ji}{}^h]B^\kappa{}_{\bar h}, \\[2mm] \overline{N}_{\mu i}{}^\kappa = B_\mu{}^{\bar j} N_{ji}{}^h B^\kappa{}_{\bar h}, \quad \overline{N}_{\mu\lambda}{}^\kappa = 0, \end{cases}$$

$$\overline{N}_{\gamma\beta} \begin{cases} \overline{N}_{ji} = (\mathcal{L}(y)N_{ji}{}^h]C_{\bar h}, \\[2mm] \overline{N}_{\mu i} = B_\mu{}^{\bar j} N_{ji}{}^h C_{\bar h}, \quad \overline{N}_{\mu\lambda} = 0, \end{cases}$$

$$\overline{N}_\beta{}^\alpha \begin{cases} \overline{N}_i{}^h = 0, \quad\quad\quad\quad \overline{N}_\lambda{}^h = 0, \\[2mm] \overline{N}_i{}^\kappa = y^j N_{ji}{}^h B^\kappa{}_{\bar h}, \quad \overline{N}_\lambda{}^\kappa = 0, \end{cases}$$

$$\overline{N}_\beta : \quad \overline{N}_i = -y^j N_{ji}{}^h C_{\bar h}, \quad \overline{N}_\lambda = 0.$$

Therefore we have the following

THEOREM 5. *Let F be an almost complex structure in a manifold M, F^c the complete lift of F to the tangent bundle $T(M)$ and $\overline{F}^c = (f, \xi, \eta)^c$ the induced almost contact structure in an S-hypersurface $S(M)$. The structure $\overline{F}^c$ is normal if and only if F is complex.*

6. The case of horizontal lifts. Let $\Gamma = (\Gamma_{ji}^h)$ be a symmetric affine connection in M. The horizontal lift of Γ to the tangent bundle $T(M)$ will be denoted by Γ^H. The components $(\widetilde{\Gamma}_{CB}^A)$ of Γ^H are given by

(6.1)
$$\begin{cases} \widetilde{\Gamma}_{ji}^h = \Gamma_{ji}^h, \quad\quad \widetilde{\Gamma}_{ji}^h = \widetilde{\Gamma}_{ji}^{\bar h} = \widetilde{\Gamma}_{ji}^{\bar h} = 0, \\[2mm] \widetilde{\Gamma}_{ji}^{\bar h} = \partial \Gamma_{ji}^h - K_{\cdot ji}{}^h, \quad \widetilde{\Gamma}_{ji}^{\bar h} = \widetilde{\Gamma}_{ji}^{\bar h} = \mathfrak{l}_{ji}^h, \widetilde{\Gamma}_{ji}^{\bar h} = 0 \end{cases}$$

with respect to an induced local coordinate system in $T(M)$, where $K_{kji}{}^h$ are the components of the curvature tensor field K of the affine connection Γ and we put $K_{\cdot ji}{}^h = y^k K_{kji}{}^h$.

The induced connection, denoted by $\overline{\Gamma}^H = (\overline{\Gamma}^\alpha_{\gamma\beta})$, in $S(M)$ from the horizontal lift Γ^H has the components

$$(6.2) \quad \begin{cases} \overline{\Gamma}^h_{ji} = \Gamma^h_{ji}, \quad \overline{\Gamma}^h_{j\lambda} = \overline{\Gamma}^h_{\mu i} = \overline{\Gamma}^h_{\mu\lambda} = 0, \\[2mm] \overline{\Gamma}^\kappa_{ji} = [\mathcal{L}(y)\Gamma^h_{ji} - K_{\cdot ji}{}^h]B^\kappa_{\bar{h}} = (\nabla_j \nabla_i y^h)B^\kappa_{\bar{h}}, \\[2mm] \overline{\Gamma}^\kappa_{\mu i} = (\partial_\mu \nabla_i y^h)B^\kappa_{\bar{h}}, \\[2mm] \overline{\Gamma}^\kappa_{\mu\lambda} = (\partial_\mu \partial_\lambda y^h)B^\kappa_{\bar{h}}. \end{cases}$$

The components of the tensor fields h, l and m in $S(M)$ defined by (2.9) are given by

$$(6.3) \quad h_{\gamma\beta} \begin{cases} h_{ji} = (\nabla_j \nabla_i y^h)C_{\bar{h}}, \\[2mm] h_{\mu i} = (\partial_\mu \nabla_i y^h)C_{\bar{h}}, \\[2mm] h_{\mu\lambda} = (\partial_\mu \partial_\lambda y^h)C_{\bar{h}}, \end{cases}$$

$$(6.4) \quad l_\gamma{}^\alpha \begin{cases} l_j{}^h = 0, \qquad l_\mu{}^h = 0, \\[2mm] l_j{}^\kappa = (\nabla_j y^h)B^\kappa_{\bar{h}}, \; l_\mu{}^\kappa = \delta^\kappa_\mu, \end{cases}$$

$$(6.5) \quad m_\gamma : \quad m_j = (\nabla_j y^h)C_{\bar{h}}, \qquad m_\mu = 0,$$

respectively.

