Tôhoku Math. Journ.
20 (1968), 257-264.

ON KILLING TENSORS IN A RIEMANNIAN SPACE

SHUN-ICHI TACHIBANA

(Received January 20, 1968)

0. Introduction. Let M^n be an n dimensional Riemannian space whose metric tensor is given by g_{ab} in terms of local coordinates $\{x^a\}$[1]. A vector field v^a is called a Killing vector if it satisfies the Killing's equation:

$$\nabla_a v_b + \nabla_b v_a = 0, \tag{0.1}$$

where $v_a = g_{ac}v^c$ and ∇_a denotes the operator of the covariant derivation. An affine Killing vector is, by definition, a vector field v^a satisfying the equation:

$$\nabla_a \nabla_b v^c + R_{eab}{}^c v^e = 0, \tag{0.2}$$

where $R_{eab}{}^c$ is the Riemannian curvature tensor. As the equation (0.2) is a consequence of (0.1), a Killing vector is necessarily an affine Killing vector. As to the converse, the following theorem is famous.

THEOREM A. (Yano [5][2]) *In a compact Riemannian space an affine Killing vector is a Killing vector.*

On the other hand, concerning the integrability condition of Killing's equation, the following theorem is well known.[3]

THEOREM B. *A necessary and sufficient condition in order that for any point p of a Riemannian space M^n and any constants C_a and C_{ab} $(=-C_{ba})$ there exists (locally) a Killing vector v^a satisfying $v_a(p)=C_a$, $(\nabla_a v_b)(p)=C_{ab}$ is that M^n is a space of constant curvature.*

As a generalization of Killing vector, K. Yano [6] has defined Killing tensor as follows:

A skew symmetric tensor $u_{a_1 \ldots a_r}$ is called a Killing tensor of order r, if it satisfies

1) Indices $a, b, \ldots$ run over the range $1, 2, \ldots, n$.
2) The number in brackets refers to the Bibliography at the end of the paper.
3) For example, L. P. Eisenhart, [1], p. 213.

(0. 3) $$\nabla_{a_1}u_{a_2\cdots a_{r+1}}+\nabla_{a_2}u_{a_1a_3\cdots a_{r+1}}=0.$$

We shall call this equation the Killing-Yano's equation.

The purpose of this paper is to obtain the corresponding theorems for Killing tensor of order 2 to the theorems stated above.

In §1 we shall deduce an equation for Killing tensor corresponding to (0. 2). §2 will be devoted to the discussion of the integrability condition of Killing-Yano's equation. T. Fukami and S. Ishihara [3] have given an example of Killing tensor of order 2 which is related to the almost complex structure on S^6 given by A. Fröhlicher [2]. Recently Y. Ogawa [4] showed that there exists Killing tensors of order odd in a Sasakian space. In §3 we shall give examples of Killing tensor of order 2 in the Euclidean space and in the sphere. About Killing tensor of order $r\,(>2)$ we shall discuss in another place.

1. Killing tensor. Let u_{ab} be a Killing tensor in an $n\,(>2)$ dimensional Riemannian space M^n. Then as we have

$$\nabla_b u_{cd}+\nabla_c u_{bd}=0,$$

it follows that

(1. 1) $$\nabla_a\nabla_b u_{cd}+\nabla_a\nabla_c u_{bd}=0.$$

Interchanging the indices in (1. 1) as $a\to b\to c\to a$, we have

(1. 2) $$\nabla_b\nabla_c u_{ad}+\nabla_b\nabla_a u_{cd}=0,$$

(1. 3) $$\nabla_c\nabla_a u_{bd}+\nabla_c\nabla_b u_{ad}=0.$$

Forming (1. 1)+(1. 2)−(1. 3), we get

(1. 4) $$\nabla_a\nabla_b u_{cd}+\nabla_b\nabla_a u_{cd}+(\nabla_a\nabla_c u_{bd}-\nabla_c\nabla_a u_{bd})+(\nabla_b\nabla_c u_{ad}-\nabla_c\nabla_b u_{ad})=0.$$

Then by virtue of the Ricci's identity:

$$\nabla_a\nabla_c u_{bd}-\nabla_c\nabla_a u_{bd}=-R_{acb}{}^e u_{ed}-R_{acd}{}^e u_{be},$$

we can write (1. 4) in the following form:

(1. 5) $$2\nabla_a\nabla_b u_{cd}+2R_{bca}{}^e u_{de}-R_{bad}{}^e u_{ce}-R_{acd}{}^e u_{be}-R_{bcd}{}^e u_{ae}=0.$$

Though (1. 5) may be the equation corresponding to (0. 2), it seems to be desirable that we have equations in the form skew symmetric with respect

to b, c and d. We shall proceed to find such equations as follows.
Interchanging the indices in (1.5) as $b\rightarrow c\rightarrow d\rightarrow b$, we have

$$2\nabla_a\nabla_c u_{db}+2R_{cda}{}^e u_{be}-R_{cab}{}^e u_{de}-R_{adb}{}^e u_{ce}-R_{cdb}{}^e u_{ae}=0\,,$$

$$2\nabla_a\nabla_d u_{bc}+2R_{dba}{}^e u_{ce}-R_{dac}{}^e u_{be}-R_{abc}{}^e u_{de}-R_{dbc}{}^e u_{ae}=0\,.$$

