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A STUDY ON CERTAIN HOMOGENEOUS SPACES

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Introduction. Let $M=G/G_0$ be a homogeneous space of a Lie group G over a closed subgroup G_0 of it. Let $\mathfrak{g}$ denote the Lie algebra of G. We suppose that there is a family $(\mathfrak{g}_i)_{i\geqq 0}$ (i : integer) of subspaces of $\mathfrak{g}$ satisfying the following conditions:

(1) $\mathfrak{g}=\Sigma\mathfrak{g}_i$ (direct sum);

(2) $[\mathfrak{g}_i,\mathfrak{g}_j]\subset\mathfrak{g}_{i+j}+\mathfrak{g}_{|i-j|}$;

(3) $\mathfrak{g}_0$ is the Lie algebra of G_0.

This homogeneous space is considered with respect to a graded Lie algebra (cf. Remark 1). In particular if $\mathfrak{g}_i=\{0\}$ for $i>1$, since conditions (1) and (2) mean

$$\mathfrak{g}=\mathfrak{g}_0+\mathfrak{g}_1 \text{ (direct sum)},\quad [\mathfrak{g}_0,\mathfrak{g}_0]\subset\mathfrak{g}_0,\quad [\mathfrak{g}_0,\mathfrak{g}_1]\subset\mathfrak{g}_1 \text{ and } [\mathfrak{g}_1,\mathfrak{g}_1]\subset\mathfrak{g}_0,$$

the homogeneous space is an extension of a symmetric homogeneous space. Our purpose is to establish the theory of the homogeneous spaces, and in this note we shall study the homogeneous spaces satisfying some additional conditions and, for the most part, ones which are homogeneous almost contact manifold that are similar to Kählerian symmetric spaces.

In §1 we shall define homogeneous spaces which are closely related with the holomorphic geometry of real submanifolds in complex manifolds which has been developed recently by N.Tanaka [8], and shall show a real submanifold of a Grassmann manifold as a typical example of it.

The notion of an almost contact structure $\Sigma=(\phi,\xi,\eta)$ was given by S.Sasaki [6]. In the previous note [4], we have shown that on a non-degenerate almost contact manifold with structure tensors (ϕ,ξ,η) there is a unique linear connection $\widetilde{\nabla}$ associated with the almost contact structure (ϕ,ξ,η). In §2 we shall consider the homogeneous spaces $M=G/G_0$ which are defined in §1 and satisfy an additional condition. The homogeneous spaces are similar to Kählerian symmetric spaces. We shall show that on the space $M=G/G_0$ a G-invariant non-degenerate almost contact structure $\Sigma=(\phi,\xi,\eta)$ can be canonically defined (Th. 1) and shall study the group of all automorphisms of (M, Σ). Next we shall show that the linear connection associated with the almost contact structure Σ is the canonical linear connection of the second kind on the homogeneous space and whose curvature tensor field can be computed (Prop. 2 and Th. 3). In the last section we shall prove the following: Let M be a non-degenerate normal almost contact manifold with structure tensors (ϕ, ξ, η). Let ∇ be the linear connection associated with the almost contact structure and R be the curvature tensor field of it. If $\nabla R = 0$, then M is locally isomorphic to the homogeneous space given in §2.

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Preliminary remark, notations. Throughout this note, we assume the differentiability of class C^∞. Let M be a differentiable manifold. For each point p of M, $T_p(M)$ denotes the tangent space of M at p and $\mathfrak{X}(M)$ denotes the Lie algebra of all vector fields on M. Let f be a differentiable mapping of a manifold into another manifold. f_* and f^* denote as usual the differential of f and the transpose of f_*. $\boldsymbol{R}$ and $\boldsymbol{C}$ denote the field of real numbers and that of complex numbers. $\boldsymbol{C}^*$ denotes the multiplicative group of non-zero complex numbers. $\boldsymbol{C}^n$ denotes the vector space of n-tuples of complex numbers. $GL(n, \boldsymbol{C})$ denotes the group of non-singular $n \times n$ complex matrices. E_n denotes the identity matrix of order n. Let g be a $p \times q$ complex matrix. ${}^t g$ and $\bar{g}$ denote as usual the transpose of g and the complex conjugate of g. Let A be an $n \times n$ complex matrix. $\mathrm{Tr}A$ denotes the trace of A.

1. On certain homogeneous spaces. Let $M = G/G_0$ be a homogeneous space of a Lie group G over a closed subgroup G_0 of G. Let $\mathfrak{g}$ denote the Lie algebra of G, the Lie algebra of all left invariant vector fields on G. We suppose that there is given a family $(\mathfrak{g}_i)_{i \geqq 0}$ (i: integers) of subspaces of $\mathfrak{g}$ satisfying the following conditions:

(1.1) $\mathfrak{g} = \Sigma \mathfrak{g}_i$ (direct sum);

(1.2) $[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j} + \mathfrak{g}_{|i-j|}$;

(1.3) $\mathfrak{g}_i = \{0\}$ if $i > 2$;

(1.4) There is an element u of $\mathfrak{g}_2$ such that
$[u, [u, X]] = -X$ for all $X \in \mathfrak{g}_1$;

(1.5) $\mathfrak{g}_0$ is the Lie algebra of G_0;

(1.6) $\mathrm{Ad}(g)\mathfrak{g}_i = \mathfrak{g}_i$, and $\mathrm{Ad}(g)u = u$ for all $g \in G_0$, where $\mathrm{Ad}(g)$ denotes the adjoint representation of G_0 in $\mathfrak{g}$.

EXAMPLE. We denote by $\boldsymbol{C}^{2n+1}$ the complex vector space of $(2n+1)$-

complex variables and by $e_1, e_2, \cdots, e_{2n+1}$ the natural basis of it. Let $H(2n+1, n)$ denote the Grassmann manifold of n-planes in $\boldsymbol{C}^{2n+1}$. We see that the projective transformation group $PL(2n+1, \boldsymbol{C})$ operates transitively on it. Let O be an n-plane spanned by the vectors $\{e_1, \cdots, e_n\}$. We have $H(2n+1, n) = PL(2n+1, \boldsymbol{C})/H$, where H is the isotropy subgroup of $PL(2n+1, \boldsymbol{C})$ at O. Next, we denote by G' the group of matrices g such that

$$g \in GL(2n+1, \boldsymbol{C}), \quad t\bar{g}kg = \lambda k, \quad \lambda \in \boldsymbol{C}^* = \boldsymbol{C} - \{0\},$$

where

$$k = \begin{array}{c} \\ n\,(\\ 1\,(\\ n\,(\end{array} \!\!\begin{array}{c} \begin{array}{ccc} \overbrace{\;n\;} & \overbrace{\;1\;} & \overbrace{\;n\;} \end{array} \\ \begin{pmatrix} 0 & 0 & E_n \\ 0 & -1 & 0 \\ E_n & 0 & 0 \end{pmatrix} \end{array} .$$