On the other hand, the components $(\widetilde{P}_{B_s \cdots B_1}{}^A)$ of the horizontal lift P^H to $T(M)$ of a $(1,s)$-tensor field $P = (P_{i_s \cdots i_1}{}^h)$ in M are given by

$$(6.6) \quad \begin{cases} \widetilde{P}_{i_s \cdots i_1}{}^h = P_{i_s \cdots i_1}{}^h, \\[2mm] \widetilde{P}_{i_s \cdots i_1}{}^{\bar{h}} = -\Gamma^h_l P_{i_s \cdots i_1}{}^l + \sum_t \Gamma^l_{i_t} P_{i_s \cdots l \cdots i_1}{}^h, \\[2mm] \widetilde{P}_{i_s \cdots \bar{i}_t \cdots i_1}{}^{\bar{h}} = P_{i_s \cdots i_t \cdots i_1}{}^h \qquad (t = 1, 2, \cdots s), \end{cases}$$

the others being zero, with respect to the induced coordinate system in $T(M)$, where we have put $\Gamma^h_i = \Gamma^h_{ij}y^j$. The components of the horizontal lift F^H of a $(1,1)$-tensor field $F = (F_i{}^h)$ in M are given by

$$(6.7) \quad \widetilde{F}_B{}^A \begin{cases} \widetilde{F}_i{}^h = F_i{}^h, \qquad\qquad \widetilde{F}_{\bar{i}}{}^h = 0, \\[2mm] \widetilde{F}_i{}^{\bar{h}} = -\Gamma^h_l F_i{}^l + \Gamma^l_i F_l{}^h, \quad \widetilde{F}_{\bar{i}}{}^{\bar{h}} = F_i{}^h, \end{cases}$$

and those of the covariant derivative $\nabla^H F^H$ of F^H with respect to Γ^H by

$$(6.8) \quad \begin{cases} \widetilde{\nabla}_j \widetilde{F}_i{}^h = \nabla_j F_i{}^h, \quad \widetilde{\nabla}_j \widetilde{F}_{\bar{i}}{}^h = \widetilde{\nabla}_{\bar{j}} \widetilde{F}_i{}^h = \widetilde{\nabla}_{\bar{j}} \widetilde{F}_{\bar{i}}{}^h = 0, \\ \widetilde{\nabla}_j \widetilde{F}_i{}^{\bar{h}} = -\Gamma_j^h \nabla_j F_i{}^l + \Gamma_i^l \nabla F_l{}^h, \\ \widetilde{\nabla}_j \widetilde{F}_{\bar{i}}{}^{\bar{h}} = \nabla_j F_i{}^h, \quad \widetilde{\nabla}_{\bar{j}} \widetilde{F}_i{}^{\bar{h}} = 0, \quad \widetilde{\nabla}_{\bar{j}} \widetilde{F}_{\bar{i}}{}^{\bar{h}} = 0. \end{cases}$$

Suppose now that M is an almost complex manifold and F is the structure. Then it is seen [9] that the horizontal lift F^H defines an almost complex structure in $T(M)$, and the components of the Nijenhuis tensor $\widetilde{N}$ of F^H are given by

$$(6.9) \quad \begin{cases} \widetilde{N}_{ji}{}^h = N_{ji}{}^h, \quad \widetilde{N}_{\bar{j}i}{}^h = \widetilde{N}_{j\bar{i}}{}^h = \widetilde{N}_{\bar{j}\bar{i}}{}^h = 0, \\ \widetilde{N}_{ji}{}^{\bar{h}} = -\Gamma_l^h N_{ji}{}^l + \Gamma_i^l (F_j{}^m \nabla_m F_l{}^h + F_l{}^m \nabla_j F_m{}^h) \\ \qquad\qquad - \Gamma_j^l (F_i{}^m \nabla_m F_l{}^h + F_l{}^m \nabla_i F_m{}^h) \\ \widetilde{N}_{j\bar{i}}{}^{\bar{h}} = F_j{}^l \nabla_l F_i{}^h + F_i{}^l \nabla_j F_l{}^h, \\ \widetilde{N}_{\bar{j}i}{}^{\bar{h}} = 0. \end{cases}$$

