Tôhoku Math. Journ.
20(1968), 368-383.

FACTORIZATIONS IN FUNCTION ALGEBRAS

SADAHIRO SAEKI

(Received April 10, 1968)

1. Introduction. In the theory of the classical Hardy classes H^p $(1 \leqq p \leqq \infty)$ there is an elegant factorization theorem of H^p-functions. A function f in H^p is a function in $L^p(-\pi,\pi)$, where we shall identify the interval $(-\pi,\pi)$ with the unit circle in the complex plane, whose negative Fourier coefficients are zero:

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{int} f(t)\, dt = 0 \qquad (n = 1, 2, 3, \cdots) . \tag{1.1}$$

Every function f in H^p produces an analytic function F in the open unit disc by the Poisson integral formula:

$$F(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\, \mathrm{Re}\left[\frac{e^{it}+z}{e^{it}-z}\right] dt . \tag{1.2}$$

If this is the case, $F(z)$ converges to the value $f(t)$ for almost all t when z approa hes non-tangentially to the point e^{it}, and it follows that

$$\lim_{r\to 1} \| F_r \|_p = \| f \|_p , \tag{1.3}$$

where F_r is the function in $L^p(-\pi,\pi)$ defined by $F_r(t) = F(re^{it})$, and $\| \cdot \|_p$ denotes the norm in $L^p(-\pi,\pi)$ normalized so that $\| 1 \|_p = 1$.

An inner function I in H^p is a function in H^p with $| I | = 1$ almost everywhere, and an outer function g in H^p is a function in H^p such that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \log|g|\, dt = \log\left|\frac{1}{2\pi}\int_{-\pi}^{\pi} g\, dt\right| > -\infty . \tag{1.4}$$

Then every non-zero function f in H^p has the form

$$f = Ig , \tag{1.5}$$

where I is an inner function in H^p and g is an outer function in H^p. In this formula I (and hence also g) is unique up to within a constant factor of modulus 1. Furthermore, every inner function I can be decomposed into the form

$$I = kBS \tag{1.6}$$

where k is a complex number with modulus 1, B is the Blaschke product induced from the zeros of I (as an analytic function in the open unit disc), and S, called a singular function, is an inner function which has the form (as a function in the open unit disc)

$$S(z) = \exp\left(-\int_{-\pi}^{\pi} \frac{e^{it}+z}{e^{it}-z}\, dm(t)\right) \qquad (|z| < 1) \tag{1.7}$$

for some non-negative singular measure m on the unit circle. About these facts, refer to Hoffman's book [4] and Privaloff's one [6].

In this paper, we are interested in the factorization formula (1.6) of inner functions. One may question what characterize Blaschke products and singular functions as functions on the unit circle. We may answer to this question as follows. In the set U of the inner functions, a linear fractional map (of the unit disc onto itself) as a function on the unit circle plays a role as a "prime" in U, since the only inner functions which divide it in U are the constant multiples of it and the constant inner functions. Then a Blaschke product may be characterized as a countable product of "primes" in U, and a singular function may be characterized as an inner function which any "prime" does not divide in U.

In this direction, we want to get the factorization formula (1.6) of inner functions in some general function algebras.

2. Notations, definitions, and a theorem. Let X be a compact Hausdorff space and A a logmodular algebra on X. As is well-known ([1], [5]), every non-zero complex homomorphism of A has a unique (non-negative) representing measure for it. Let m be a probability measure on X which is multiplicative on A:

$$\int fg\,dm = \left(\int f\,dm\right)\left(\int g\,dm\right) \qquad (f, g \in A). \tag{2.1}$$

Then m satisfies Jensen's inequality

$$\log\left|\int f\,dm\right| \leqq \int \log|f|\,dm \qquad (f \in A), \tag{2.2}$$

and is an Arens-Singer measure:

$$\log\left|\int f\,dm\right| = \int \log|f|\,dm \qquad (f\in A^{-1}), \tag{2.3}$$

where A^{-1} denotes the set of those functions in A which have an inverse in A.

Suppose now that m is an arbitrary probability measure on X. Then, A_m is the space of all functions f in A with $\int f\,dm=0$. For each positive number $p\geqq 1$, $H^p(dm)$ denotes the closure of A in the normed space $L^p(dm)$, and $H^\infty(dm)$ denotes the set of all h in $L^\infty(dm)$ such that $\int fh\,dm = 0$ for all f in A_m. It follows that $H^\infty(dm)$ is a weak-star closed subspace of $L^\infty(dm)$. An outer function g in $H^p(dm)$ is a function in $H^p(dm)$ such that

$$\log\left|\int g\,dm\right| = \int \log|g|\,dm > -\infty, \tag{2.4}$$

and an inner function I in $H^p(dm)$ is a function in $H^p(dm)$ with $|I| = 1$ almost everywhere.

