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ON A DUALITY FOR LOCALLY COMPACT GROUPS

KAZUYUKI SAITÔ

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At earlier time, W. F. Stinespring proved in [8] an operator algebraic version of duality theorem for locally compact unimodular groups as an application of non-commutative integration theory. Recently, a duality theorem for locally compact groups was established by N. Tatsuuma as a generalization of the so-called Tannaka duality theorem [4, 11, 14]. In this paper, we shall prove the operator algebraic duality for not necessarily unimodular locally compact groups as an extension of [8].

After writing this paper, the author found Eymard's paper [L'algèbre de Fourier d'un groupe localement compact, Bull. Soc. math. France, 92 (1964), 181-236, Theorem 3.34], in which he proved the analogous result with the main theorem of this paper. His notation and method of proof are different from that used in this paper.

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Let $\mathfrak{G}$ be a locally compact group with left-Haar measure μ. From the theory of Haar measure, we know that there is a continuous positive-valued function $\Delta(x)$, defined on $\mathfrak{G}$, called the modular function, satisfying $\Delta(xy) = \Delta(x)\cdot\Delta(y)$ and for all $f\in L^1(\mathfrak{G})$ (the set of all complex-valued μ-integrable functions on $\mathfrak{G}$) the following properties;

$$\int_{\mathfrak{G}} f(xy)\,d\mu(x) = \Delta(y^{-1})\cdot\int_{\mathfrak{G}} f(x)\,d\mu(x). \tag{1}$$

$$\int_{\mathfrak{G}} f(x^{-1})\cdot\Delta(x^{-1})\,d\mu(x) = \int_{\mathfrak{G}} f(x)\,d\mu(x). \tag{2}$$

We define convolution in $L^1(\mathfrak{G})$:

$$f*g(x) = \int_{\mathfrak{G}} f(y)\,g(y^{-1}x)\,d\mu(y). \tag{3}$$

Define f^* for $f\in L^1(\mathfrak{G})$ by $f^*(x)=\overline{f(x^{-1})}\cdot\Delta(x^{-1})$, where $\bar{a}$ is the complex conjugate of the complex number a. Then under convolution as multiplication, $L^1(\mathfrak{G})$ forms a Banach algebra having a natural involution $f \to f^*$. [1, 5, 13].

Let $s \in \mathfrak{G} \to \lambda(s)$ denote the left regular representation of $\mathfrak{G}$, which is defined by

$$(\lambda(s)f)(x) = f(s^{-1}x)$$

for every $f \in L^2(\mathfrak{G})$, $s, x \in \mathfrak{G}$, where $L^2(\mathfrak{G})$ is the Hilbert space of all complex-valued μ-square integrable functions on $\mathfrak{G}$.

Similarly, the left regular representation of the Banach algebra $L^1(\mathfrak{G})$ is defined by

$$(\lambda(f)\, g)(x) = \int_{\mathfrak{G}} f(s)\, g(s^{-1}x)\, d\mu(s)$$

for every $f \in L^1(\mathfrak{G})$, $g \in L^2(\mathfrak{G})$ and $x \in \mathfrak{G}$. Then, the mapping $f \to \lambda(f)$ is to be thought of as a "global" Fourier transform. A greater formal analogy with the abelian case is manifest in the formula

$$\lambda(f) = \int_{\mathfrak{G}} f(x) \cdot \lambda(x)\, d\mu(x), \qquad f \in L^1(\mathfrak{G})$$

where the integral is interpreted in the σ-weak sense.

Let M be the von Neumann algebra generated by all the $\lambda(a)$, with a in $\mathfrak{G}$. Then, the operators $\lambda(f)$ are in M, and M is the von Neumann algebra they generate.

For a little while, suppose $\mathfrak{G}$ is abelian. Then, the spectrum of $L^1(\mathfrak{G})$ becomes a locally compact abelian group $\widehat{\mathfrak{G}}$, which is called the dual group of $\mathfrak{G}$, and M is spatially isomorphic to the von Neumann algebra $L^\infty(\widehat{\mathfrak{G}})$ of all complex-valued essentially bounded measurable functions over $\widehat{\mathfrak{G}}$ with the Haar measure of $\widehat{\mathfrak{G}}$, which are represented as multiplication operation on $L^2(\widehat{\mathfrak{G}})$. Therefore, once we have identified M with $L^\infty(\widehat{\mathfrak{G}})$ by extended Fourier transform, the Fourier transform of $L^1(\mathfrak{G})$ becomes the canonical imbedding of $L^1(\mathfrak{G})$ into M $(=L^\infty(\widehat{\mathfrak{G}}))$. Thus, we get the following schema of the Fourier transform $\mathfrak{F}$ and the back transform $\widehat{\mathfrak{F}}$:

$$(1) \qquad \begin{array}{c} L^1(\mathfrak{G}) \xrightarrow{\ \mathfrak{F}\ } L^\infty(\widehat{\mathfrak{G}}) \\ L^\infty(\mathfrak{G}) \xleftarrow{\ \widehat{\mathfrak{F}}\ } L^1(\widehat{\mathfrak{G}})\,. \end{array}$$