Similarly to §5, we consider an S-hypersurface $S(M)$ and denote by $\overline{F}^H = (f, \xi, \eta)^H$ the induced almost contact structure in $S(M)$ from the horizontal lift F^H to $T(M)$. The components of the tensor fields f, ξ and η of the structure $\overline{F}^H$ are given by

$$(6.10) \quad f_\beta{}^\alpha \begin{cases} f_i{}^h = F_i{}^h, \qquad\qquad F_\lambda{}^h = 0, \\ f_i{}^\kappa = (-\Gamma_l^h F_i{}^l + \Gamma_i^l F_l{}^h + B_i{}^{\bar{l}} F_l{}^h - F_i{}^l B_l{}^{\bar{h}}) B_{\bar{h}}^\lambda \\ \qquad = (-F_i{}^l \nabla_l y^h + F_l{}^h \nabla_i y^l) B_{\bar{h}}^\kappa, \\ f_\lambda{}^\kappa = B_\lambda{}^{\bar{i}} F_i{}^h B_{\bar{h}}^\kappa, \end{cases}$$

$$(6.11) \quad \xi^\alpha \begin{cases} \xi^h = 0, \\ \xi^\kappa = y^i F_i{}^h B_{\bar{h}}^\kappa, \end{cases}$$

$$(6.12) \quad \eta_\beta \begin{cases} \eta_i = -(-F_i{}^l \nabla_l y^h + F_l{}^h \nabla_i y^l) C_{\bar{h}}, \\ \eta_\lambda = -B_\lambda{}^{\bar{i}} F_i{}^h C_{\bar{h}}, \end{cases}$$

respectively.

Substituting (6.4), (6.5), (6.8), (6.10), (6.11) and (6.12) into the four kinds of the components, $L_i{}^h, L_\lambda{}^h, L_i{}^\kappa, L_\lambda{}^\kappa$, of the tensor field L defined by (2.16), we see that

THEOREM 6. *For the induced almost contact structure $\overline{F}^H$ of an S-hypersurface from the horizontal lift F^H to $T(M)$ of an almost complex structure F in M, the tensor field L defined by (2.16) vanishes identically.*

Hence the Nijenhuis tensor $\overline{N}$ and the others given by (2.17) of the induced almost contact structure $\overline{F}^H$ in $S(M)$ have components

$$\overline{N}_{\gamma\beta}{}^\alpha \begin{cases} \overline{N}_{ji}{}^h = N_{ji}{}^h, \qquad \overline{N}_{\mu i}{}^h = \overline{N}_{j\lambda}{}^h = \overline{N}_{\mu\lambda}{}^h = 0, \\[2mm] \overline{N}_{ji}{}^\kappa = [-N_{ji}{}^l \nabla_l y^h + (\nabla_i y^l)(F_j{}^m \nabla_m F_l{}^h + F_l{}^m \nabla_j F_m{}^h) \\[2mm] \qquad\qquad -(\nabla_j y^l)(F_i{}^m \nabla_m F_l{}^h + F_l{}^m \nabla_i F_m{}^h)]B^\kappa{}_{\bar{h}}, \\[2mm] \overline{N}_{j\lambda}{}^\kappa = B_\lambda{}^{\bar{i}}(F_j{}^l \nabla_l F_i{}^h + F_i{}^l \nabla_j F_l{}^h)B^\kappa{}_{\bar{h}}, \\[2mm] \overline{N}_{\mu\lambda}{}^\kappa = 0, \end{cases}$$

$$\overline{N}_{\gamma\beta} \begin{cases} \overline{N}_{ji} = [-N_{ji}{}^l \nabla_l y^h + (\nabla_i y^l)(F_j{}^m \nabla_m F_l{}^h + F_l{}^m \nabla_j F_m{}^h) \\[2mm] \qquad\qquad -(\nabla_j y^l)(F_i{}^m \nabla_m F_l{}^h + F_l{}^m \nabla_i F_m{}^h)]C_{\bar{h}}, \\[2mm] \overline{N}_{j\lambda} = B_\lambda{}^{\bar{i}}(F_j{}^l \nabla_l F_i{}^h + F_i{}^l \nabla_j F_l{}^h)C_{\bar{h}}, \\[2mm] \overline{N}_{\mu\lambda} = 0, \end{cases}$$

$$\overline{N}_\beta{}^\alpha \begin{cases} \overline{N}_i{}^h = \overline{N}_\lambda{}^h = 0, \\[2mm] \overline{N}_i{}^\kappa = -y^j(F_i{}^l \nabla_l F_j{}^h + F_j{}^l \nabla_i F_l{}^h)B^\kappa{}_{\bar{h}}, \\[2mm] \overline{N}_\lambda{}^\kappa = 0, \end{cases}$$

$$\overline{N}_\beta \begin{cases} \overline{N}_i = y^j(F_i{}^l \nabla_l F_j{}^h + F_j{}^l \nabla_i F_l{}^h)C_{\bar{h}}, \\[2mm] \overline{N}_\lambda = 0. \end{cases}$$