These notations and definitions are used throughout this paper.

THEOREM 2.1. *Let m be a probability measure on X which is multiplicative on A. If m is not a point mass on X, then m is a continuous measure. If this is the case, $H^\infty(dm)$ contains a non-constant inner function I and a non-constant outer function g such that Ig coincides with a function in A except for a set of measure zero.*

We need a lemma.

LEMMA 2.2. *Let m be an arbitrary probability measure on X, and h any function in $L^1(dm)$ with $\int h\,dm = d > 0$. Suppose that*

$$\log\left|\int (h-z)\,dm\right| \geqq \int \log|h-z|\,dm \tag{2.5}$$

for all complex numbers z such that $|z| < d$. Then we have $h = d$ almost everywhere.

PROOF. We shall first show that for each real number $r>0$, the function

$\log|h-r\exp[it]|$ is absolutely summable on the product space $X\times(-\pi,\pi)$ with respect to the product measure $dm\,dt$, where dt is the Lebesgue measure on the interval $(-\pi,\pi)$ normalized so that $\int_{-\pi}^{\pi} dt = 1$. Since

$$|h-re^{it}| - \log|h-re^{it}| > 0,$$

Fubini's theorem applies, and we have

$$\begin{aligned}\int_X\int_{-\pi}^{\pi} &\{|h-re^{it}| - \log|h-re^{it}|\}\,dm\,dt \\ &\leqq \int_X |h|\,dm + r - \int_X\left\{\int_{-\pi}^{\pi}\log|h-re^{it}|\,dt\right\}dm \\ &= \int_X |h|\,dm + r - \int_X \max(\log|h|, \log r)\,dm \\ &\leqq \int_X |h|\,dm + r - \log r < \infty,\end{aligned}$$

for all $r>0$. This clearly implies that the function $\log|h-r\exp[it]|$ is absolutely summable for all $r>0$.

Suppose that h satisfies the hypotheses in the Lemma. Then we have

$$\log|d-re^{it}| \geqq \int \log|h-re^{it}|\,dm \qquad (0<r<d) \tag{2.6}$$

for all real numbers t. Since $\log|h-r\exp[it]|$ is absolutely summable as a function of two variables, integrating both sides of (2.6) with respect to t, we see that

$$\begin{aligned}\log d &\geqq \int_X\left\{\int_{-\pi}^{\pi}\log|h-re^{it}|\,dt\right\}dm \\ &= \int_X \max(\log|h|, \log r)\,dm \\ &= \int_{E_r}\log|h|\,dm + m(E_r')\log r \qquad (0<r<d),\end{aligned}$$

where E_r is the subset of X on which $|h|\geqq r$ and E_r' is its complement. Letting $r\to d$, it follows that

$$\log d \geqq \int_{E_d} \log|h|\, dm + m(E_d') \log d\,,$$

or equivalently, that

$$m(E_d) \log d \geqq \int_{E_d} \log|h|\, dm\,.$$

This implies that $|h| = d$ almost everywhere on E_d, and hence that $|h| \leqq d$ almost everywhere on X. Since $\int hdm = d$, we must have that $h=d$ almost everywhere. This completes the proof.

PROOF OF THEOREM 2.1. Let m be as in the Theorem. The first statement there is well-known, and is essentially contained in [5]. (See also [8].) We shall omit the proof.

In order to prove the last assertion, let K be the closed support of m. Since m is a non-zero continuous measure, K is a perfect set. Since A separates the points of X, A contains a function f such that the set $f(K)$ is not entirely contained in any circle in the complex plane. Then, Lemma 2.2, combined with Jensen's inequality (2.2), shows that

$$\int \log|f-z|\, dm > \log\left|\int (f-z)\, dm\right| > -\infty \tag{2.6}$$

for some complex number z. Putting $h=f-z$, we see that h is in A and that $\int hdm \neq 0$. Therefore, h has the form $h=Ig$, where I is an inner function in $H^\infty(dm)$ and g is an outer function in $H^\infty(dm)$ [5]. h is not an outer function by (2.6), and hence I can not be constant. Since $|g| = |h|$, and since $|h|$ is a non-constant continuous function on K, g is also non-constant. This establishes the Theorem.