It is worth noticing that both the systems $\{L^1(\mathfrak{G}), L^\infty(\mathfrak{G})\}$ and $\{L^1(\widehat{\mathfrak{G}}), L^\infty(\widehat{\mathfrak{G}})\}$ are duality systems as Banach spaces. Then, the Pontryagin's duality theorem says that the space of all self-adjoint characters of $L^1(\widehat{\mathfrak{G}})$ (Note that $L^1(\widehat{\mathfrak{G}})$ is the pre-dual of $L^\infty(\widehat{\mathfrak{G}})$ [7]), with respect to the conjugation, is homeomorphic to the originally given locally compact abelian group $\mathfrak{G}$. The set of all self-adjoint characters of $L^1(\widehat{\mathfrak{G}})$ is the dual group $\widehat{\widehat{\mathfrak{G}}}$ of $\widehat{\mathfrak{G}}$.

Returning to the general situation and without the commutativity assumption for a given group, we cannot give the dual object $\widehat{\mathfrak{G}}$ as a group. However, we can realize a similar situation as the schema (1). In fact, M is considered as the non-commutative L^∞-space and by making use of the tensor power of the regular representation, we shall make the predual M_* of M an involutive commutative semi-simple Banach algebra and denote it by $L^1(M)$.

The representation $\lambda(x) \otimes \lambda(x)$ (tensor power of $\lambda(x)$) of $\mathfrak{G}$ is multiple of the left regular representation $\lambda(x)$; in fact, $\lambda(x) \otimes \lambda(x)$ is an $\aleph$-fold copy of $\lambda(x)$ where $\aleph$ is the dimension of $L^2(\mathfrak{G})$. This means that the representation $x \to \lambda(x) \otimes 1$ where 1 is the identity operator on $L^2(\mathfrak{G})$ is unitarily equivalent to the representation $\lambda(x) \otimes \lambda(x)$. A particular unitary operator which implements this equivalence is the operator w on $L^2(\mathfrak{G} \times \mathfrak{G}) = L^2(\mathfrak{G}) \otimes L^2(\mathfrak{G})$ defined by

$$(wf)(x, y) = f(x, xy) \quad \text{for all } f \in L^2(\mathfrak{G} \times \mathfrak{G}) .$$

Let $\Phi(t) = w^{-1}(t \otimes 1) w$, for t in M, then, Φ is a $*$-isomorphism of M into $M \otimes M$ (W^*-tensor product), such that

$$\Phi(\lambda(a)) = \lambda(a) \otimes \lambda(a) \text{ for } a \in \mathfrak{G} .$$

In the case of an abelian group $\mathfrak{G}$, $f \in L^1(\mathfrak{G})$, the operator $\lambda(f)$ corresponds to the multiplication by Fourier transform $\hat{f}$ of f on $L^2(\widehat{\mathfrak{G}})$. An easy computation shows that $\Phi(\lambda(f))$ corresponds to the mutiplication by the function $\hat{f}(\hat{x}\hat{y})$ of two variables $\hat{x}$ and $\hat{y}$ on $L^2(\widehat{\mathfrak{G}} \times \widehat{\mathfrak{G}})$. If F and H are functions in $L^1(\widehat{\mathfrak{G}})$, then, their convolution is the function $F * H$ satisfying the equation

$$\int_{\hat{\mathfrak{G}}} (F * H)(\hat{x}) \hat{f}(\hat{x}) \, d\hat{\mu}(\hat{x}) = \iint_{\hat{\mathfrak{G}} \times \hat{\mathfrak{G}}} F(\hat{x}) H(\hat{y}) \hat{f}(\hat{x}\hat{y}) \, d\hat{\mu}(\hat{x}) \, d\hat{\mu}(\hat{y}),$$

where $\hat{\mu}$ is the Haar measure on $\widehat{\mathfrak{G}}$. Thus, when $\mathfrak{G}$ is an arbitrary locally compact group, we are led to the following definition of convolution in M_*. [8, p. 48].

DEFINITION 1. If F and H are in M_*, we define $F*H$ to be the unique element of M_* such that $(F*H)(t)=(F\otimes H)(\Phi(t))$ for all t in M, where $F\otimes H$ is in $M_*\widehat{\otimes}_{\alpha_0^*}M_*$, and $\alpha_0{}^*$ means the adjoint cross norm of the α_0-norm in the sense of Turumaru [12].

Along with the convolution in M_* just defined, there is a companion involution. Let C denote complex conjugation in $L^2(\mathfrak{G})$, i.e., $(Cf)(x)=\overline{f(x)}$. If t is any operator on $L^2(\mathfrak{G})$, we define $\tilde{t}$ to be CtC. It is easy to see that $\widetilde{\lambda(a)}=\lambda(a)$ for $a\in\mathfrak{G}$, and therefore $t\to\tilde{t}$ is a conjugate linear $*$-automorphism of M.

DEFINITION 2. If F is in M_*, we define $\tilde{F}$ in M_* such that $\tilde{F}(t)=\overline{F(\tilde{t})}$.