Now we have the following

THEOREM 7. *Let M be an almost complex manifold with structure F and symmetric affine connection Γ. In order that the induced almost contact structure $\overline{F}^H = (f, \xi, \eta)^H$ in an S-hypersurface $S(M)$ from the horizontal lift F^H to $T(M)$ of F is normal, it is necessary and sufficient that the*

structure F is complex and the affine connection Γ *satisfies the equation*

(6. 13) $$F_j{}^l\nabla_l F_i{}^h + F_i{}^l\nabla_j F_l{}^h = 0.$$

PROOF. It is known that $\overline{N}_{\gamma\beta}{}^\alpha=0$ implies $\overline{N}_{\gamma\beta}=\overline{N}_\beta{}^\alpha=\overline{N}_\beta=0$. It follows from the first components $\overline{N}_{ji}{}^h=0$ of $\overline{N}$ that the Nijenhuis tensor N vanishes and F is complex. Moreover, it follows from $\overline{N}_{j\lambda}{}^\kappa=0$ and $\overline{N}_{j\lambda}=0$ that

$$B_\lambda{}^i(F_j{}^l\nabla_l F_i{}^h + F_i{}^l\nabla_j F_l{}^h) = 0$$

and from $\overline{N}_i{}^\kappa=0$ and $\overline{N}_i=0$ that

$$y^i(F_j{}^l\nabla_l F_i{}^h + F_i{}^l\nabla_j F_l{}^h) = 0.$$

By use of these equations, we obtain the equation (6.13).   Q.E.D.

If Γ is the symmetric affine connection which leaves F invariant, $\nabla F=0$ and which exists in a complex manifold, then the equation (6.13) is satisfied.

7. Tangent sphere bundle of a Riemannian manifold.

Let M be a Riemannian manifold with metric tensor $g=(g_{ji})$ and $\Gamma=(\Gamma_{ji}^h)$ the Riemannian connection of $\widetilde{g}$. Putting

(7. 1) $$\delta y^h = dy^h + \Gamma_j^h dx^j, \qquad \Gamma_j^h = \Gamma_{ji}^h y^i,$$

many authors consider the Riemannian metric $\widetilde{g}=(\widetilde{g}_{CB})$ in the tangent bundle $T(M)$ defined by

(7. 2) $$\widetilde{g}_{CB}dx^C dx^B = g_{ji}dx^j dx^i + g_{ji}\delta y^j \delta y^i$$

with respect to an induced coordinate system. The components of the metric tensor $\widetilde{g}$ are

(7. 3) $$\widetilde{g}_{CB} \begin{cases} \widetilde{g}_{ji} = g_{ji} + g_{ts}\Gamma_j^t\Gamma_i^s, & \widetilde{g}_{i\bar{j}} = \Gamma_{ij}, \\ \widetilde{g}_{\bar{i}j} = \Gamma_{ji}, & \widetilde{g}_{\bar{j}\bar{i}} = g_{ji}, \end{cases}$$

and its contravariant components are

(7. 4) $$\widetilde{g}^{BA} \begin{cases} \widetilde{g}^{\bar{i}\bar{h}} = g^{ih}, & \widetilde{g}^{\bar{i}h} = -\Gamma_t^i g^{th}, \\ \widetilde{g}^{i\bar{h}} = -\Gamma_t^h g^{it}, & \widetilde{g}^{ih} = g^{ih} + g^{ts}\Gamma_t^i\Gamma_s^h, \end{cases}$$

where we have put

$$\Gamma_{ji} = \Gamma_j^h g_{hi}.$$

The components of the Riemannian connection, denoted by $\Gamma^M = (\widetilde{\Gamma}_{CB}^A)$, of the metric $\widetilde{g}$ in $T(M)$ are given by