3. Factorizations of $H^1(dm)$-functions. Hereafter, m denotes a fixed probability measure on X which satisfies (2.1). To avoid trivialities, we shall assume that m is not a point mass on X. Thus $H^\infty(dm)$ contains at least one inner function which is not constant. We shall often identify two inner functions I and J in $H^\infty(dm)$ (and write $I=J$) if I is a constant multiple of J. If I is not a constant multiple of J, we say that I and J are essentially distinct.

DEFINITION 3.1. For any function f in $H^p(dm)$ and any inner function

I in $H^\infty(dm)$ such that $f\bar{I}$ is also in $H^p(dm)$, we say that I divides f or that f dominates I, and write $I < f$ or $f > I$.

Let U denote the set of all inner functions in $H^\infty(dm)$. The following theorem is an easy consequence of the above definition, and we omit the proof.

THEOREM 3.2. *The relation '$<$" restricted in the set U is transitive, and has the properties*:

(i) *For any I and J in U, we have $I < J$ and $J < I$ if and only if $I = J$ (that is, if and only if I is a constant multiple of J)*;

(ii) *If I_1, I_2, J_1 and J_2 are in U, the statement that $I_1 < I_2$ and $J_1 < J_2$ implies that $I_1 J_1 < I_2 J_2$.*

Thus, in particular, U is a partially ordered set with respect to the order "$<$". To state our main theorem, we need some definitions.

DEFINITION 3.3. An LF-function L is a function in U such that any I in U which divides L is either 1 or L. A singular function S is a function in U such that the only LF-function which divides S is 1.

DEFINITIONS 3.4. A finite Blaschke product B is a function in U which is expressible in the form

$$B = I_1^{p_1} \cdots I_N^{p_N}, \tag{3.1}$$

where $\{p_n\}$ is a set of positive integers, and $\{I_n\}$ is a set of essentially distinct LF-funtions. An infinite Blaschke product B is a function in U which is expressible in the form

$$B = I_1^{p_1} \cdots I_n^{p_n} \cdots, \tag{3.2}$$

where $\{p_n\}$ is a sequence of positive integers, $\{I_n\}$ is a sequence of essentially distinct LF-funtions, and the convergence of the infinite product in the right-hand side of (3.2) is that in the weak-star topology of $L^\infty(dm)$.

Using these definitions, our main theorem is stated as follows.

THEOREM 3.5. *Every function f in $H^1(dm)$ with $\int f dm \neq 0$ has the form $f = BSg$, where B is a (finite or infinite) Blaschke product, S is a* *singular function, and g is an outer funtion in $H^1(dm)$. This factorization of f is unique up to a constant factor of modulus* 1.

The proof will be accomplished in a series of theorems. The following theorem will be frequently used in the remainder part of this paper.

THEOREM A. *Let T be a closed subspace of $H^2(dm)$ which is invariant under multiplication by the functions in A. Suppose that $\int f\,dm \neq 0$ for some f in T. Then T has the form $T=IH^2(dm)$ for some function I in U.*

This theorem is originally due to Beurling [2]; see also [3], [7], and [5].

THEOREM 3.6. *Let E be a non-empty subset of $H^2(dm)$, and suppose that there exists an element I_0 in U such that*:

(a) *I_0 divides all the functions in E*;

(b) *$\int h_0 \bar{I}_0\, dm \neq 0$ for some h_0 in E.*

Then we can find an element in U which divides the functions in E and which is the largest one in U with this property.

PROOF. Let T be the smallest closed subspace of $H^2(dm)$ that contains E and is invariant under multiplication by the functions in A. Since I_0 divides all the members in E by (a), $\bar{I}_0 T$ is a closed invariant subspace of $H^2(dm)$. By (b), $\bar{I}_0 T$ contains $h_0 \bar{I}_0$, $\int h_0 \bar{I}_0\, dm \neq 0$, and hence $\bar{I}_0 T$ must have the form $\bar{I}_0 T = GH^2(dm)$ for some G in U by Theorem A. Thus $T = GI_0 H^2(dm)$, and we see that GI_0 is a function in U which divides the functions in E.

If I is any element in U which divides all the members of E, it is trivially true that $IH^2(dm)$ contains T. But, this implies that I divides GI_0, and hence that GI_0 is the largest divisor of E. This establishes the Theorem.