THEOREM (Duality theorem). *The space of all self-adjoint characters of M_*, with respect to the conjugation, is homeomorphic to the originally given locally compact group. The set of all elements u of M with $\Phi(u)=u\otimes u$ and $\tilde{u}=u$ becomes a locally compact group with respect to the multiplication and the relative topology as a subset of M, which is isomorphic to $\mathfrak{G}$ under the map: $x\in\mathfrak{G}\to\lambda(x)\in M$.*

In order to prove the theorem, we need following lemmas.

LEMMA 1. *Under the convolution as multiplication, M_* becomes a commutative Banach algebra having the isometric involution $F\to\tilde{F}$.*

PROOF. As the set $\left\{\sum_{i=1}^{n}\alpha_i\lambda(x_i)\,;\ \alpha_i \text{ is a complex number and } x_i\in\mathfrak{G}\right\}$ is σ-weakly dense in M, the assertion is clear from Definitions 1 and 2.

LEMMA 2. *Let $F\in M_*$ and set $\hat{F}(x)=F(\lambda(x))$ for $x\in\mathfrak{G}$. Then, $\hat{F}(x)$ is a bounded continuous function on $\mathfrak{G}$ and $\overline{\hat{F}(x)}=\hat{\tilde{F}}(x)$.*

REMARK. The function $\hat{F}(x)$ on $\mathfrak{G}$ is the back transform of F in M_* $(=L^1(M))$.

LEMMA 3. *Let F and H be in $L^1(M)$. Set $\hat{F}(x)=F(\lambda(x))$ and $\hat{H}(x)=H(\lambda(x))$ for $x\in\mathfrak{G}$. Then $\hat{F}(x)\hat{H}(x)=F*H(\lambda(x))$.*

LEMMA 4. *The commutative involutive Banach algebra $L^1(M)$ is semi-simple and there is a one to one correspondence between operators in M*

which satisfy the equations

$$s \otimes s = \Phi(s) \quad and \quad \tilde{s} = s,$$

and $*$-homomorphisms σ of $L^1(M)$ into the complex numbers (i.e., self-adjoint characters); the correspondence being given by $\sigma(F)=F(s)$ for all F in $L^1(M)$.

PROOF. Suppose that σ is a self-adjoint character of $L^1(M)$. By [4], σ is automatically continuous linear functional on $L^1(M)$, hence $\sigma(F) = F(s)$ for some s in M. Then, for all F and H in $L^1(M)$,

$$\begin{aligned}(F\otimes H)(\Phi(s)) &= (F*H)(s) = \sigma(F*H) \\ &= \sigma(F)\cdot\sigma(H) = F(s)H(s) \\ &=(F\otimes H)(s\otimes s).\end{aligned}$$

Therefore $\Phi(s) = s\otimes s$ and also for all F in $L^1(M)$,

$$F(\tilde{s}) = \overline{\tilde{F}(s)} = \overline{\sigma(\tilde{F})} = \sigma(F) = F(s).$$

This means that $\tilde{s} = s$.

Conversely, suppose $s\otimes s=\Phi(s)$ and $s = \tilde{s}$. Set $\sigma(F) = F(s)$. Then, we have

$$\begin{aligned}\sigma(F*H) &= (F*H)(s) = (F\otimes H)(\Phi(s)) \\ &= (F\otimes H)(s\otimes s) = F(s)H(s) = \sigma(F)\cdot\sigma(H)\end{aligned}$$

for all F and H in $L^1(M)$ and

$$\sigma(\tilde{F}) = \tilde{F}(s) = \overline{F(\tilde{s})} = \overline{F(s)} = \overline{\sigma(F)}$$

for all $F\in L^1(M)$.

If $\sigma(F) = \sigma(H)$ for each self-adjoint character, where F and H are in $L^1(M)$, then considering that $\Phi(\lambda(x)) = \lambda(x)\otimes\lambda(x)$ and $\widetilde{\lambda(x)}=\lambda(x)$, it follows that $F(\lambda(x)) = H(\lambda(x))$ for all $x\in\mathfrak{G}$. Since $\left\{\sum_{i=1}^{n}\alpha_i\lambda(x_i);\ \alpha_i \text{ is a complex number and } x_i\in\mathfrak{G}\right\}$ is σ-weakly dense in M, $F=H$ and $L^1(M)$ is semi-simple. The lemma follows.

LEMMA 5. *Let $L_2^1(M)$ be the set $\{F;\ F\in L^1(M)$ and $\widehat{F}(x)\in L^2(\mathfrak{G})\}$ and $\mathscr{B}$ be the set of back transforms of the elements of $L_2^1(M)$. Then $\mathscr{B}$ is norm-dense in $L^2(\mathfrak{G})$ and is weakly dense in $L^\infty(\mathfrak{G})$, where $L^\infty(\mathfrak{G})$ is the set of all complex-valued μ-essentially bounded measurable functions on $\mathfrak{G}$.*

PROOF. Define elements $W_{f,g}$ in $L^1(M)$ by;

$$W_{f,g}(s) = (sf, g),$$

for $s\in M$ and any pair f, g in $C_c(\mathfrak{G})$, where $C_c(\mathfrak{G})$ is the set of all complex-valued continuous functions on $\mathfrak{G}$ with compact supports. Then, an easy calculation shows that $\widehat{W_{f^*,g}(x)} = (\bar{g} * (\overline{\Delta\cdot f}))(x)$ for $x\in\mathfrak{G}$, and $W_{f^*,g}\in L_2^1(M)$. Therefore $\{f*g;\ f, g\in C_c(\mathfrak{G})\}\subset\mathscr{B}$. Hence $\mathscr{B}$ is norm-dense in $L^2(\mathfrak{G})$.