(7.5)
$$
\begin{cases}
\widetilde{\Gamma}_{ji}^h = \Gamma_{ji}^h + \frac{1}{2}(K_{.sj}{}^h\Gamma_i^s + K_{.si}{}^h\Gamma_j^s), \\[2mm]
\widetilde{\Gamma}_{\bar{j}i}^h = \frac{1}{2}K_{.ji}{}^h, \ \widetilde{\Gamma}_{j\bar{i}}^h = \frac{1}{2}K_{.ij}{}^h, \ \widetilde{\Gamma}_{\bar{j}\bar{i}}^h = 0, \\[2mm]
\widetilde{\Gamma}_{ji}^{\bar{h}} = \partial\Gamma_{ji}^h - \frac{1}{2}[K_{.ji}{}^h + K_{.ij}{}^h + (K_{.sj}{}^l\Gamma_i^s + K_{.si}{}^l\Gamma_j^s)\Gamma_l^h], \\[2mm]
\widetilde{\Gamma}_{\bar{j}i}^{\bar{h}} = \Gamma_{ji}^h - \frac{1}{2}K_{.ji}{}^l\Gamma_l^h, \\[2mm]
\widetilde{\Gamma}_{j\bar{i}}^{\bar{h}} = 0.
\end{cases}
$$

In this and the next paragraphs, we consider the tangent sphere bundle $S(M)$, which consists of the unit spheres defined by

(7.6)
$$g_{ih}(x)y^i y^h = 1$$

with respect to the metric $g(x)$ in the tangent space $T_x(M)$ at each point x. The sphere bundle $S(M)$ is represented by parametric equations

(7.7)
$$x^h = x^h, \quad x^{\bar{h}} = y^h = y^h(x^i, u^\lambda)$$

satisfying the equation (7.6). Differentiating (7.6) covariantly in x^j and partially in u^λ, we have

(7.8)
$$g_{ih}(\nabla_j y^i)y^h = g_{ih}(B_j{}^{\bar{i}} + \Gamma_j^i)y^h = g_{ih}B_j{}^{\bar{i}}y^h + \Gamma_{jh}y^h = 0$$

and

(7.9)
$$g_{ih}B_\lambda{}^{\bar{i}}y^h = 0,$$

respectively.

By means of (7.8) and (7.9), the vector field C, having the components

$\begin{pmatrix} C^h \\ C^{\bar{h}} \end{pmatrix} = \begin{pmatrix} 0 \\ y^h \end{pmatrix}$, on $S(M)$ satisfies the equations

(7.10) $\widetilde{g}_{BA}B_j{}^B C^A = (\widetilde{g}_{i\bar{h}}B_j{}^i + \widetilde{g}_{i\bar{h}}B_j{}^{\bar{i}})y^h = (\Gamma_{jh} + g_{ih}B_j{}^{\bar{i}})y^h = 0,$

$\widetilde{g}_{BA}B_\lambda{}^B C^A = g_{\bar{i}\bar{h}}B_\lambda{}^{\bar{i}}y^h = g_{ih}B_\lambda{}^{\bar{i}}y^h = 0$

and

(7.11) $\widetilde{g}_{BA}C^B C^A = \widetilde{g}_{i\bar{h}}y^i y^h = g_{ih}y^i y^h = 1.$

Therefore the vector field $C = \begin{pmatrix} 0 \\ y^h \end{pmatrix}$ is the unit normal to the tangent sphere bundle $S(M)$ with respect to the metric $\widetilde{g}$. The covariant components of C are equal to

(7.12) $C_B : C_i = \Gamma_{ih}y^h, \qquad C_{\bar{i}} = g_{ih}y^h = y_i.$

The induced metric $\overline{g} = (\overline{g}_{\gamma\beta})$ in the tangent sphere bundle $S(M)$ from the metric $\widetilde{g}$ is given by

$$\overline{g}_{\gamma\beta} = \widetilde{g}_{CB}B_\gamma{}^C B_\beta{}^B$$

and have covariant components

(7.13) $\begin{cases} \overline{g}_{ji} = g_{ji} + g_{ts}\Gamma_j^t\Gamma_i^s + \Gamma_{jh}B_i{}^{\bar{h}} + \Gamma_{ih}B_j{}^{\bar{h}} + g_{ih}B_j{}^{\bar{t}}B_i{}^{\bar{h}} \\ \quad = g_{ji} + g_{ts}(\nabla_j y^t)(\nabla_i y^s), \\ \overline{g}_{\mu i} = \Gamma_{ij}B_\mu{}^{\bar{j}} + g_{js}B_\mu{}^{\bar{j}}B_i{}^{\bar{s}} = B_\mu{}^{\bar{j}}\nabla_i y_j = g_{ts}(\partial_\mu y^t)(\nabla_i y^s), \\ \overline{g}_{\mu\lambda} = g_{ji}B_\mu{}^{\bar{j}}B_\lambda{}^{\bar{i}} = g_{ji}(\partial_\mu y^j)(\partial_\lambda y^i), \end{cases}$

and contravariant components

(7.14) $\begin{cases} \overline{g}^{ih} = g^{ih}, \\ \overline{g}^{\lambda h} = - g^{hl}(\nabla_l y^i)B_{\bar{i}}^\lambda, \\ \overline{g}^{\lambda\kappa} = [g^{ih} + g^{ts}(\nabla_t y^i)(\nabla_s y^h)]B_{\bar{i}}^\lambda B_{\bar{h}}^\kappa. \end{cases}$