DEFINITION 3.7. Suppose that E is a non-empty subset of U. The G.C.D. of E is defined to be the (essentially unique) largest lower bound of it with respect to the order "$<$", provided that it exists. If E consists of the finite elements $I_1, \cdots, I_N$, we denote the G.C.D. of it by the notation $I_1 \wedge \cdots \wedge I_N$.

Thus, Theorem 3.6 gives a sufficient condition for existence of G.C.D. We list below some rather trivial properties of the operation "$\wedge$".

THEOREM 3.8. *Let I, J, and K be any elements in U. Then we have*:

(i) *$I \wedge J$ exists, provided that $\int I dm \neq 0$ or that $I > K$, $J > K$, and at least one of the integrals $\int I\bar{K}dm$ and $\int J\bar{K}dm$ is different from 0*;

(ii) *$(I \wedge J) \wedge K = I \wedge (J \wedge K) = I \wedge J \wedge K$, if all of them exist*;

(iii) *If $I < J$, and if both $I \wedge K$ and $J \wedge K$ exist, then $I \wedge K < J \wedge K$.*

PROOF. To prove (i), it suffices to take 1 or K as I_0 in Theorem 3.6. The statements (ii) and (iii) are trivial and we omit the proofs.

THEOREM 3.9. *Let N be a natural number, and let I and I_k $(k = 1, \cdots, N)$ be any elements in U such that $I > I_k$ for all k. Then we have $I > I_1 \cdots I_N$, provided that*:

(a) *$I_j \wedge I_k = 1$ for all j, k with $j \neq k$ $(j, k = 1, \cdots, N)$*;

(b) *$\int I\bar{I}_k dm \neq 0$ for some k $(k = 1, \cdots, N)$.*

PROOF. We may assume that $N \geqq 2$ and that $\int I\bar{I}_1 dm \neq 0$. Let T be the intersection of $H^2(dm)$ and $I\bar{I}_1 \cdots \bar{I}_N H^2(dm)$. T is a closed invariant subspace of $H^2(dm)$, and contains all the elements $I\bar{I}_k$, $k = 1, \cdots, N$. Since $\int I\bar{I}_1 dm \neq 0$, we have $T = GH^2(dm)$ for some G in H^2, by Theorem A. Since G is in $I\bar{I}_1 \cdots \bar{I}_N H^2(dm)$, we must have

$$G = I\bar{I}_1 \cdots \bar{I}_N G' \tag{3.3}$$

for some G' in $H^2(dm)$. We must also have

$$I\bar{I}_k = GI'_k \qquad (k = 1, \cdots, N) \tag{3.4}$$

for some I'_k in $H^2(dm)$, because $I\bar{I}_k$ is contained in T for every $k = 1, \cdots, N$. Note that G' and all the I'_k's are actually functions in U. Let $J = I_1 \cdots I_N$, and let $J_k = J\bar{I}_k$ for $k = 1, \cdots, N$. It follows from (3.3) and (3.4) that

$$J_k = I'_k G' \qquad (k = 1, \cdots, N). \tag{3.5}$$

In case $N = 2$, (3.5) becomes

(3.6) $$I_1 = I_1' G' \quad \text{and} \quad I_2 = I_2' G',$$

which implies that G' is a constant, since $I_1 \wedge I_2 = 1$ by (a). This combined with (3.3) shows that $I\bar{I}_1\bar{I}_2$ is in U, or equivalently, that $I > I_1 I_2$, and we get the desired conclusion.

Returning to the general case, we have that

(3.7) $$\int J_1\, dm = \int I_2 \cdots I_N\, dm = \left(\int I_2\, dm\right) \cdots \left(\int I_N\, dm\right) \neq 0 .$$

To see this, applying the result in case $N=2$, we have $I > I_1 I_k$ for all $k = 2, \cdots, N$. Since $\int I\bar{I}_1\, dm \neq 0$, it follows that

$$0 \neq \int I\bar{I}_1\, dm = \left(\int I\bar{I}_1\bar{I}_k\, dm\right)\left(\int I_k\, dm\right) \quad (k=2, \cdots, N),$$

and hence (3.7) is true. Therefore, Theorem 3.6 guarantees the existence of the G.C.D. $J_1 \wedge \cdots \wedge J_N$, and (3.5) shows that $J_1 \wedge \cdots \wedge J_N > G'$. But it is easy to see that $J_1 \wedge \cdots \wedge J_N = 1$, using (a), (3.7), and the definitions of the J_k's. Thus $G'=1$, and the fact $I > I_1 \cdots I_N$ follows from (3.3). This completes the proof.