Let $\mathscr{B}_0$ be the algebra of continuous functions on $\mathfrak{G}$ generated by $\{f*g;\ f, g\in C_c(\mathfrak{G})\}$. Then, $\mathscr{B}_0$ is self-adjoint and separates points of $\mathfrak{G}$ and $_0$ is uniformly dense in $C_\infty(\mathfrak{G})$, where $C_\infty(\mathfrak{G})$ is the set of all complex-valued continuous functions on $\mathfrak{G}$ vanishing at infinity. [4]. On the other hand, $C_\infty(\mathfrak{G})$ is weakly dense in $L^\infty(\mathfrak{G})$, hence $\mathscr{B}$ is weakly dense in $L^\infty(\mathfrak{G})$. This completes the proof.

REMARK. Since $L_2^1(M) = \{F;\ F\in L^1(M),\ |F(\lambda(f))|\leqq c\|f\|_2$ for some positive constant c and all $f\in L^1(\mathfrak{G})\cap L^2(\mathfrak{G})\}$, putting $\|F\|_2 = \sup\{|F(\lambda(f))|;\ f\in L^1(\mathfrak{G})\cap L^2(\mathfrak{G})\}$, the completion $L^2(M)$ of $L_2^1(M)$ under the norm $\|\ \ \|_2$ becomes a Hilbert space and the back transform can be extended to a unitary operator of $L^2(M)$ onto $L^2(\mathfrak{G})$. [5, p. 145, 36. D].

In fact, as $F(\lambda(f)) = \int_{\mathfrak{G}} f(x)\widehat{F}(x)\,d\mu(x)$ for all $f\in L^1(\mathfrak{G})\cap L^2(\mathfrak{G})$, it is clear from the above lemma.

LEMMA 6. *Let F be in $L_2^1(M)$, then for any s in M, putting $Fs(t) = F(st)$ for $a\in M$, $\widehat{Fs}\in L^2(\mathfrak{G})$ and $\widehat{Fs}=\tilde{s}^*\widehat{F}$ μ-a.e..*

PROOF. For any $g\in L^1(\mathfrak{G})\cap L^2(\mathfrak{G})$, we have

$$(\tilde{s}^*\widehat{F}, g) = (\widehat{F}, \tilde{s}g) = \int_{\mathfrak{G}} \widehat{F}(x)\overline{(\tilde{s}g)(x)}\,d\mu(x)$$

$$= \int_{\mathfrak{G}} \widehat{F}(x)(s\bar{g})(x)\,d\mu(x).$$

By the density theorem of Kaplansky, we can choose a directed system $\{f_\alpha\}$

of elements of $L^1(\mathfrak{G})$ such that $\lambda(f_\alpha) \to s$, σ-weakly and $\| \lambda(f_\alpha) \| \leqq \| s \|$. Therefore,

$$
\begin{aligned}
(\tilde{s} * \widehat{F}, g) &= \lim_\alpha \int_{\mathfrak{G}} \widehat{F}(x)((\lambda(f_\alpha))\bar{g})(x)\, d\mu(x) \\
&= \lim_\alpha \int_{\mathfrak{G}} \widehat{F}(x)(f_\alpha * \bar{g})(x)\, d\mu(x) \\
&= \lim_\alpha F(\lambda(f_\alpha * \bar{g})) = F(s \cdot \lambda(\bar{g})) \\
&= \int_{\mathfrak{G}} F(s \cdot \lambda(x))\, \bar{g}(x)\, d\mu(x) \\
&= (\widehat{Fs}, g),
\end{aligned}
$$

for all g in $L^1(\mathfrak{G}) \cap L^2(\mathfrak{G})$. Now $\widehat{Fs}$ is uniformly bounded and continuous on $\mathfrak{G}$ and $\tilde{s} * \widehat{F} \in L^2(\mathfrak{G})$, hence $\widehat{Fs} \in L^2(\mathfrak{G})$ and $\widehat{Fs} = \tilde{s} * \widehat{F}$ μ-a.e.. The lemma follows.