We set $G = G'/Z \cap G'$, where Z is the center of $GL(2n+1, \boldsymbol{C})$. The group G is a Lie subgroup of $PL(2n+1, \boldsymbol{C})$. Let M be a G-orbit through O, i.e., the set of points $g \cdot O$, $g \in G$. We have known that M is represented as a homogeneous space G/G_0, where G_0 is the isotropy subgroup of G at O and is a real submanifold of the Grassmann manifold $H(2n+1, n)$. Let K' denote the group of matrices g such that

$$g \in GL(2n+1, \boldsymbol{C}), t\bar{g}jg = \lambda j, \ \lambda \in \boldsymbol{C}^*,$$

where

$$j = \begin{pmatrix} E_n & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & E_n \end{pmatrix} .$$

We set $K = G' \cap K'/Z \cap G' \cap K'$, where Z is the center of $GL(2n+1, \boldsymbol{C})$. The group K is the maximal compact analytic subgroup of G. A K-orbit through O, K/K_0 (where K_0 is the isotropy subgroup of K at O), is a compact submanifold of $M = G/G_0$. Since M is connected and $\dim M (= \dim G/G_0) = \dim K/K_0$, we have $M = K/K_0$.

The Lie algebra $\mathfrak{k}$ of K may be identified with the Lie algebra of matrices of the form

$$\begin{pmatrix} A & -{}^t\bar{\xi} & B \\ \xi & \alpha & -\xi \\ B & {}^t\bar{\xi} & -{}^t\bar{A} \end{pmatrix},$$

where $A, B \in \mathfrak{u}(n)$, $\alpha = -2\mathrm{Tr}A$ and $\xi \in \boldsymbol{C}^n$. Now we define a family $(\mathfrak{k}_i)_{i \geqq 0}$ of subspaces of $\mathfrak{k}$ as follows:

$$\mathfrak{k}_0 = \left\{ \begin{pmatrix} A & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & -{}^t\bar{A} \end{pmatrix}, \quad A \in \mathfrak{u}(n) \text{ and } \alpha = -2\mathrm{Tr}A \right\},$$

$$\mathfrak{k}_1 = \left\{ \begin{pmatrix} 0 & -{}^t\bar{\xi} & 0 \\ \xi & 0 & -\xi \\ 0 & {}^t\bar{\xi} & 0 \end{pmatrix}, \quad \xi \in \boldsymbol{C}^n \right\},$$

$$\mathfrak{k}_2 = \left\{ \begin{pmatrix} 0 & 0 & B \\ 0 & 0 & 0 \\ B & 0 & 0 \end{pmatrix}, \quad B \in \mathfrak{u}(n) \right\}$$

and $\mathfrak{k}_i = \{0\}$ if $i > 2$. Then, the family $(\mathfrak{k}_i)_{i \geqq 0}$ of subspaces of $\mathfrak{k}$ defined above satisfies conditions (1.1)$\sim$(1.6).

Finally we notice that the above homogeneous space $M = G/G_0 = K/K_0$, a real submanifold of Grassmann manifold $H(2n+1, n)$, is a space in the holomorphic geometry of real submanifolds in complex manifolds (cf.[8]).

REMARK 1. The space M is represented as two homogeneous spaces G/G_0 and K/K_0. We shall consider the relation between G/G_0 and K/K_0. The Lie algebra $\mathfrak{g}$ of G may be identified with the Lie algebra of matrices of the form

$$\begin{pmatrix} A & {}^t\bar{\eta} & C \\ \xi & \alpha & \eta \\ B & {}^t\bar{\xi} & -{}^t\bar{A} \end{pmatrix},$$

where A, B and $C \in \mathfrak{u}(n)$, $\alpha = -2\mathrm{Tr}A$ and $\xi, \eta \in \boldsymbol{C}^n$. We define a family $(\mathfrak{g}_i)$ of subspaces as follows:

$$\mathfrak{g}_{-2}=\left\{\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ B & 0 & 0 \end{pmatrix},\ B\in\mathfrak{u}(n)\right\},\quad \mathfrak{g}_{-1}=\left\{\begin{pmatrix} 0 & 0 & 0 \\ \xi & 0 & 0 \\ 0 & {}^t\bar{\xi} & 0 \end{pmatrix},\ \xi\in \boldsymbol{C}^n\right\},$$

$$\mathfrak{g}_{2}=\left\{\begin{pmatrix} 0 & 0 & C \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},\ C\in\mathfrak{u}(n)\right\},\quad \mathfrak{g}_{1}=\left\{\begin{pmatrix} 0 & {}^t\bar{\eta} & 0 \\ 0 & 0 & \eta \\ 0 & 0 & 0 \end{pmatrix},\ \eta\in \boldsymbol{C}^n\right\},$$

$$\mathfrak{g}_{0}=\left\{\begin{pmatrix} A & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & -{}^t\bar{A} \end{pmatrix},\ A\in\mathfrak{u}(n) \text{ and } \alpha=-2\operatorname{Tr}A\right\}, \text{ and}$$

$$\mathfrak{g}_i=\{0\} \text{ if } |i|>2.$$

Then, the Lie algebra $\mathfrak{g}$ has a graded Lie algebra structure, i.e.,

$$\mathfrak{g}=\Sigma\mathfrak{g}_i \quad \text{(direct sum)}; \tag{1.7}$$

$$[\mathfrak{g}_i, \mathfrak{g}_j]\subset\mathfrak{g}_{i+j}. \tag{1.8}$$

We denote by θ the involutive automorphism of $\mathfrak{g}$ defined by:

$$\theta X=J^{-1}XJ \qquad \text{for } X\in\mathfrak{g}$$

where

$$J=\begin{pmatrix} 0 & 0 & -E_n \\ 0 & 1 & 0 \\ -E_n & 0 & 0 \end{pmatrix}.$$

Then, we have that $\theta\mathfrak{g}_i=\mathfrak{g}_{-i}$, and $\mathfrak{k}_i=\mathfrak{g}(\theta)\cap(\mathfrak{g}_i+\mathfrak{g}_{-i})$ for all $i\geqq 0$, where $\mathfrak{g}(\theta)$ is the subalgebra of $\mathfrak{g}$ consisting of all elements $X\in\mathfrak{g}$ such that $\theta X=X$ (cf. [8]).

We now remark that we have generally the following: Let $(\mathfrak{g},(\mathfrak{g}_i))_{i\in Z}$, Z being the additive group of integers, be a graded Lie algebra over $\boldsymbol{R}$, that is, the family $(\mathfrak{g}_i)$ of subspaces of $\mathfrak{g}$ satisfying conditions (1.7) and (1.8). We assume the following:

(1) $\mathfrak{g}$ is finite dimensional and simple;

(2) $\mathfrak{g}_i=0$ if $|i|>2$ and $[\mathfrak{g}_{-1},\mathfrak{g}_{-1}]\neq\{0\}$.