COROLLARY 3.10. *Let N be a natural number, and let J and I_k ($k=1, \cdots, N$) be any elements in U such that $\int J\, dm \neq 0$ and such that $I_1 \cdots I_N > J$. If the set $\{I_k\}$ satisfies condition* (a) *in Theorem* 3.9, *then we have $J > J_1 \cdots J_N$, where $J_k = I_k \wedge J$ for $k = 1, \cdots, N$. If, in addition, $\int I_1 \cdots I_N\, dm \neq 0$, then we have $J = J_1 \cdots J_N$.*

PROOF. Since $\int J\, dm \neq 0$, all the J_k's exist. Since the set $\{I_k\}$ satisfies condition (a) in Theorem 3.9, it is easy to see that the set $\{J_k\}$ also does that. Thus it is clear that all the conditions in Theorem 3.9 are satisfied for J and $\{J_k\}$, and hence we have the first desired conclusion. To prove the last statement, let $J' = J\bar{J}_1 \cdots \bar{J}_N$, and let $I = (I_1 \bar{J}_1) \cdots (I_N \bar{J}_N)$. It is trivial that J' and I are in U, that

(3.8) $$I > J', \quad I > I_k \bar{J}_k \qquad (k = 1, \cdots, N)$$

and that

$$(3.9)\qquad \begin{aligned} &J' \wedge I_k\bar{J}_k = 1 && (k=1,\cdots,N) \\ &I_j\bar{J}_j \wedge I_k\bar{J}_k = 1 && (j,k=1,\cdots,N;\ j\neq k)\,. \end{aligned}$$

Thus, Theorem 3.9 applies since $\int I\,dm \neq 0$, and we see that $I > J'I$, i.e., that $1 > J'$. Hence $J'=1$, and this completes the proof.

THEOREM 3.11. *Let M and N are two natural numbers, and let I_i $(i=1,\cdots,M)$ and J_j $(j=1,\cdots,N)$ be essentially distinct non-constant LF-functions such that*

$$(3.10)\qquad \int I_1\cdots I_M\,dm \neq 0\,, \quad \textit{and} \quad \int J_1\cdots J_N\,dm \neq 0.$$

Suppose that both the sets $\{I_i\}$ and $\{J_j\}$ satisfy condition (a) *in Theorem* 3.9, *and that*

$$(3.11)\qquad I_1^{p_1}\cdots I_M^{p_M}S = J_1^{q_1}\cdots J_N^{q_N}T\,,$$

where S and T are singular functions, p_i and q_j are positive integers for all i and j. Then we have $M=N$, $S=T$, and $J_1,\cdots,J_N$ is a permutation of $I_1,\cdots,I_N$.

PROOF. We shall first show that if I and J are essentially distinct LF-functions, and if $\int IJ\,dm\neq 0$, then $I^p \wedge J^q = 1$ for all positive integers p and q. To do this, let $K=I^p\wedge J^q$ for fixed p and q. Since I is an LF-funtion, $I\wedge K=1$ or I. In case $I\wedge K=1$, we have $I^p > IK$ (i.e., $I^{p-1} > K$) by Theorem 3.9, and inductively we see that $1 > K$, i.e., that $K=1$. In case $I\wedge K=I$, we have $I<K<J^q$. Since I and J are essentially distinct LF-functions, we have $J^q > JI$ by applying Theorem 3.9. By induction, we see $1 > I$, i.e., $I=1$. Therefore, $K=I^p\wedge J^q=1\wedge J^q=1$, and this gives the required conclusion.

Now put $I=I_1^{p_1}\cdots I_M^{p_M}$; we claim that $K=I\wedge T=1$. In fact, let $K_i=K\wedge I_i^{p_i}$ for all i. As shown above, $I_i^{p_i}\wedge I_k^{p_k}=1$ for all i, k with $i\neq k$, since I_i and I_k are essentially distinct LF-functions if $i\neq k$; thus Corollary 3.10 applies, and we see that $K=K_1\cdots K_M$. Since $K_i < K < T$, and since T is a singular function, each K_i must be a singular function. But, $I_i^{p_i} > K_i$, and each I_i is an LF-funtion; it is easy to see (by using the preceding arguments) that $K_i=1$ for all i, and hence that $K=1$. Now since $I\wedge T=1$, and since both I and T divide $IS=JT$, we can apply the Theorem 3.9, and we see that $IS>IT$. Therefore $S > T$, and similarly $S < T$; hence $S=T$.