Let Γ be the set $\{s\,;\, s \in M,\ \Phi(s) = s \otimes s,\ \tilde{s} = s\}$, then, Γ is σ-weakly closed subset of the unit sphere of M, and therefore is σ-weakly compact. Hence, putting $\hat{\hat{\mathfrak{G}}} = \Gamma - \{0\}$, $\hat{\hat{\mathfrak{G}}}$ is a σ-weakly locally compact subset of Γ and for every pair f, g in $L^2(\mathfrak{G})$, there exists a unique finite Radon measure $\mu_{f,g}$ on $\hat{\hat{\mathfrak{G}}}$ such that $(\widehat{F}f, g) = \int_{\hat{\hat{\mathfrak{G}}}} F(s)\, d\mu_{f,g}(s)$ for all $F \in L^1(M)$. In fact, from Lemma 4, $L^1(M)$ can be represented as a dense subalgebra C of $C_\infty(\hat{\hat{\mathfrak{G}}})$, where $C_\infty(\hat{\hat{\mathfrak{G}}})$ is the set of all complex-valued continuous functions on $\hat{\hat{\mathfrak{G}}}$ vanishing at infinity and the restriction of the represented function of $L^1(M)$ to $\{\lambda(x),\ x \in \mathfrak{G}\}$ is the back transform of it. Putting $L(F) = (\widehat{F}f, g)$, if F_n converges uniformly to F on $\hat{\hat{\mathfrak{G}}}$, then, $\widehat{F_n}$ converges to $\widehat{F}$ uniformly in $L^\infty(\mathfrak{G})$ and hence $L(F_n)$ tends to $L(F)$ as $n \to \infty$. Therefore, $L(F)$ can be uniquely extended to a uniformly continuous linear functional on $C_\infty(\hat{\hat{\mathfrak{G}}})$ and there is a finite Radon measure $\mu_{f,g}$ on $\hat{\hat{\mathfrak{G}}}$, such that

$$
(\widehat{F}f, g) = \int_{\hat{\hat{\mathfrak{G}}}} F(s)\, d\mu_{f,g}(s)
$$

for all $F \in L^1(M)$.

LEMMA 7. *For E, F U and V in $L^1_2(M)$,*

$$E(s)\,\overline{F(s)}\,d\mu_{\hat{U},\hat{V}}(s) = U(s)\,\overline{V(s)}\,d\mu_{\hat{E},\hat{F}}(s)\,.$$

PROOF. For any H in $L^1(M)$, we have

$$\begin{aligned}\int_{\hat{\mathfrak{G}}} H(s)\,U(s)\,\overline{V(s)}\,d\mu_{\hat{E},\hat{F}}(s) &= \int_{\hat{\mathfrak{G}}} (H*U*\widetilde{V})(s)\,d\mu_{\hat{E},\hat{F}}(s) \\ &= (\widehat{H*U*\widetilde{V}}\cdot\widehat{E},\widehat{F}) = (\widehat{H}\cdot\widehat{U}\cdot\overline{\widehat{V}}\cdot\widehat{E},\widehat{F}) \\ &= (\widehat{H}\cdot\widehat{E}\cdot\overline{\widehat{F}}\cdot\widehat{U},\widehat{V}) \\ &= \int_{\hat{\mathfrak{G}}} H(s)\,E(s)\,\overline{F(s)}\,d\mu_{\hat{U},\hat{V}}(s)\,.\end{aligned}$$

The semi-simplicity of $L^1(M)$ shows that $U(s)\overline{V(s)}\,d\mu_{\hat{E},\hat{F}}(s) = E(s)\,\overline{F(s)}\,d\mu_{\hat{U},\hat{V}}(s)$. Thus the lemma follows.

LEMMA 8. *There exists a positive Radon measure* $\hat{\hat{\mu}}$ *on* $\hat{\hat{\mathfrak{G}}}$ *with the following properties*;

(i) $(\widehat{E},\widehat{F}) = \int_{\hat{\mathfrak{G}}} E(s)\,\overline{F(s)}\,d\hat{\hat{\mu}}(s)$ *for every pair of* $E, F \in L^1_2(M)$.

(ii) *For any* H *in* $L^1(M)$ *and any pair* E, F *in* $L^1_2(M)$,

$$\int_{\hat{\mathfrak{G}}} H(s)\,d\mu_{\hat{E},\hat{F}}(s) = \int_{\hat{\mathfrak{G}}} H(s)\,E(s)\,\overline{F(s)}\,d\hat{\hat{\mu}}(s)\,.$$

PROOF. For each $F \in L^1_2(M)$, let U_F be the set $\{s;\ s\in\hat{\hat{\mathfrak{G}}}, F(s)\neq 0\}$. Then U_F is an open subset of $\hat{\hat{\mathfrak{G}}}$. Since $L^1_2(M)$ is weakly dense in $L^1(M)$, the family $\{U_F;\ F\in L^1_2(M)\}$ forms an open covering of $\hat{\hat{\mathfrak{G}}}$. For each $f \in C_c(U_F)$, where $C_c(U_F)$ is the set of all complex-valued continuous functions on U_F with compact supports,

$$(*)\qquad \mu_F(f) = \int_{\hat{\mathfrak{G}}} \frac{f(s)}{|F(s)|^2}\,d\mu_{\hat{F},\hat{F}}(s)$$

defines a positive Radon measure μ_F on U_F. If a continuous function f on $\hat{\hat{\mathfrak{G}}}$ has the compact support contained in $U_E \cap U_F$ for some $E, F \in L^1_2(M)$, then $\mu_E(f) = \mu_F(f)$ by Lemma 7. Therefore, the system $\{U_F, \mu_F\}$ defines a unique positive Radon measure $\hat{\hat{\mu}}$ on $\hat{\hat{\mathfrak{G}}}$ whose restriction to U_F coincides with μ_F by

[2, Chap. III, §3, Prop. 1, p. 68]. Also equation (∗) implies that $d\mu_{\hat{E},\hat{F}}(s) = E(s)\overline{F(s)}\, d\hat{\hat{\mu}}(s)$ for every pair E, F in $L_2^1(M)$, which means equation (ii).