Let B be the Killing form of $\mathfrak{g}$. Then there is an involutive automorphism

θ of $\mathfrak{g}$ such that the quadratic form $\mathfrak{g} \ni X \to B(X, \theta X) \in \boldsymbol{R}$ is negative definite [8].

Let $\mathfrak{g}(\theta)$ denote the subalgebra of $\mathfrak{g}$ consisting of all elements $X \in \mathfrak{g}$ such that $\theta X = X$. We set $\mathfrak{k} = \mathfrak{g}(\theta)$ and $\mathfrak{k}_i = \mathfrak{g}(\theta) \cap (\mathfrak{g}_i + \mathfrak{g}_{-i})$ for $i \geqq 0$. Then we can see that the family $(\mathfrak{k}_i)_{i \geqq 0}$ of subspaces of $\mathfrak{k}$ satisfies conditions (1.1) and (1.2).

2. On certain almost contact homogeneous spaces. In this section we shall consider only the homogeneous spaces which are defined in §1 and satisfy the following condition:

$$\dim \mathfrak{g}_2 = 1 . \tag{2.1}$$

To begin with we explain the definitions and the notations which will be required for the later treatment. Let M be a homogeneous space G/G_0 which satisfies conditions (1.1) $\sim$ (1.6) and (2.1). We set $\mathfrak{m} = \mathfrak{g}_1 + \mathfrak{g}_2$, the subspace of $\mathfrak{g}$. It follows from conditions (1.1) and (1.6) that

$$\mathfrak{g} = \mathfrak{g}_0 + \mathfrak{m} \text{ (direct sum) and } \mathrm{Ad}(G_0)\mathfrak{m} \subset \mathfrak{m}. \tag{2.2}$$

Let π be the natural projection of G onto $M = G/G_0$. We set $0 = \pi(e)$, where e is the identity element of G. The mapping $\mathfrak{g} \ni X \to \pi_*(X_e) \in T_0(M)$ gives a linear isomorphism of $\mathfrak{m}$ onto the tangent space $T_0(M)$ at 0. We shall identify with $\mathfrak{m}$ and $T_0(M)$ by this isomorphism.

Let M be a differentiable manifold of dimension $2n+1$ $(n \geqq 1)$. An almost contact structure on M is, by definition [6], a triple $\Sigma = (\phi, \xi, \eta)$, where ϕ is a tensor field of type $(1, 1)$ on M, ξ is a vector field on M and η is a 1-form on M, which satisfies the following conditions:

$$\phi^2 X = -X + \eta(X) \cdot \xi \qquad \text{for all } X \in \mathfrak{X}(M); \tag{2.3}$$

$$\eta(\xi) = 1. \tag{2.4}$$

An almost contact structure $\Sigma = (\phi, \xi, \eta)$ on M is called normal if the following tensor field N of type $(1, 2)$ vanishes:

$$\begin{aligned} N(X, Y) \equiv & -\phi^2[X, Y] + \phi[\phi X, Y] + \phi[X, \phi Y] - [\phi X, \phi Y] \\ & - 2d\eta(X, Y) \cdot \xi \quad \text{for all} \quad X, Y \in \mathfrak{X}(M) \text{ (cf. [7])}. \end{aligned} \tag{2.5}$$

With these preparations, we shall prove the following:

THEOREM 1. *Let $M = G/G_0$ be a homogeneous space of a Lie group G over a closed subgroup G_0 of G, and assume that the Lie algebra $\mathfrak{g}$ of G*

has a family $(\mathfrak{g}_i)_{i\geq 0}$ *of subspaces of* $\mathfrak{g}$ *satisfying conditions* (1.1) $\sim$ (1.6) and (2.1). *Let* u *be an element of* $\mathfrak{g}_2$ *satisfying condition* (1.4). *Then, there exists a unique* G*-invariant normal almost contact structure* $\Sigma = (\phi, \xi, \eta)$ *which satisfies the initial conditions*:

$$\phi_0 = \mathrm{ad}_{\mathfrak{m}} u, \ \xi_0 = u \ \textit{and} \ \eta_0 = u^*,$$

where $\mathrm{ad}_{\mathfrak{m}} u$ *is the restriction of* $\mathrm{ad} u$ *on* $\mathfrak{m}$, *and* u^* *is a* 1*-form on* $\mathfrak{m}$ *defined by* $u^*(u) = 1$ *and* $u^*(X) = 0$ *for all* $X \in \mathfrak{g}_1$.

To prove Th.1 we shall first establish the following lemmas.

LEMMA 2.1. *For all* $a \in G_0$ *and for all* $X \in \mathfrak{g}_1$, *we have*

$$\mathrm{Ad}_{\mathfrak{m}}(a)[u, X] = [u, \mathrm{Ad}_{\mathfrak{m}}(a)X]. \tag{2.6}$$

$$\mathrm{Ad}_{\mathfrak{m}}(a)u = u. \tag{2.7}$$

PROOF. (2.7) follows from condition (1.6). From equality (2.7), we have

$$\mathrm{Ad}_{\mathfrak{m}}(a)[u, X] = [\mathrm{Ad}_{\mathfrak{m}}(a)u, \mathrm{Ad}_{\mathfrak{m}}(a)X] = [u, \mathrm{Ad}_{\mathfrak{m}}(a)X].$$

Q.E.D.

Let U be an open neighbourhood of 0 in $\mathfrak{m}$. We set

$$N = \exp U = \{\exp X, X \in U\},$$

and

$$N^* = \pi(N) = \{p = a \cdot 0, a \in N\}.$$

If U is sufficiently small, N is a submanifold of G, N^* is an open neighbourhood of 0 in M, and the restriction of the projection π of G onto M gives a diffeomorphism of N onto N^*. For $X \in \mathfrak{m}$, we define a vector field X^* in N^* by $(X^*)_p = (\tau_a)_* X$, where a is a unique element of N satisfying $p = \pi(a)$ and τ_a is the mapping $gG_0 \to agG_0$ of $M = G/G_0$ onto itself.

Then, we have

LEMMA 2.2 ([5]). $[X^*, Y^*]_0 = [X, Y]_{\mathfrak{m}}$ *for all* $X, Y \in \mathfrak{m}$, *where* $[X, Y]_{\mathfrak{m}}$ *denotes the* $\mathfrak{m}$*-component of the element* $[X, Y]$.