Since the Riemannian connection $\overline{\Gamma}^{M} = (\overline{\Gamma}_{\gamma\beta}^{\alpha})$ of the induced metric $\overline{g}$ are related to the Riemannian connection Γ^{M} in $T(M)$ by the equation (2.8), we obtain by straightforward computations

(7. 15)
$$
\begin{cases}
\overline{\Gamma}_{ji}^{h} = \Gamma_{ji}^{h} + \dfrac{1}{2}(K_{\cdot sj}{}^{h}\nabla_{i}y^{s} + K_{\cdot si}{}^{h}\nabla_{j}y^{s}), \\[2mm]
\overline{\Gamma}_{\mu i}^{h} = \dfrac{1}{2}K_{\cdot si}{}^{h}B_{\mu}{}^{\bar{s}}, \\[2mm]
\overline{\Gamma}_{\mu\lambda}^{h} = 0, \\[2mm]
\overline{\Gamma}_{ji}^{\kappa} = [\nabla_{j}\nabla_{i}y^{h} + \dfrac{1}{2}K_{\cdot ji}{}^{h} - \dfrac{1}{2}K_{\cdot ij}{}^{h} \\[2mm]
\qquad\quad - \dfrac{1}{2}(K_{\cdot sj}{}^{l}\nabla_{i}y^{s} + K_{\cdot si}{}^{l}\nabla_{j}y^{s})\nabla_{l}y^{h}]B_{\bar{h}}^{\kappa}, \\[2mm]
\overline{\Gamma}_{\mu i}^{\kappa} = (\partial_{\mu}\nabla_{i}y^{h} - \dfrac{1}{2}K_{\cdot si}{}^{l}B_{\mu}{}^{s}\nabla_{l}y^{h})B_{\bar{h}}^{\kappa}, \\[2mm]
\overline{\Gamma}_{\mu\lambda}^{\kappa} = (\partial_{\mu}\partial_{\lambda}y^{h})B_{\bar{h}}^{\kappa}.
\end{cases}
$$

The second fundamental tensor h in $S(M)$ has components

(7. 16)
$$
h_{\gamma\beta}\begin{cases}
h_{ji} = y_{h}\nabla_{j}\nabla_{i}y^{h} = -(\nabla_{j}y_{h})(\nabla_{i}y^{h}), \\[2mm]
h_{\mu i} = y_{h}\partial_{\mu}\nabla_{i}y^{h} = -(\partial_{\mu}y_{h})(\nabla_{i}y^{h}), \\[2mm]
h_{\mu\lambda} = y_{h}\partial_{\mu}\partial_{\lambda}y^{h} = -(\partial_{\mu}y_{h})(\partial_{\lambda}y^{h}).
\end{cases}
$$

8. Contact structure in a tangent sphere bundle.

In the tangent bundle $T(M)$ of a Riemannian manifold M, there exists the 1-from

(8. 1)
$$
\theta = y_{i}dx^{i} = g_{ji}y^{j}dx^{i}
$$

and the derived form $d\theta$ is equal to

(8. 2)
$$
\begin{aligned}
d\theta &= [(\partial_{j}g_{ki})y^{k}dx^{j} + g_{ji}dx^{j}]\wedge dx^{i} \\[2mm]
&= [(\Gamma_{jk}^{l}g_{li} + \Gamma_{ji}^{l}g_{lk})y^{k}dx^{j} + g_{ji}dy^{j}]\wedge dx^{j} \\[2mm]
&= \frac{1}{2}[(\Gamma_{ji} - \Gamma_{ij})dx^{j}\wedge dx^{i} + g_{ji}dy^{j}\wedge dx^{i} - g_{ji}dx^{i}\wedge dy^{j}].
\end{aligned}
$$