The finiteness of the Radon measure $\mu_{\hat{E},\hat{F}}$ on $\hat{\hat{\mathfrak{G}}}$ yields the square integrability of E with respect to $\hat{\hat{\mu}}$ for every $E \in L_2^1(M)$.

By the semi-simplicity of $L^1(M)$, there is a directed system $\{H_\alpha\}$ in $L^1(M)$ such that $H_\alpha \to 1$ for compact convergence topology of $\hat{\hat{\mathfrak{G}}}$ and $\|H_\alpha\|_\infty \leqq 2$. By the equation (ii), we have

$$\int_{\hat{\hat{\mathfrak{G}}}} H_\alpha(s) |F(s)|^2 \, d\hat{\hat{\mu}}(s) = \int_{\hat{\hat{\mathfrak{G}}}} H_\alpha(s)\, d\mu_{\hat{F},\hat{F}}(s)$$

$$= \int_{\mathfrak{G}} \hat{H}_\alpha(x) |\hat{F}(x)|^2 \, d\mu(x) \,.$$

Hence, we have $(\hat{F}, \hat{F}) = \int_{\hat{\hat{\mathfrak{G}}}} |F(s)|^2 \, d\hat{\hat{\mu}}(s)$, and via the polarization, equation (i) is valid. Thus the lemma follows.

REMARK. The proof of Lemma 8 is the modification of the argument used in [9, Lemma 5, 6, p. 13].

LEMMA 9. *There is no non-trivial projection in* Γ, *that is, if* $e \in \Gamma$, *and is a projection, then* $e=1$ *or* 0.

PROOF. Since M' (the commutant of $M) \supset \{\rho(x)\,;\, x \in \mathfrak{G},\ \rho$ is the right regular representation of $\mathfrak{G}\}$, putting $(1-e)L^2(\mathfrak{G}) = \mathfrak{M}$, the closed linear manifold $\mathfrak{M}$ is invariant under right translation.

Next we show $L^\infty(\mathfrak{G})\mathfrak{M} \subset \mathfrak{M}$. As $\mathscr{B}$ is weakly dense in $L^\infty(\mathfrak{G})$, it is sufficient to prove that $\mathscr{B}\mathfrak{M} \subset \mathfrak{M}$. For F in $L_2^1(M)$, and e in Γ noting that $e = \widetilde{e^*}$, by Lemma 6 $\widehat{eF} = \widehat{Fe}$ μ-a.e. and $\widehat{eF} \in L^\infty(\mathfrak{G})$. For $H \in L_2^1(M)$, we have

$$e(\hat{F} \cdot \hat{H}) = (e\hat{F})(e\hat{H}) \quad \mu\text{-a.e.}.$$

In fact,

$$\begin{aligned}(F * H)\, e(\lambda(x)) &= (F * H)(e \cdot \lambda(x)) \\ &= (F \otimes H)(\Phi(e \cdot \lambda(x))) \\ &= (F \otimes H)((e \cdot \lambda(x)) \otimes (e \cdot \lambda(x))) \\ &= F(e \cdot \lambda(x))\, H(e \cdot \lambda(x)) \,.\end{aligned}$$

Therefore $e(\widehat{F}\cdot\widehat{H})=(e\widehat{F})(e\widehat{H})$ μ-a.e.. On the other hand, as $\mathcal{B}$ is strongly dense in $L^2(\mathfrak{G})$, $e(\widehat{F}g)=(e\widehat{F})eg$ in $L^2(\mathfrak{G})$ for all $g\in L^2(\mathfrak{G})$. Thus, if $g\in\mathfrak{M}$, then $e\widehat{F}g=0$, that is, $\mathcal{B}\mathfrak{M}\subset\mathfrak{M}$.

By [13, Chap. III, p. 42], $e=1$ or 0. This completes the proof.

REMARK. In the above lemma, we can drop the condition that $\widetilde{e}=e$. In fact, by Lemma 6 and the same reason as the proof of the above lemma, we have

$$\widetilde{e}(\widehat{F}g)=\widetilde{e}\widehat{F}\,\widetilde{e}g$$

in $L^2(\mathfrak{G})$ for all $g\in L^2(\mathfrak{G})$ and F in $L^1_2(M)$. Thus, if $\widetilde{\mathfrak{M}}=(1-\widetilde{e})L^2(\mathfrak{G})$, then $\widetilde{\mathfrak{M}}$ is invariant under right translation and $L^\infty(\mathfrak{G})\widetilde{\mathfrak{M}}\subset\widetilde{\mathfrak{M}}$. Hence by the same reason, $\widetilde{e}=1$ or 0. This means that $e=1$ or 0.