To complete the proof, we note that the set $\{I_i^{p_i}\}$ satisfies condition (a) in

Theorem 3.9. Since $I=I_1^{p_1}\cdots I_M^{d_M}=J_1^{q_1}\cdots J_N^{q_N}$, we have $I>J_1^{q_1}$; thus Corollary 3.10 applies, and we get $J_1^{q_1}=(J_1^{q_1}\wedge I_1^{p_1})\cdots(J_1^{q_1}\wedge I_M^{p_M})$. Repeating the arguments which we have often used above, we see that $J_1=I_i$ and $q_1=p_i$ for some i. By induction, we have the desired conclusion.

THEOREM 3.12. *Let D be a non-empty subset of U which is directed with respect to the order "$<$", that is, which has the property: for each pair I and J in D, we can find a K in D such that $I<K$ and $J<K$. Suppose that*

$$\inf\left\{\left|\int I\,dm\right| : I\in D\right\}>0\,. \tag{3.12}$$

Then we can find a J in U which dominates all the functions in D and which is the smallest one in U with this property.

PROOF. For each I_1 in D, let $D(I_1)$ be the set of all I in D with $I_1<I$. Since D is directed, it is easy to see that the family of all the sets $D(I)$, I in D, has the finite intersection property. Since the closed unit ball of $H^\infty(dm)$ is compact in the weak-star topology of $L^\infty(dm)$, this assures that there exists a J in the unit ball of $H^\infty(dm)$ which lies in the weak-star closure of $D(I)$ for every I in D. We want to show that this J has all the required properties. Let I_0 be an arbitrary element in D. Take any function f in A_m and any real number $\varepsilon>0$. Since $f\bar{I}_0$ is a function in $L^1(dm)$, and since J is in the weak-star closure of $D(I_0)$, there exists an I_1 in D with $I_0<I_1$ such that

$$\left|\int f\bar{I}_0(J-I_1)\,dm\right|<\varepsilon\,.$$

Since $I_1\bar{I}_0$ is in $H^\infty(dm)$ and since f is in A_m, we see that

$$\int fI_1\bar{I}_0\,dm=\left(\int f\,dm\right)\left(\int I_1\bar{I}_0\,dm\right)=0\,.$$

Therefore we have

$$\left|\int fJ\bar{I}_0\,dm\right|\leqq\left|\int f\bar{I}_0(J-I_1)\,dm\right|+\left|\int fI_1\bar{I}_0\,dm\right|<\varepsilon\,.$$

But since ε was arbitrary, we must have

$$\int fJ\bar{I}_0\,dm=0\,. \tag{3.13}$$

This holds for every f in A_m, and hence $J\bar{I}_0$ is a function in $H^\infty(dm)$. Since I_0 is an arbitrary element in D, it follows that J dominates all the members in D. In order to prove that $|J|=1$ almost everywhere, we note that

$$\left|\int J\,dm\right| \geqq d > 0 \tag{3.14}$$

for some d, as is easily seen from (3.12). Now for given $\varepsilon > 0$, we can find an I in D such that

$$\left|\int (J-I)\,dm\right| < \varepsilon\,, \quad \left|\int J(\bar{I}-\bar{J})\,dm\right| < \varepsilon\,, \tag{3.15}$$

because J is in the weak-star closure of D. Thus we have

$$\begin{aligned}\int |J|^2\,dm &\geqq \left|\int J\bar{I}\,dm\right| - \left|\int J(\bar{I}-\bar{J})\,dm\right| \\ &\geqq \left|\int J\bar{I}\,dm\right| - \varepsilon = \left|\int J\,dm\right| \Big/ \left(\left|\int I\,dm\right|\right) - \varepsilon \\ &\geqq \left|\int J\,dm\right| \Big/ \left(\left|\int J\,dm\right| + \varepsilon\right) - \varepsilon\,.\end{aligned} \tag{3.16}$$

Since ε can be taken arbitrarily small, and since $\int J\,dm \neq 0$ by (3.14), it must be

$$\int |J|^2\,dm \geqq 1\,. \tag{3.17}$$

But, since $|J| \leqq 1$ almost everywhere, this implies that $|J| = 1$ almost everywhere. Thus we conclude that J is an inner function in $H^\infty(dm)$ which dominates all the members in D.