Adding these equations to (1.5) and taking account of the skew symmetric property of $\nabla_b u_{cd}$, we can get

$$\nabla_a\nabla_b u_{cd}+(1/2)(R_{cda}{}^e u_{be}+R_{bca}{}^e u_{de}+R_{dba}{}^e u_{ce})=0, \tag{1.6}$$

or equivalently

$$\nabla_a\nabla_b u_{cd}+(1/2)(R_{aecd}u_b{}^e+R_{eabc}u^e{}_d+R_{eabd}u_c{}^e)=0.$$

It seems to be suitable to consider (1.6) as the equation corresponding to (0.2).

REMARK. In a space of constant curvature, as we have

$$R_{abc}{}^d=k(g_{bc}\delta_a{}^d-g_{ac}\delta_b{}^d) \tag{1.7}$$

where k is constant, the equation (1.5) reduces to the following form:

$$\nabla_a\nabla_b u_{cd}=-k(g_{ab}u_{cd}+g_{ac}u_{db}+g_{ad}u_{bc}). \tag{1.8}$$

For the discussion in the next section we prepare the following

LEMMA. *If there exists (locally) a Killing tensor u_{ab} satisfying $u_{ab}(p)=C_{ab}$ for any point p of M^n and any constants $C_{ab}\,(=-C_{ba})$, then M^n is a space of constant curvature.*

PROOF. Forming $(1.6)\times 2-(1.5)$, we can get

$$R_{cba}{}^e u_{de}-R_{cbd}{}^e u_{ae}+R_{adc}{}^e u_{be}-R_{adb}{}^e u_{ce}=0\,,$$

i.e.,

$$(R_{cba}{}^e\delta_d{}^f-R_{cbd}{}^e\delta_a{}^f+R_{adc}{}^e\delta_b{}^f-R_{adb}{}^e\delta_c{}^f)u_{fe}=0.$$

By the assumption, as the skew symmetric part of coefficients of u_{fe} are zero, we have

$$R_{cba}{}^{e}\delta_{d}{}^{f}-R_{cbd}{}^{e}\delta_{a}{}^{f}+R_{adc}{}^{e}\delta_{b}{}^{f}-R_{adb}{}^{e}\delta_{c}{}^{f}$$
$$-R_{cba}{}^{f}\delta_{d}{}^{e}+R_{cbd}{}^{f}\delta_{a}{}^{e}-R_{adc}{}^{f}\delta_{b}{}^{e}+R_{adb}{}^{f}\delta_{c}{}^{e}=0.$$

By contraction with respect to d and f it follows that

$$(n-1)R_{cba}{}^{e}=-R_{ac}\delta_{b}{}^{e}+R_{ab}\delta_{c}{}^{e}.$$

Transvecting this with g^{ba}, we have $R_{c}{}^{e}=(R/n)\delta_{c}{}^{e}$, where R means the scalar curvature. Thus we can obtain the equation of the form (1.7). Q.E.D.

Now we shall call a skew symmetric tensor u_{ab} *an affine Killing tensor* (of order 2) if it satisfies (1.6). Then a Killing tensor is an affine Killing tensor. Conversely, we can obtain the following theorem corresponding to Theorem A.

THEOREM 1. *In a compact Riemannian space, an affine Killing tensor (of order* 2 *) is a Killing tensor.*

PROOF. Transvecting (1.6) with g^{ab}, we have

$$\nabla^{a}\nabla_{a}u_{cd}+(1/2)(R_{d}{}^{e}u_{ce}+R_{c}{}^{e}u_{ed}+R_{cdae}u^{ae})=0, \tag{1.9}$$

where $R_{ce}=g^{ab}R_{cabe}$ means the Ricci tensor.

Next from transvection (1.6) with g^{bc}, it follows that $\nabla_{a}\nabla^{b}u_{bd}=0$, so we get

$$(\nabla^{a}u_{a}{}^{d})(\nabla^{b}u_{bd})=\nabla^{a}(u_{a}{}^{d}\nabla^{b}u_{bd}). \tag{1.10}$$

Without loss of generality we may assume that M^{n} is orientable and then applying the Green's theorem to (1.10) we have

$$\nabla^{b}u_{bd}=0. \tag{1.11}$$

Thus we know that an affine Killing tensor satisfies (1.9) and (1.11). On the other hand, it is known that these equations are a sufficient condition for a skew symmetric tensor u_{ab} to be a Killing tensor.[4] Thus the theorem is proved. Q.E.D.

2. Integrability condition of Killing-Yano's equation[5]. In a Riemannian space M^{n} $(n>2)$ we consider Killing-Yano's equation as a system of partial

4) K. Yano, [6]. K. Yano and S. Bochner, [7] p. 76.