Next lemma, which we prove by making use of an argument of Takesaki [9, Lemma 7, p. 14], shows that $\hat{\hat{\mathfrak{G}}}$ is contained in the unitary part of M and is a locally compact group, that is,

LEMMA 10. *$\hat{\hat{\mathfrak{G}}}$ is a locally compact group for σ-weak topology, and $\hat{\hat{\mu}}$ is its left Haar measure.*

PROOF. Since, for any s in $\hat{\hat{\mathfrak{G}}}$ and any pair E, F in $L^1_2(M)$,

$$\int_{\hat{\hat{\mathfrak{G}}}} E(st)\,\overline{F(t)}\,d\hat{\hat{\mu}}(t)=(\widehat{Es},\widehat{F})=(\widetilde{s^*}\widehat{E},\widehat{F})=(\widehat{E},s\widehat{F})$$

$$=\int_{\hat{\hat{\mathfrak{G}}}} E(t)\,\overline{F(s^*t)}\,d\hat{\hat{\mu}}(t)\qquad\text{by Lemma 4,}$$

it follows that

$$(**)\qquad\int_{\hat{\hat{\mathfrak{G}}}} f(st)\,g(t)\,d\hat{\hat{\mu}}(t)=\int_{\hat{\hat{\mathfrak{G}}}} f(t)\,g(s^*t)\,d\hat{\hat{\mu}}(t),$$

for all f and g in $C_c(\hat{\hat{\mathfrak{G}}})$.

For a little while, we shall assume that the set $H=\{t\,;\,t\in\hat{\hat{\mathfrak{G}}}\ st\in K\}$ is compact for every compact subset K of $\hat{\hat{\mathfrak{G}}}$. Then there exists, for an $f\in C_c(\hat{\hat{\mathfrak{G}}})$ with compact support K, a g in $C_c(\mathfrak{G})$ with $g(t)=1$ if $st\in K$ and $g(s^*t)=1$ if $t\in K$, which implies by (**) that

$$\int_{\hat{\hat{\mathfrak{G}}}} f(st)\,d\hat{\hat{\mu}}(t)=\int_{\hat{\hat{\mathfrak{G}}}} f(t)\,d\hat{\hat{\mu}}(t)$$

for every $f \in C_c(\hat{\mathfrak{G}})$. Therefore, the Radon measure $\hat{\mu}$ is invariant under the left translation: $t \in \hat{\mathfrak{G}} \to st \in \hat{\mathfrak{G}}$, so that, for every pair $E, F \in L^1_2(M)$ and $s \in \hat{\mathfrak{G}}$,

$$(s^*\hat{E}, s^*\hat{F}) = \int_{\hat{\mathfrak{G}}} E(st)\,\overline{F(st)}\,d\hat{\mu}(t) = \int_{\hat{\mathfrak{G}}} E(t)\,\overline{F(t)}\,d\hat{\mu}(t)$$
$$= (\hat{E}, \hat{F}).$$

From Lemma 5, $(ss^*f, g)=(f, g)$ for any pair f, g in $L^2(\mathfrak{G})$, which means that $ss^*=1$, and by the same reason, it follows that $s^*s=1$. Therefore s is a unitary element of M and s^* is the inverse of s. Hence, $\hat{\mathfrak{G}}$ is a σ-weakly locally compact Hausdorff group with the left Haar measure $\hat{\mu}$.

To complete the proof, it is sufficient to show the fact assumed in the last paragraph. First, we show that the map: $t \in M \to st \in M$ is one to one and is σ-weakly continuous for every $s \in \hat{\mathfrak{G}}$. Suppose $st = 0$ for some $t \in M$. Replacing s with s^*s, we may assume that s is a positive element of M. For each positive integer n, $s^{1/n}$ is positive n-th root of s, and $s^{1/n}$ is also in $\hat{\mathfrak{G}}$. The sequence $\{s^{1/n}\}$ converges strongly to the support projection e of s, so e belongs to $\hat{\mathfrak{G}}$. Hence by Lemma 9, $e=1$, this implies $t=0$. Since the σ-weak-continuity of this map is clear, the map: $t \in \hat{\mathfrak{G}} \cup \{0\} \to st \in s\hat{\mathfrak{G}} \cup \{0\}$ is one to one and is σ-weakly continuous and hence $sH = (s\hat{\mathfrak{G}} \cup \{0\}) \cap K$ is σ-weakly compact subset of $\hat{\mathfrak{G}} \cup \{0\}$. Therefore H is compact in $\hat{\mathfrak{G}} \cup \{0\}$. On the other hand if $t_\alpha \in H$ converges to 0 for σ-weak topology, then st_α converges to 0. But it is impossible, because st_α is contained in the compact set sH. This means the compactness of H in $\hat{\mathfrak{G}}$. The lemma follows.

Now we are in the position to prove the main theorem.