Finally, let I' be a function in U such that $I < I'$ for all I in D. Let f in A_m and $\varepsilon > 0$ be arbitrary. We can find an I_1 in D such that

$$\left|\int f I'(\bar{J}-\bar{I}_1)\,dm\right| < \varepsilon\,,$$

since fI' is a function in $L^1(dm)$. Noting that $I'\bar{I}_1$ belongs to $H^\infty(dm)$, we see that

$$\left|\int fI'\bar{J}\,dm\right| \leqq \left|\int fI'(\bar{J}-\bar{I}_1)\,dm\right| + \left|\int fI'\bar{I}_1\,dm\right| < \varepsilon + 0 = \varepsilon\,.$$

Since ε can be taken arbitrarily small, this implies that $I'\bar{J}$ is an $H^\infty(dm)$-function, that is, that $J < I'$. It follows that J is the smallest upper bound of D.

The Theorem is now completely established.

COROLLARY 3.13. *Let $I_1, I_2, \cdots, I_n, \cdots$ be a sequence of inner functions in $H^\infty(dm)$. Then we have:*

(i) *If $I_1 < I_2 < \cdots < I_n < \cdots$, and if for some natural number N the limit*

$$\lim_{n\to\infty} \int I_n \bar{I}_N\, dm \tag{3.18}$$

exists and is different from 0, then the sequence $\{I_n\}$ converges to an inner function in $H^\infty(dm)$ in the weak-star topology of $L^\infty(dm)$. The limit point of this sequence is the smallest upper bound in U of the set $\{I_n\}$;

(ii) *If there exists a natural number N such that the limit*

$$\lim_{n\to\infty} \int I_N I_{N+1} \cdots I_n\, dm \tag{3.19}$$

exists and is different from 0, then the infinite product

$$I_1 I_2 \cdots I_n \cdots \tag{3.20}$$

converges to an inner function in $H^\infty(dm)$ in the weak-star topology of $L^\infty(dm)$. The limit point of this product is the smallest upper bound in U of the set of all the partial products of the infinite product in (3.20).

PROOF. We note, in general, that a sequence $\{h_n\}$ in $L^\infty(dm)$ converges to an h in $L^\infty(dm)$ if and only if the sequence $\{h_n f\}$ converges to hf for every f in $L^\infty(dm)$. Hence we may assume that $N=1$, $I_N=1$ in (i) and that $N=1$ in (ii).

Now suppose that the sequence $\{I_n\}$ satisfies the hypotheses in (i). Then, Theorem 3.12 assures that the set $\{I_n\}$ has a (the) smallest upper bound J in U, and its proof shows that a subsequence of $\{I_n\}$ converges to J. If K is any cluster point of the sequence $\{I_n\}$, it is easy to see that K is in U and is also a smallest upper bound of the set $\{I_n\}$. Hence K is a constant multiple of J. But, from (3.18), it is clear that

$$\int J\,dm = \int K\,dm = \lim_{n\to\infty}\int I_n\,dm \neq 0.$$

Therefore J and K coincide as functions of $H^\infty(dm)$. Thus J is the only cluster point of the sequence $\{I_n\}$, and it is easy to see that the $\{I_n\}$ converges to J.

To prove (ii), let $J_n = I_1 \cdots I_n$ for $n = 1, 2, 3, \cdots$. Then (i) applies, and we are done.

PROOF OF THEOREM 3.5. Let f be any function in $H^1(dm)$ with $\int f\,dm \neq 0$. f has the form $f = Ig$, where I is an inner function in $H^\infty(dm)$ and g is an outer function in $H^1(dm)$. This factorization of f is unique up to a constant factor of modulus 1. $\int I\,dm \neq 0$, because $\int f\,dm \neq 0$. Let D be the set consisting of all essentially distinct LF-functions which divide I. We first show that D contains at most countably many functions. In fact, let t be any real number such that $0 < t < 1$, and let $I_1, \cdots, I_N$ be any finite distinct functions in D such that $\left|\int I_k\,dm\right| < t$. Since $\{I_k\}$ satisfies condition (a) in Theorem 3.9, and since $\int I\,dm \neq 0$, we have $I > I_1 \cdots I_N$. Thus we must have

$$0 < \left|\int I\,dm\right| \leqq \left|\int I_1 \cdots I_N\,dm\right| < t^N. \tag{3.21}$$

This implies that D contains at most finitely many functions J such that $\left|\int J\,dm\right| < t$. Since t was arbitrary, and since the only inner functions J such that $\left|\int J\,dm\right| = 1$ are the constant inner functions, we conclude that D contains at most countably infinite functions.