Putting

(8. 3)
$$
d\theta = \frac{1}{2}\widetilde{F}_{CB}dx^{C}\wedge dx^{B},
$$

we have the skew-symmetric tensor field

$$(8.4) \qquad \widetilde{F}_{CB} \begin{cases} \widetilde{F}_{ji} = \Gamma_{ji} - \Gamma_{ij}, & \widetilde{F}_{\bar{j}i} = g_{ji}, \\ \widetilde{F}_{j\bar{i}} = -g_{ji}, & \widetilde{F}_{\bar{j}\bar{i}} = 0, \end{cases}$$

and the $(1, 1)$-tensor field $F^M = (\widetilde{F}_B{}^A)$ defined by

$$\widetilde{F}_B{}^A = \widetilde{F}_{BC} g^{CA},$$

whose components are

$$(8.5) \qquad \widetilde{F}_B{}^A \begin{cases} \widetilde{F}_i{}^h = \Gamma_i^h, & \widetilde{F}_{\bar{i}}{}^h = \delta_i^h, \\ \widetilde{F}_i{}^{\bar{h}} = -\Gamma_i^l \Gamma_l^h - \delta_i^h, & \widetilde{F}_{\bar{i}}{}^{\bar{h}} = -\Gamma_i^h. \end{cases}$$

The tensor field F^M satisfies the equation

$$(8.6) \qquad \widetilde{F}_C{}^B \widetilde{F}_B{}^A = -\delta_C^A,$$

that is, it defines an almost Hermitian structure in $T(M)$ with the metric $\widetilde{g}$. Since the fundamental form $d\theta$ is closed, the structure $(F^M, \widetilde{g})$ is almost Kählerian.

Furthermore, since $(\widetilde{F}_{BA})$ is skew symmetric and we have

$$C^B \widetilde{F}_B{}^A C_A = \widetilde{F}_{BA} C^B C^A = 0,$$

the tangent sphere bundle $S(M)$ is an S-hypersurface in $T(M)$ with respect to the almost Kählerian structure $(F^M, \widetilde{g})$. Hence, to the tangent sphere bundle $S(M)$, the contact metric structure, denoted by $(\overline{F}, \overline{g})^M = (f, \xi, \eta, \overline{g})^M$ is induced, from the almost Kählerian structure in $T(M)$, and the tensor fields f, ξ and η have components

$$(8.7) \qquad f_\beta{}^\alpha \begin{cases} f_i{}^h = \nabla_i y^h, & f_\lambda{}^h = B_\lambda{}^{\bar{h}}, \\ f_i{}^\kappa = -[(\nabla_i y^l)(\nabla_l y^h) + \delta_i^h] B^\kappa{}_{\bar{h}}, \\ f_\lambda{}^\kappa = -B_\lambda{}^{\bar{i}}(\nabla_i y^h) B^\kappa{}_{\bar{h}}, \end{cases}$$

$$(8.8) \qquad \xi^\alpha \begin{cases} \xi^h = y^h, \\ \xi^\kappa = -y^i (\nabla_i y^h) B^\kappa{}_{\bar{h}}, \end{cases}$$

$$(8.9) \qquad \eta_\beta \begin{cases} \eta_i = y_i, \\ \eta_\lambda = 0. \end{cases}$$

The covariant components of $f=(f_\beta{}^\alpha)$ are

(8. 10)
$$f_{\gamma\beta} \begin{cases} f_{ji} = \nabla_j y_i - \nabla_i y_j, \\ f_{\mu i} = -f_{i\mu} = \partial_\mu y_i, \\ f_{\mu\lambda} = 0, \end{cases}$$

and we have

(8. 11)
$$f_{\gamma\beta} = \partial_\gamma \eta_\beta - \partial_\beta \eta_\gamma.$$

Therefore the induced structure $(\overline{F}, \overline{g})^M$ in the tangent sphere bundle $S(M)$ is a contact metric structure.

Now we prove the following

THEOREM 8. *Let M be a Riemannian manifold with metric g and $(\overline{F}, \overline{g})^M = (f, \xi, \eta, \overline{g})^M$ the induced contact metric structure defined above in the tangent sphere bundle $S(M)$. In order that the structure $(\overline{F}, \overline{g})^M$ is K-contact, that is, the vector yeld ξ in $S(M)$ is an infinitesimal isometry with respect to the induced metric $\overline{g}$, it is necessary and sufficient that the manifold M is of positive constant curvature. Then the induced structure $(\overline{F}, \overline{g})^M$ is Sasakian.*