PROOF OF THEOREM 1. Let p be an arbitrary point of M. Choose $g \in G$ such that $g \cdot 0 = p$. From Lemma 2.1, $(\tau_g)_*(\mathrm{ad}_m u)\,(\tau_{g^{-1}})_*$, is a linear mapping of the tangent space $T_p(M)$ at p, is independent of the choice of g, and so we set $\phi_p = (\tau_g)_*(\mathrm{ad}_{\mathfrak{m}} u)(\tau_{g^{-1}})_*$. Hence we have a G-invariant tensor field ϕ of type $(1, 1)$ such that $\phi_0 = \mathrm{ad}_{\mathfrak{m}} u$. Similarly we can define a G-invariant vector field ξ and a G-invariant 1-form η on M such that $\xi_0 = u$ and $\eta_0 = u^*$. Then, we can show that a triple of the tensor fields $\Sigma = (\phi, \xi, \eta)$ satisfies conditions (2.1) and (2.2). Hence we obtain a unique G-invariant almost contact structure (ϕ, ξ, η) which satisfies the initial conditions

$$\phi_0 = \mathrm{ad}_{\mathfrak{m}} u, \ \xi_0 = u \text{ and } \eta_0 = u^*.$$

Finally, we shall show that the almost contact structure (ϕ, ξ, η) is normal. Owing to the homogeneity of M, it is sufficient to verify that $N = 0$ at the origin 0. And since N is a tensor field, it is sufficient to verify that $N(X^*, Y^*)_0 = 0$ for all $X, Y \in \mathfrak{m}$. On the other hand, ϕ and η are G-invariant, we have

$$\phi(X^*) = (\phi_0(X))^* \text{ and } \eta(X^*) = \text{constant on } N^*. \tag{2.8}$$

And from condition (1.6), we have

$$[u, \mathfrak{g}_0] = \{0\}. \tag{2.9}$$

By Lemma 2.2 and equalities (2.8) and (2.9), we have

$$\begin{aligned} N(X^*, Y^*)_0 &= -\phi_0^2[X^*, Y^*]_0 + \phi_0[\phi X^*, Y^*]_0 + \phi_0[X^*, \phi Y^*]_0 \\ &\quad - [\phi X^*, \phi Y^*]_0 - 2d\eta(X^*, Y^*)_0 \cdot \xi_0 \\ &= -\phi_0^2[X^*, Y^*]_0 + \phi_0[(\phi_0 X)^*, Y^*]_0 + \phi_0[X^*, (\phi_0 Y)^*]_0 \\ &\quad - [(\phi_0 X)^*, (\phi_0 Y)^*] + \eta_0([X^*, Y^*]) \cdot \xi_0 \\ &= -[u, [u, [X, Y]_{\mathfrak{m}}]] + [u, [[u, X], Y]_{\mathfrak{m}}] + [u, [X, [u, Y]]_{\mathfrak{m}}] \\ &\quad - [[u, X], [u, Y]]_{\mathfrak{m}} + u^*([X, Y]_{\mathfrak{m}}) \cdot u \\ &= -[u, [u, [X, Y]] - [[u, X], Y] - [X, [u, Y]]] \\ &\quad - [[u, X], [u, Y]]_{\mathfrak{m}} + u^*([X, Y]_{\mathfrak{m}}) \cdot u \\ &= -[[u, X], [u, Y]]_{\mathfrak{m}} + u^*([X, Y]_{\mathfrak{m}}) \cdot u\,. \end{aligned}$$

Now we write $X = X_1 + X_2$ and $Y = Y_1 + Y_2$, where $X_1, Y_1 \in \mathfrak{g}_1$ and $X_2, Y_2 \in \mathfrak{g}_2$. Then, by conditions (1.1), (1.4) and equality (2.9), we have

$$\begin{aligned} N(X^*, Y^*)_0 &= -[[u, X_1], [u, Y_1]]_{\mathfrak{m}} + u^*([X_1, Y_1]_{\mathfrak{m}}) \cdot u \\ &= -[u, [X_1, [u, Y_1]]]_{\mathfrak{m}} + [X_1, [u, [u, Y_1]]]_{\mathfrak{m}} \end{aligned}$$

$$+ u^*([X_1, Y_1]_{\mathfrak{m}})\cdot u$$
$$= - [X_1, Y_1]_{\mathfrak{m}} + u^*([X_1, Y_1]_{\mathfrak{m}})\cdot u = 0 .$$

This completes the proof of Th. 1.

REMARK. Suppose that the homogeneous space G/G_0 in Th. 1 satisfies the following condition:

(2.10) The bilinear mapping $\mathfrak{g}_1 \times \mathfrak{g}_1 \ni (X, Y) \to [X, Y]_{\mathfrak{g}_2} \in \mathfrak{g}_2$ is non-degenerate, where $[X, Y]_{\mathfrak{g}_2}$ is the $\mathfrak{g}_2$-component of $[X, Y]$.

Then, the almost contact structure $\Sigma = (\phi, \xi, \eta)$ given in Th. 1 is non-degenerate (cf.[4]). In particular, η is a contact form on M and $M = G/G_0$ is a homogeneous contact manifold (cf.[1]).

Next, we assume that the Lie group G is simple and compact. Since $\mathfrak{g}$ is simple and compact, the Killing form B of $\mathfrak{g}$ is a negative definite symmetric bilinear form on $\mathfrak{g}$. Since

$$B(\mathrm{Ad}(a)X, \mathrm{Ad}(a)Y) = B(X, Y) \tag{2.11}$$

for all $a \in G$ and for all $X, Y \in \mathfrak{g}$, there is a unique G-invariant Riemannian metric g on M which satisfies the initial condition:

$$g_0 = - \frac{1}{2n} B_{\mathfrak{m}} , \tag{2.12}$$

where $2n = \dim \mathfrak{g}_1$ and $B_{\mathfrak{m}}$ is the restriction of B on $\mathfrak{m}$.

Then, we obtain the following

PROPOSITION 1. *Let $\Sigma = (\phi, \xi, \eta)$ be the G-invariant (normal) almost contact structure on M in Theorem 1, and let g be the G-invariant Riemannian metric on M defined as above.*

Then the Riemannian metric g is determined by the almost contact structure Σ. More precisely, we have the equality:

$$g(X, Y) = - 2d\eta(X, \phi Y) + \eta(X)\eta(Y) \tag{2.13}$$

for all $X, Y \in \mathfrak{X}(M)$.

Using the above notations, we have

LEMMA 2.3.

$$g_0(X,u) = \eta_0(X), \tag{2.14}$$

$$g_0(X,Y) = \eta_0([X,\phi_0 Y]_{\mathfrak{m}}) + \eta_0(X)\eta_0(Y) \tag{2.15}$$

for all $X, Y \in \mathfrak{m}$.