Let $D = \{I_0, I_1, \cdots, I_n, \cdots\}$. We may assume that $I_0 = 1$. Arrange each I_n so that $\int I_n\,dm > 0$, and let p_n be the largest positive integer such that $I > I_n^{p_n}$ for all $n \geqq 1$. As we saw in the proof of Theorem 3.11, we have $I_j^{p_j} \wedge I_k^{p_k} = 1$ if $j \neq k$. Thus Theorem 3.9 applies, and we have

(3.22) $$I > I_0 I_1^{p_1} \cdots I_n^{p_n} \quad \text{(for all } n\text{)}.$$

In case D consists of the finite elements $I_0, I_1, \cdots, I_N$, put

(3.23) $$B = I_0 I_1^{p_1} \cdots I_N^{p_N}.$$

B is a finite Blaschke product, and it is easy to see that $S = I\bar{B}$ is a singular function. Thus we have $I = BS$, and this factorization of I is unique up to a constant multiple of modulus 1 by Theorem 3.11.

In case D consists of the countably infinite elements $I_0, I_1, \cdots, I_n, \cdots$, note that the numerical sequence

$$\left\{\int I_1^{p_1} \cdots I_n^{p_n}\, dm\right\}$$

is a decreasing sequence of positive numbers, since $0 < \int I_n\, dm < 1$ for all $n = 1, 2, 3, \cdots$, and is bounded from 0 by (3.22). Hence, by virtue of part (ii) of Corollary 3.13, the infinite product

(3.24) $$I_1^{p_1} \cdots I_n^{p_n} \cdots$$

converges to an inner function B in $H^\infty(dm)$. B is an infinite Blaschke product by the definition. It is easy to see that I dominates B and that $S = I\bar{B}$ is a singular functions. Thus we have the desired factorization $I = BS$. To prove the uniqueness of this factorization, let I have another form $B'S'$, where B' is a Blaschke product and S' is a singular function. Let J be any LF-function, and p any positive integer. It is easy to see that J^p divides B' if and only if it divides $B'S'(=I)$. But this last condition is equivalent to that J^p divides B. Since B is the smallest upper bound of the set consisting of all the partial products of the infinite product in (3.24) by (ii) of Corollary 3.13, we see that $B < B'$. On the other hand, B dominates each partial product of the infinite product that represents B', and hence it follows that $B' < B$. Therefore, B' is a constant multiple of B, and so the factorization of I is unique up to a constant factor of modulus 1. This completes the proof of Theorem 3.5.

REMARKS. (a) In Theorem 3.5, the condition $\int f\, dm \neq 0$ cannot be dropped. See [3; p. 176]. (b) The author does not know whether the condition that m is a continuous measure implies that $H^\infty(dm)$ contains at least one inner function which is a non-constant LF-function. But this is the case, if m is contained in a non-trivial Gleason part. In fact, in this case, H^2_m (the space of all the functions h in $H^2(dm)$ such that $\int hdm=0$) has the form $ZH^2(dm)$ for some inner function Z [5; p. 309]. Then it is trivial that Z is an LF-function.

REFERENCES

[1] R. ARENS AND I. M. SINGER, Function values as boundary integrals, Proc. Amer. Math. Soc., 5(1954), 735-745.
[2] A. BEURLING, On two problems concerning linear transformations in Hilbert space, Acta Math., 81(1949), 239-255.
[3] H. HELSON AND D. LOWDENSLAGER, Prediction theory and Fourier series in several variables, Acta Math., 99(1958), 165-202.
[4] K. HOFFMANN, Banach spaces of analytic functions, Prentice-Hall, Englewood Cliffs, 1962.
[5] K. HOFFMANN, Analytic functions and logmodular Banach algebras, Acta Math., 108 (1962), 271-317.
[6] I. PRIVALOFF, Randeigenschaften analytischer Funktionen, Deutscher Verlag der Wiss., Berlin, 1956.
[7] W. RUDIN, Fourier analysis on groups, Interscience, 1962.
[8] J. WERMER, Seminar über Funktionen-Algebren, Lecture notes in Math. No. 1, Springer, 1964.

DEPARTMENT OF MATHEMATICS
TOKYO METROPOLITAN UNIVERSITY
TOKYO, JAPAN