PROOF. The Killing equation of ξ in $S(M)$ is now written as

(8. 12)
$$\mathcal{L}(\xi)\overline{g}_{\gamma\beta} = \overline{\nabla}_\gamma \eta_\beta + \overline{\nabla}_\beta \eta_\gamma = \partial_\gamma \eta_\beta + \partial_\beta \eta_\gamma - 2\overline{\Gamma}^\alpha_{\gamma\beta}\eta_\alpha = 0,$$

where $\overline{\nabla}$ indicates covariant differentiation with respect to the metric $\overline{g}$. Substituting the expressions (7.15) and (8.9) of the components η_β and $\overline{\Gamma}^\alpha_{\gamma\beta}$ into (8.12), we have the equations

(8. 13)
$$\begin{cases} \mathcal{L}(\xi)\overline{g}_{ji} = \nabla_j y_i + \nabla_i y_j - (K._{sj}.\nabla_i y^s + K._{si}.\nabla_j y^s) = 0, \\ \mathcal{L}(\xi)\overline{g}_{\mu i} = \partial_\mu y_i - K._{si}.B_{\mu.}^{\overline{s}} = 0, \\ \mathcal{L}(\xi)\overline{g}_{\mu\lambda} = 0. \end{cases}$$

It follows from the second equation that

$$B_\mu^{\bar{j}}(g_{ji} - K_{\cdot ji \cdot}) = 0$$

and, taking account of the symmetry of $g_{ji} - K_{\cdot ji \cdot}$ in i and j, we may put

$$g_{ji} - K_{\cdot ji \cdot} = \alpha y_j y_i.$$

Contracting this expression with y^i, we see that $\alpha = 1$ and hence we have the equation

(8. 14) $$g_{ji} - K_{\cdot ji \cdot} = y_j y_i.$$

By means of (7.11), this equation is written as

$$K_{kjih} y^k y^h = g_{ji} g_{kh} y^k y^h - g_{jk} g_{ih} y^k y^h.$$

Since this equation is valid for an arbitrary tangent vector y, we have the equation

$$K_{kjih} + K_{hjik} = 2g_{ji} g_{kh} - g_{jk} g_{ih} - g_{jh} g_{ik}.$$

By interchanging the indices k and j, taking the difference and using Bianchi's identity, we obtain the equation

(8 .15) $$K_{kjih} = g_{kh} g_{ji} - g_{jh} g_{ki},$$

which means for M to be of positive constant curvature. We see that the first equation of (8.13) is satisfied by (8.15). Thus the first half of the theorem has been established.

When the curvature tensor of M has the form (8.15), by substituting (7.15) and (8.10) into the components of the covariant derivative

$$\overline{\nabla}_\gamma f_{\beta\alpha} = \partial_\gamma f_{\beta\alpha} - \overline{\Gamma}_{\gamma\beta}^\varepsilon f_{\varepsilon\alpha} - \overline{\Gamma}_{\gamma\alpha}^\varepsilon f_{\beta\varepsilon},$$

and comparing the results with (7.13) and (8.9), we obtain the equations

$$\overline{\nabla}_j f_{ih} = \frac{1}{2}\left[y_i(g_{jh} + (\nabla_j y^t)(\nabla_h y_t)) - y_h(g_{ji} + (\nabla_j y^t)(\nabla_i y_t)) \right]$$

$$= \frac{1}{2}(\eta_i \overline{g}_{jh} - \eta_h \overline{g}_{ji}),$$

$$\overline{\nabla}_j f_{i\kappa} = \frac{1}{2} y_i (\nabla_j y^t)(\partial_\kappa y_t)$$

$$= \frac{1}{2} \eta_i \overline{g}_{jk},$$

$$\overline{\nabla}_j f_{\lambda \kappa} = 0,$$

$$\overline{\nabla}_\mu f_{ih} = \frac{1}{2} \Big[y_i (\partial_\mu y^l)(\nabla_h y_l) - y_h (\partial_\mu y^l)(\nabla_i y_l) \Big] = \frac{1}{2} (\eta_i \overline{g}_{\mu h} - \eta_h \overline{g}_{\mu i}),$$

$$\overline{\nabla}_\mu f_{i\kappa} = \frac{1}{2} y_i (\partial_\mu y^l)(\partial_\kappa y_l) = \frac{1}{2} \eta_i \overline{g}_{\mu \kappa},$$

$$\overline{\nabla}_\mu f_{\lambda \kappa} = 0.$$

These equations are combined up to the tensor equation

$$2\overline{\nabla}_\gamma f_{\beta \alpha} = \eta_\beta \overline{g}_{\gamma \alpha} - \eta_\alpha \overline{g}_{\gamma \beta},$$

which means that the induced structure $(\overline{F}, \overline{g})^M$ is Sasakian.   Q.E.D.

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