Combining above results, it is sufficient to show that the mapping $x \in \mathfrak{G} \to \lambda(x) \in \hat{\mathfrak{G}}$ is an isomorphism of $\mathfrak{G}$ onto $\hat{\mathfrak{G}}$. The mapping $x \in \mathfrak{G} \to \lambda(x) \in \hat{\mathfrak{G}}$ is continuous from $\mathfrak{G}$ into $\hat{\mathfrak{G}}$. To show that it is a homeomorphism we use the fact that the σ-weak topology on $\hat{\mathfrak{G}}$ is the same as the strong topology. Let V be any compact neighborhood of the unit of $\mathfrak{G}$. Choose $f \in L^2(\mathfrak{G})$ so that $\|f\|_2=1$ and the support K of f is such that $KK^{-1} \subset V$. Suppose $\|\lambda(a)f-f\|_2 < 1$ for some $a \in \mathfrak{G}$. Then $a \in V$. [7, Corollary 10.3].

To complete the proof, it suffices to show that this homeomorphism is onto. Let $\mathfrak{G}'$ be the image of $\mathfrak{G}$ under the map $x \to \lambda(x)$, then $\mathfrak{G}'$ is a closed subgroup of $\hat{\mathfrak{G}}$, and there is a positive Radon measure ν on $\hat{\mathfrak{G}}$ which is concentrated on $\mathfrak{G}'$ and is the Haar measure of $\mathfrak{G}'$. In fact, if f is a continuous function with compact support on $\hat{\mathfrak{G}}$, then

$$\int_{\mathfrak{G}} f(s)\, d\nu(s) = \int_{\mathfrak{G}} f(\lambda(x))\, d\mu(x)\,.$$

Let C be the algebra of all functions f on $\hat{\hat{\mathfrak{G}}}$ of the form $f(s)=F(s)$ for some $F\in L_2^1(M)$. Consider C^2 which consists of all sums of products of functions in C. If $f\in C^2$, then $f=\sum_{j=1}^{n} g_j\bar{h}_j$ where $g_j(s)=G_j(s)$ and $h_j(s)=H_j(s)$ for $s\in\hat{\hat{\mathfrak{G}}}$ with G_j, H_j in $L_2^1(M)$. Then by the definition of $L_2^1(M)$, $f\in L^1(\hat{\hat{\mathfrak{G}}},\nu)$, and

$$\begin{aligned}\int_{\hat{\hat{\mathfrak{G}}}} f(s)\, d\nu(s) &= \int_{\mathfrak{G}} f(\lambda(x))\, d\mu(x)\\ &= \sum_{j=1}^{n}\int_{\mathfrak{G}} \widehat{G}_j(x)\overline{\widehat{H}_j(x)}\, d\mu(x)\\ &= \sum_{j=1}^{n}(\widehat{G}_j, \widehat{H}_j)\,.\end{aligned}$$

Furthermore, for $t\in\hat{\hat{\mathfrak{G}}}$, and $E\in L_2^1(M)$, by Lemma 6, $\widehat{Et^*}=t\widehat{E}$ μ-a.e., hence we have

$$\begin{aligned}\int_{\hat{\hat{\mathfrak{G}}}} f(t^*s)\, d\nu(s) &= \sum_{j=1}^{n}(t\widehat{G}_j, t\widehat{H}_j)\\ &= \sum_{j=1}^{n}(\widehat{G}_j, \widehat{H}_j)\\ &= \int_{\hat{\hat{\mathfrak{G}}}} f(s)\, d\nu(s)\,.\end{aligned}$$

By applying [7, Corollary 1.4] to ν and its left translates, we conclude that ν is left Haar measure on $\hat{\hat{\mathfrak{G}}}$. In other words $\hat{\hat{\mathfrak{G}}}=\mathfrak{G}'$, and $\mathfrak{G}\cong\hat{\hat{\mathfrak{G}}}$. This completes the proof of the theorem.

COROLLARY. *Let $F\in L^1(M)$ and set $\widehat{F}(x)=F(\lambda(x))$ for $x\in\mathfrak{G}$. Then $\widehat{F}$ vanishes at infinity on $\mathfrak{G}$.*

PROOF. Since $\mathfrak{G}'\cup\{0\}$ is the one-point compactification of $\mathfrak{G}'$, the function $f(s)=F(s)$ vanishes at infinity on $\mathfrak{G}'$. [8, Corollary 10.5].

REMARK. Thus, we get the following schema of the "global" Fourier transform λ and the back transform $\widehat{\lambda}$:

$$
(2) \qquad \begin{array}{ccc} L^1(\mathfrak{G}) & \xrightarrow{\lambda} & M \\ L^\infty(\mathfrak{G}) & \xrightarrow{\hat{\lambda}} & L^1(M)\,. \end{array}
$$

It is worth noticing that the the two systems $\{L^1(\mathfrak{G}), L^\infty(\mathfrak{G})\}$, $\{L^1(M), M\}$ are both duality systems as Banach spaces.

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MATHEMATICAL INSTITUTE
TÔHOKU UNIVERSITY
SENDAI, JAPAN