PROOF. From condition (2.1), we have $B(X,u) = \mathrm{Tr}(\mathrm{ad}\,X \cdot \mathrm{ad}\,u) = 0$ for $X \in \mathfrak{g}_1$. And since $[u,[u,X]] = -X$ for $X \in \mathfrak{g}_1$, we have

$$B(u,u) = \mathrm{Tr}(\mathrm{ad}\,u \cdot \mathrm{ad}\,u) = -\dim \mathfrak{g}_1 = -2n.$$

Hence (2.14) follows. Next, we notice that $B(\mathfrak{g}_0, \mathfrak{g}_2) = 0$. In fact, since G is compact and simple, it is known that $\mathfrak{g}_0$ is semisimple (cf.[2]). So we have

$$\begin{aligned} B(\mathfrak{g}_0, \mathfrak{g}_2) &= B([\mathfrak{g}_0, \mathfrak{g}_0], \mathfrak{g}_2) = B(\mathfrak{g}_0, [\mathfrak{g}_0, \mathfrak{g}_2]) \\ &= B(\mathfrak{g}_0, \{0\}) = 0. \end{aligned}$$

If both X and Y are in $\mathfrak{g}_1$, using Lemma 2 and the above notice, we have

$$\begin{aligned} g_0(X,Y) &= -\frac{1}{2n} B_{\mathfrak{m}}(X,Y) = \frac{1}{2n} B(X,[u,[u,Y]]) = -\frac{1}{2n} B([u,X],[u,Y]) \\ &= -\frac{1}{2n} B(u,[X,[u,Y]]) = -\frac{1}{2n} B_{\mathfrak{m}}(u,[X,[u,Y]]_{\mathfrak{m}}) \\ &= g_0(u,[X,[u,Y]]_{\mathfrak{m}}) = \eta_0([X,[u,Y]]_{\mathfrak{m}}), \end{aligned}$$

which yields (2.15). If X or Y is in $\mathfrak{g}_2$, (2.15) follows from equality (2.14). Q.E.D.

PROOF OF PROPOSITION 1. Since g, ϕ and η are G-invariant, it is sufficient to verify

$$g_0(X^*, Y^*) = -2(d\eta)_0(X^*, \phi Y^*) + \eta_0(X^*)\eta_0(Y^*) \tag{2.16}$$

for all $X, Y \in \mathfrak{m}$. By equality (2.8) and Lemma 2.2, we have

$$\begin{aligned} 2(d\eta)_0(X^*, \phi Y^*) &= X_0^*\eta(\phi Y^*) - (\phi Y^*)_0 \cdot \eta(X^*) - \eta_0([X^*, \phi Y^*]) \\ &= X_0^*\eta((\phi_0 Y)^*) - (\phi_0 Y)_0^*\eta(X^*) - \eta_0([X^*, (\phi_0 Y)^*]) \\ &= -\eta_0([X, \phi_0 Y]_{\mathfrak{m}}). \end{aligned} \tag{2.17}$$

Hence (2.16) follows from equalities (2.15) and (2.17). Q.E.D.

Next, we assume, hereafter, that G is simply-connected. We set $\mathfrak{h} = \mathfrak{g}_0 + \mathfrak{g}_2$. By condition (1.2), $\mathfrak{h}$ is a subalgebra of $\mathfrak{g}$. Let H denote the Lie subgroup of G with the algebra $\mathfrak{h}$. We denote by s the involutive automorphism defined by: $s(X) = X$ if $X \in \mathfrak{h}$ and $s(X) = -X$ if $X \in \mathfrak{g}_1$. Since G is simply-connected, there exists an analytic homomorphism σ of G into itself such that $(\sigma_*)_e = s$. Since s is an involutive automorphism, the same is true of σ. The group H is the identity component of the group of fixed point of σ. In particular, H is closed in G.

LEMMA 2.4. *A pair (G, H) is a Riemannian symmetric pair.*(1)

Let B denote the Killing form of $\mathfrak{g}$. We have a G-invariant Riemannian structure Q such that $Q_0 = B_{\mathfrak{g}_1}$, where $B_{\mathfrak{g}_1}$ denotes the restriction of B to $\mathfrak{g}_1$.

From Lemma 2.1 and condition (1.4), we have

$$\mathrm{ad}u \cdot \mathrm{Ad}(h) = \mathrm{Ad}(h) \cdot \mathrm{ad}u, \tag{2.18}$$

$$B(\mathrm{ad}(u)X, \mathrm{ad}(u)Y) = B(X, Y) \tag{2.19}$$

for all $h \in H$ and for all $X, Y \in \mathfrak{g}_1$. Hence we obtain a G-invariant almost complex structure J such that $J_0 = \mathrm{ad}u$. In [2], we have shown that the structure Q is Kählerian, J is integrable, and with the corresponding complex structure, G/H is a Kählerian symmetric space.

PROPOSITION 2 (cf. [1]). *The space G/G_0 is a principal circle bundle over the Kählerian symmetric space.*

Now, we shall study on the group of automorphisms of G/G_0.

DEFINITION ([3]). Let M (resp. M') be an almost contact manifold with structure tensors $\Sigma = (\phi, \xi, \eta)$ (resp. $\Sigma' = (\phi', \xi', \eta')$). A diffeomorphism f of M onto M' is called an *isomorphism of* (M, Σ) *onto* (M', Σ') if it satisfies the following conditions:

$$\phi' \cdot f_* = f_* \cdot \phi\,; \tag{2.20}$$

$$f^* \eta' = \eta\,. \tag{2.21}$$

(1) Let G be a connected Lie group and H be a closed subgroup of it. The pair (G, H) is called a Riemannian symmetric pair if the group $\mathrm{Ad}_G(H)$ is compact and there exists an involutive analytic automorphism σ of G such that $(H_\sigma)_0 \subset H \subset H_\sigma$, where H_σ is the set of fixed points of σ and $(H_\sigma)_0$ is the identity component of H_σ.

If, moreover, $M = M'$ and $\Sigma = \Sigma'$, f is called an *automorphism of* (M, Σ). The set of all automorphisms of (M, Σ) forms a group of transformations of M. We denote it by $A(M)$ and $A_\Sigma(M)$.

REMARK [3]. If f is an isomorphism of (M, Σ) onto (M', Σ'), then the following equality also holds:

$$f_*\xi = \xi'. \tag{2.22}$$

Let M be the above homogeneous space G/G_0. By Theorem 1, M has the G-invariant almost contact structure $\Sigma = (\phi, \xi, \eta)$. By Proposition 1, we see that automorphisms of (M, Σ) are isometries of the Riemannian manifold M with structure g. The group $I(M)$ of all isometries of M is a Lie transformation group with respect to the compact open topology. Since the group $A(M)$ is closed in $I(M)$, $A(M)$ is also a Lie transformation group of M. Let $A_0(M)$ denote the identity component of it.

Then we have

THEOREM 2. $A_0(M) = G \times U(1)$, *the product group of* G *and* $U(1)$, *where* $U(1)$ *denotes the multiplicative group of complex numbers of absolute value* 1.

PROOF. Let $f \in A_0(M)$. Since the vector field ξ generates the right translations of the bundle and $f_*\xi = \xi$, f commutes with all right translations of it. Hence, f is a homomorphism of the principal circle bundle and induces a mapping of the base space $N = G/H$ onto itself, which will be denoted by $\bar{f}$. Recalling the definition of ϕ, g, J and Q, we can see that $\bar{f}$ is a holomorphic isometry of the Kählerian symmetric space N. Let $A_0(N)$ denote the identity component of the group of all holomorphic isometries of N. It follows that the mapping $I: A_0(M) \ni f \rightarrow \bar{f} \in A_0(N)$ gives a continuous homomorphism of $A_0(M)$. Since the pair (G, H) is a Riemannian symmetric pair and G is simple and acts effectively on the coset space $N = G/H$, by facts on symmetric spaces (cf. Th. 1 (ch. V, p. 207) and Lemma 4.3 (ch. VIII, p. 303) in [2]), we have $A_0(N) = G$. Since $A_0(M) \supset G$, acting on the left on $M = G/G_0$, I is surjective. Next, let $f \in$ the kernel of I. Since η defines a connection in the bundle and $f^*\eta = \eta$, there exists a unique element a of $U(1)$ such that $f = R_a$, where R_a denotes the right translation of the bundle. Hence, Theorem 2 is proved.

EXAMPLE 1. The $(2n+1)$-dimensional sphere S^{2n+1} may be naturally represented by the homogeneous space $SU(n+1)/SU(n)$. We set $G = SU(n+1)$ and $G_0 = SU(n)$. The Lie algebra $\mathfrak{g}$ of G may be identified with the Lie algebra

$\mathfrak{su}(n+1)$. Now, define a family $(\mathfrak{g}_i)_{i\geqq 0}$ of subspaces of $\mathfrak{g}$ as follows:

$$\mathfrak{g}_0 = \left\{\begin{pmatrix} 0 & 0 \\ 0 & \alpha \end{pmatrix},\ \alpha \in \mathfrak{su}(n+1)\right\},$$

$$\mathfrak{g}_1 = \left\{\begin{pmatrix} 0 & -{}^t\bar{\xi} \\ \xi & 0 \end{pmatrix},\ \xi \in \boldsymbol{C}^n\right\},$$

$$\mathfrak{g}_2 = \left\{\lambda\begin{pmatrix} n\sqrt{-1} & 0 \\ 0 & -\sqrt{-1}E_n \end{pmatrix},\ \lambda \in \boldsymbol{R}\right\}.$$

Then the family $(\mathfrak{g}_i)_{i\geqq 0}$ thus obtained satisfies conditions $(1.1)\sim(1.6), (2.1)$ and (2.10). Th.1 and remark show that there is a G-invariant (non-degenerate normal) almost contact structure $\Sigma = (\phi, \xi, \eta)$ on S^{2n+1}. By Th.2, the group of automorphisms of (S^{2n+1}, Σ) is $U(n+1) = SU(n+1)\times U(1)$.

EXAMPLE 2. We now consider an irreducible compact Kählerian symmetric homogeneous space G/H. Let $\mathfrak{g}$ and $\mathfrak{h}$ denote the Lie algebra of G and H, respectively. There exists a family $(\mathfrak{g}_i)_{i\geqq 0}$ of $\mathfrak{g}$ which satisfies conditions (1.1) $\sim(1.4)$ and the conditions:

$$[\mathfrak{g}_0, \mathfrak{g}_2] = 0, \text{ and } \mathfrak{h} = \mathfrak{g}_0 + \mathfrak{g}_2 \quad \text{(cf. [2])}.$$

Let G_0 be the Lie subgroup of G with Lie algebra $\mathfrak{g}_0$. Then it can be proved that G_0 is a closed subgroup of G and that the homogeneous space G/G_0 together with the family $(\mathfrak{g}_i)$ satisfies conditions $(1.1)\sim(1.6)$ and (2.1). The homogeneous almost contact manifold G/G_0 with the structure tensor fields $\Sigma = (\phi, \xi, \eta)$ given in Th. 1 is a principal circle bundle over the Kählerian symmetric space G/H (cf.[1]).

Now, let M be the homogeneous space G/G_0 of a connected Lie group G over a closed subgroup G_0 of it which satisfies conditions $(1.1)\sim(1.6)$, (2.1) and (2.10). By Theorem 1, we have a G-invariant non-degenerate normal almost contact structure $\Sigma = (\phi, \xi, \eta)$ on M, and so we have the G-invariant (pseudo) Riemannian structure g given by:

$$g(X, Y) = -2d\eta(X, \phi Y) + \eta(X)\eta(Y) \tag{2.23}$$

for all $X, Y \in \mathfrak{X}(M)$. Hence, by Theorem A in §3, we have the linear connection $\tilde{\nabla}$ which is uniquely determined by the almost contact structure $\Sigma = (\phi, \xi, \eta)$.

We shall study the linear connection $\tilde{\nabla}$ on the space G/G_0. Hereafter we shall use the notations given in the beginning of this section. In [5], we have shown that there is one-to-one correspondence between the set of G-invariant linear connection ∇ and the set of all bilinear mappings α of $\mathfrak{m}\times\mathfrak{m}$ into $\mathfrak{m}$ which are invariant by $\mathrm{Ad}(G_0)$, that is,

$$\mathrm{Ad}(g)\alpha(X, Y) = \alpha(\mathrm{Ad}(g)X, \mathrm{Ad}(g)Y) \tag{2.24}$$

for all $X, Y\in\mathfrak{m}$ and for all $g\in G_0$, and the correspondence is given as follows:

$$(\nabla_{X^*}Y^*)_0 = \alpha(X, Y) \tag{2.25}$$

for all $X, Y\in\mathfrak{m}$. Let T and R denote the torsion tensor field and the curvature tensor field of the linear connection ∇ corresponding to α. Then, at the origin 0, we have

$$T_0(X, Y) = \alpha(X, Y) - \alpha(Y, X) - [X, Y]_{\mathfrak{m}}; \tag{2.26}$$

$$\begin{aligned} R_0(X, Y)Z = \alpha(X, \alpha(Y, Z)) - \alpha(Y, \alpha(X, Z)) - \alpha([X, Y]_{\mathfrak{m}}, Z) \\ - [[X, Y]_{\mathfrak{g}_0}, Z] \end{aligned} \tag{2.27}$$

for all $X, Y, Z\in\mathfrak{m}$, where $[\ \ ]_{\mathfrak{g}_0}$ denotes the $\mathfrak{g}_0$-component of the element $[X, Y]$. In particular, the invariant linear connection ∇ corresponding to $\alpha\equiv 0$, is called the canonical linear connection of the second kind, satisfies $\nabla T=0$ and $\nabla R=0$, where T and R denote the torsion tensor field and the curvature tensor field of it. The canonical linear connection of the second kind is of fundamental importance to represent locally a manifold as a homogeneous space [5].

Turning now to study the linear connection $\tilde{\nabla}$, clearly it is a G-invariant linear connection.

Then, we have

PROPOSITION 3. *The linear connection $\tilde{\nabla}$ on the space G/G_0 is identical with the canonical linear connection of the second kind on it.*

PROOF. Let α denote the bilinear mapping of $\mathfrak{m}\times\mathfrak{m}$ into $\mathfrak{m}$ corresponding to the G-invariant linear connection $\tilde{\nabla}$. We shall show $\alpha\equiv 0$. Since $\tilde{\nabla}g=0$, we have

$$X\cdot g(Y, Z) = g(\tilde{\nabla}_X Y, Z) + g(Y, \tilde{\nabla}_X Z) \tag{2.28}$$

for all $X, Y, Z\in\mathfrak{X}(M)$. Since g is G-invariant, it follows from the correspondence between $\tilde{\nabla}$ and α that, at the origin $0\in G/G_0$, (2.29) $$g_0(\alpha(X,Y),Z)+g_0(Y,\alpha(X,Z))=0$$

for all $X,Y,Z\in\mathfrak{m}$. In fact, we have

$$g_0(\alpha(X,Y),Z)+g_0(Y,\alpha(X,Z))=g_0((\nabla_{X^*}Y^*)_0,Z_0^*)+g_0(Y_0^*,(\nabla_{X^*}Z^*)_0)$$
$$=X_0^*\cdot g(Y^*,Z^*)=0.$$

On the other hand, the torsion tensor field $\tilde{T}$ of the linear connection $\tilde{\nabla}$ is given by:

(2.30) $$\tilde{T}(X,Y)=2d\eta(X,Y)\cdot\xi+\eta(Y)\phi X-\eta(X)\phi Y$$

for all $X,Y\in\mathfrak{X}(M)$. Then, we have

(2.31) $$\tilde{T}_0(X,Y)=-[X,Y]_{\mathfrak{m}}$$

for all $X,Y\in\mathfrak{m}$. In fact, we have

$$\begin{aligned}\tilde{T}_0(X,Y)&=\tilde{T}_0(X^*,Y^*)\\&=2(d\eta)_0(X^*,Y^*)\cdot\xi_0+\eta_0(Y^*)\cdot\phi_0X^*-\eta_0(X^*)\phi_0Y^*\\&=-\eta_0([X^*,Y^*])\cdot\xi_0+\eta_0(Y^*)\cdot(\mathrm{ad}u(X))_0^*-\eta_0(X^*)(\mathrm{ad}u(Y))_0^*\\&=-u^*([X,Y]_{\mathfrak{m}})\cdot u+u^*(Y)[u,X]-u^*(X)[u,Y]\\&=-[X,Y]_{\mathfrak{g}_2}-[X,Y]_{\mathfrak{g}_1}=-[X,Y]_{\mathfrak{m}},\end{aligned}$$

where $[\ \ ]_{\mathfrak{g}_i}(i=1,2)$ denotes the $\mathfrak{g}_i$-component of the element $[X,Y]$.

Hence, by equalities (2.26) and (2.31), we have

(2.32) $$\alpha(X,Y)=\alpha(Y,X)$$

for all $X,Y\in\mathfrak{m}$.

Therefore, by equalities (2.29) and (2.32), we have, for any $X,Y,Z\in\mathfrak{m}$,

$$\begin{aligned}g_0(\alpha(X,Y),Z)&=-g_0(Y,\alpha(X,Z))=-g_0(Y,\alpha(Z,X))=g_0(\alpha(Z,Y),X)\\&=g_0(\alpha(Y,Z),X)=-g_0(Z,\alpha(Y,X))=-g_0(Z,\alpha(X,Y))\\&=-g_0(\alpha(X,Y),Z),\end{aligned}$$

which yields $\alpha\equiv 0$, because g_0 is a non-degenerate bilinear form on the tangent space at 0. This completes the proof of Proposition 3.

From Proposition 3 and equalities (2.26) and (2.27), we have

THEOREM 3. *Let $\tilde{T}$ and $\tilde{R}$ denote the torsion tensor field and the curvature tensor field of the linear connection $\tilde{\nabla}$ on the space G/G_0. Then, at the origin $0 \in G/G_0$,*

$$\tilde{T}(X, Y) = -[X, Y]_{\mathfrak{m}}, \tag{2.33}$$

$$\tilde{R}(X, Y)Z = -[[X, Y]_{\mathfrak{g}_0}, Z] \tag{2.34}$$

for all $X, Y, Z \in \mathfrak{m}$.

3. On the almost contact manifold which is locally representable as a homogeneous space. To begin with we shall explain the definition and the fact which were established in the previous note [4]. Let M be an almost contact manifold with structure tensors $\Sigma = (\phi, \xi, \eta)$. Let us consider the tensor field g of type (0,2) on M defined by:

$$g(X, Y) = -2d\eta(X, \phi Y) + \eta(X)\eta(Y) \tag{3.1}$$

for all $X, Y \in \mathfrak{X}(M)$. We shall say that the almost contact structure $\Sigma = (\phi, \xi, \eta)$ is non-degenerate, if the tensor field g is a (pseudo-) Riemannian structure on M.

We have shown

THEOREM A [4]. Let M be an almost contact manifold with structure tensors $\Sigma = (\phi, \xi, \eta)$. If the almost contact structure $\Sigma = (\phi, \xi, \eta)$ is non-degenerate and normal, then there exists a unique linear connection ∇ on M such that ϕ, ξ, η and g are parallel with respect to it, and whose torsion tensor field T is given by

$$T(X, Y) = 2d\eta(X, Y)\cdot\xi + \eta(Y)\phi X - \eta(X)\phi Y \tag{3.2}$$

for all $X, Y \in \mathfrak{X}(M)$.

It is well known that a locally symmetric Riemannian manifold is locally isomorphic to a homogeneous Riemannian symmetric space. We shall now establish such a fact on almost contact manifolds.

THEOREM 4. *Let M be a non-degenerate normal almost contact* *manifold. Let ∇ be the linear connection given by Theorem* A *and R be the curvature tensor field of it. If $\nabla R=0$, then there exists a homogeneous space G/G_0 of a connected Lie group G over a closed subgroup G_0 of it which satisfies the following conditions*:

(1) *There exists a family $(\mathfrak{g}_i)_{i\geqq 0}$ of subspaces of the Lie algebra of G which satisfies conditions* (1.1)$\sim$(1.6) *and* (2.1),

(2) *For each point p of M, there exists an isomorphism f of some open neighbourhood of p in M onto an open neighbourhood of the origin* 0 *of the homogeneous space G/G_0.*

Let $\mathfrak{m}$ denote the tangent space $T_p(M)$. If A is an endomorphism of $\mathfrak{m}$, then A can be uniquely extended to the tensor algebra over $\mathfrak{m}$ as a derivation, preserving type of tensors and commuting with contractions.

LEMMA 3.1 (cf. [2], [5]). *Let $\mathfrak{g}_0$ denote the set of all endomorphisms of $\mathfrak{m}$ which, when extended to the tensor algebra as above, anihilate $\phi_p, \xi_p, \eta_p, T_p$ and R_p. Then $\mathfrak{g}_0$ is a Lie algebra with the bracket $[A,B]=AB-BA$, furthermore, $R_p(X,Y)\in\mathfrak{g}_0$ for all $X,Y\in\mathfrak{m}$.*

We set $\mathfrak{g}=\mathfrak{g}_0+\mathfrak{m}$ (direct sum). We now introduce a bracket operation in $\mathfrak{g}$ as follows:

For $X,Y\in\mathfrak{m}$, $[X,Y]=-T_p(X,Y)-R_p(X,Y)$.

For $X\in\mathfrak{m}$, $A\in\mathfrak{g}_0$, $[A,X]=-[X,A]=AX$ (A operating on X).

For $A,B\in\mathfrak{g}_0$, $[A,B]=AB-BA$.

Since $\nabla T=0$ and $\nabla R=0$, we have

LEMMA 3.2 [5]. *The bracket operation above turns $\mathfrak{g}$ into a Lie algebra.*

Let $\mathfrak{g}_1$(resp. $\mathfrak{g}_2$) denote the subspace of $\mathfrak{m}$ which is spanned by the elements $\{-\phi_p^2X, X\in\mathfrak{m}\}$ (resp. $\{\xi_p\}$). We set $\mathfrak{g}_i=\{0\}$ if $i>2$.

We can show the following

LEMMA 3.3. *The family $(\mathfrak{g}_i)_{i\geqq 0}$ of subspaces of $\mathfrak{g}$ satisfies conditions* (1.1)$\sim$(1.4).

PROOF. We shall first show the followng equality:

(3.3) $$R(X,\xi)=0 \text{ for all } X\in\mathfrak{X}(M).$$

In fact, we have shown in [4] the following equality:

$$R(\phi X, \phi Y) = R(X, Y) \text{ for all } X, Y \in \mathfrak{X}(M). \tag{3.4}$$

Hence, it yields (3.3).

From the definition of the bracket operation, putting $u = \xi_p$, we can show that conditions (1.1) $\sim$ (1.4) follow.

PROOF OF THEOREM 4. It can be shown that there exists the connected Lie group G whose Lie algebra is $\mathfrak{g}$ and that the analytic subgroup G_0 of G corresponding to $\mathfrak{g}_0$ is closed (cf. [2]). Then, we can show that the homogeneous space G/G_0 satisfies conditions (1.1) $\sim$ (1.6), (2.1) and (2.10). In fact, by Lemma 3.3, conditions (1.1) $\sim$ (1.5) follow. By the definition of the bracket oparation, we have

$$[\mathfrak{g}_0, \mathfrak{g}_i] \subset \mathfrak{g}_i \text{ and} \tag{3.5}$$

$$[\mathfrak{g}_0, u] = \{0\}, \text{ where } u = \xi_p. \tag{3.6}$$

Since the analytic subgroup G_0 is connected, the above conditions (3.5) and (3.6) yield condition (1.6). Next, let $X, Y \in \mathfrak{g}_1$. Then, by the definition of the bracket operation, we have

$$[X, Y]_{\mathfrak{g}_2} = (-T(X, Y) - R(X, Y))_{\mathfrak{g}_2} = -2d\eta(X, Y) \cdot \xi.$$

Since the almost contact structure is non-degenerate, we can show that condition (2.10) follows. Hence, by Th.1 and remark, we obtain a G-invariant non-degenerate normal almost contact structure $\Sigma' = (\phi', \xi', \eta')$ on the homogeneous space G/G_0.

Let π be the natural projection of G onto G/G_0. Let e be the identity element of G. We put $\pi(e) = 0$. Then, the mapping $\mathfrak{g} \ni X \to (\pi_*)_e(X_e) \in T_0(G/G_0)$ gives a linear isomorphism of $T_p(M)$ onto the tangent space $T_0(G/G_0)$. We denote it by A. By Prop.1, the canonical linear connection of the second kind on the space G/G_0 is the linear connection ∇' obtained in Th. A. Let T' and R' denote the torsion tensor field and the curvature tensor field of the linear connection. By Th.3 and the bracket operation of $\mathfrak{g}$, we have $T_p = T'_0$ and $R_p = R'_0$. Since $\nabla T = \nabla R = 0$ and $\nabla' T' = \nabla' R' = 0$, it follows from the equivalence theorem (cf. Lemma 1.2 (ch.IV, p.165) in [2]) that there exists an affine transformation f of some open neighbourhood of p in M onto an open neighbourhood of the origin 0 of G/G_0 such that $f(p) = 0$ and $(f_*)_p = A$. Then, we have

$$\phi'_0 \cdot A = A \cdot \phi_p \text{ and } \eta'_0 \cdot A = \eta_p. \tag{3.7}$$

Therefore, Theorem 4 will thus follow from

LEMMA 3.4. *Let* M *(resp.* M'*) be a manifold with a non-degenerate (normal) almost contact structure* $\Sigma = (\phi, \xi, \eta)$ *(resp.* $\Sigma' = (\phi', \xi', \eta')$*) and the linear connection given by Th. A. Let* f *be an affine transformation of* M *onto* M'*. Suppose that for some point* $p \in M$*, the linear mapping* $(f_*)_p$: $T_p(M) \to T_{f(p)}(M')$ satisfies

$$\phi'_{f(p)} \cdot (f_*)_p = (f_*)_p \cdot \phi_p \text{ and } \eta'_{f(p)} \cdot (f_*)_p = \eta_p.$$

If M *is connected, then* f *is an isomorphism of* (M, Σ) *onto* (M', Σ').

PROOF. Let $q \in M$ and $X \in T_q(M)$. We join q and p by a curve γ. Let τ denote the parallel translation from q to p along γ and τ_f denote the parallel translation from $f(q)$ to $f(p)$ along $f \cdot \gamma$. Since f is an affine transformation, we have

$$(f_*)_p(\tau X) = \tau_f \cdot (f_*)_q(X).$$

Hence, we have

$$\begin{aligned}(f_*)_q \cdot (\phi_q X) &= \tau_f^{-1} \cdot (f_*)_p \cdot \tau \cdot \phi_q X = \tau_f^{-1} \cdot (f_*)_p \cdot \phi_p \cdot (\tau X)\\ &= \tau_f^{-1} \cdot \phi'_{f(p)} \cdot (f_*)_p(\tau X) = \tau_f^{-1} \cdot \phi'_{f(p)} \cdot \tau_f \cdot (f_*)_q(X)\\ &= \phi'_{f(q)} \cdot (f_*)_q(X).\end{aligned}$$

In a similar way, we have

$$\eta'_{f(p)} \cdot (f_*)_p = \eta_p.$$

This completes the proof.

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