5) In this section, we shall assume that M^{n} and all quantities are real analytic.

differential equations of unknown functions u_{ab}. This system is equivalent to the following system of partial differential equations with unknown functions u_{ab} and u_{abc}:

$$u_{bc}+u_{cb}=0, \tag{2.1}$$

$$u_{bcd}+u_{cbd}=0, \tag{2.2}$$

$$u_{bcd}+u_{bdc}=0, \tag{2.3}$$

$$\nabla_b u_{cd}=u_{bcd}, \tag{2.4}$$

$$\nabla_a u_{bcd}=-R_{bca}{}^e u_{de}+(1/2)(R_{bad}{}^e u_{ce}+R_{acd}{}^e u_{be}+R_{bcd}{}^e u_{ae}). \tag{2.5}$$

We shall discuss the integrability condition of this system.

Assume that the system (2.1)～(2.5) is completely integrable, so for any point p of M^n and any constant C_{ab} $(=-C_{ba})$ and C_{bcd} $(=-C_{cbd}=-C_{bdc})$ there exists (locally) a Killing tensor u_{ab} satisfying $u_{ab}(p)=C_{ab}$ and $(\nabla_b u_{cd})(p)=C_{bcd}$. Then, by Lemma in §1, we know that M^n is a space of constant curvature.

Conversely, we shall show that the system is compltely integrable if M^n is of constant curvature.

From our assumtion, we can replace (2.5) by the following equation:

$$\nabla_a u_{bcd}=-k(g_{ab}u_{cd}+g_{ac}u_{db}+g_{ad}u_{bc}), \tag{2.5$'$}$$

on taking account of (1.8).

The equation obtained from (2.1) by differentiation:

$$\partial_a u_{bc}+\partial_a u_{cb}=0, \qquad (\partial_a=\partial/\partial x^a),$$

are satisfied identically by (2.4), (2.3) and (2.1). Next consider the equations obtained from (2.2) by differentiation:

$$\partial_a u_{bcd}+\partial_a u_{cbd}=0. \tag{2.6}$$

We can see easily that (2.6) is satisfied identically by (2.5)′, (2.1) and (2.3). The equation obtained from (2.3):

$$\partial_a u_{bcd}+\partial_a u_{bdc}=0$$

are satisfied identically too by virtue of (2.1), (2.5)′ and (2.3).

Next consider the integrability condition of (2.4):

$$\nabla_a \nabla_b u_{cd} - \nabla_b \nabla_a u_{cd} = -R_{abc}{}^e u_{ed} - R_{abd}{}^e u_{ce}. \tag{2.7}$$

Taking account that M^n is of constant curvature, we know that (2.7) is an algebraic consequence of (2.4), (2.5)′ and (2.1).

The integrability condition of (2.5)′ is

$$\nabla_f \nabla_a u_{bcd} - \nabla_a \nabla_f u_{bcd} = -R_{fab}{}^e u_{ecd} - R_{fac}{}^e u_{bed} - R_{fad}{}^e u_{bce},$$

which follows from (2.5)′, (2.4), (2.2) and (2.3).

Thus the system (2.1) ~ (2.4) and (2.5)′ is completely integrable. Hence we get

THEOREM 2. *A necessary and sufficient condition in order that Killing-Yano's equation is completely integrable is that the Riemannian space M^n $(n>2)$ is a space of constant curvature.*

3. Examples of Killing tensors. (i) Let E^{n+1} be a Euclidean space and $\{y^\lambda\}$ $(\lambda=1,\cdots,n+1)$ an orthogonal coordinate system. A Killing tensor in E^{n+1} is a skew symmetric tensor $u_{\lambda\mu}$ such that

$$\partial_\lambda u_{\mu\nu} + \partial_\mu u_{\lambda\nu} = 0, \quad (\partial_\lambda = \partial/\partial y^\lambda). \tag{3.1}$$

For such a tensor, we have by virtue of (1.5)

$$\partial_\lambda \partial_\mu u_{\nu\omega} = 0.$$

Integrating the last equation we get as the general solution of (3.1)

$$u_{\nu\omega} = y^\alpha a_{\alpha\nu\omega} + b_{\nu\omega}, \tag{3.2}$$

where $a_{\alpha\nu\omega}$ and $b_{\nu\omega}$ are skew symmetric constant tensors.

(ii) Consider an $n+1$ dimensional Riemannian space M^{n+1} and a hypersurface M^n represented locally by $y^\lambda = y^\lambda(x^a)$ in terms of local coordinates $\{y^\lambda\}$ in M^{n+1} and $\{x^a\}$ in M^n. Putting $B_a^\lambda = \partial y^\lambda/\partial x^a$, the induced metric g_{ab} is given by $g_{ab} = G_{\lambda\mu} B_a^\lambda B_b^\mu$, where $G_{\lambda\mu}$ means the Riemannian metric of M^{n+1}. The second fundamental tensor $H_{ab}{}^\lambda$ is defined by

$$H_{ab}{}^\lambda \equiv \nabla_a B_b^\lambda \equiv \partial B_b^\lambda/\partial x^a - \left\{ {c \atop ab} \right\} B_c^\lambda + \left\{ {\lambda \atop \mu\nu} \right\} B_a^\mu B_b^\nu,$$

where $\left\{ {c \atop ab} \right\}$ means the Christoffel's symbols of g_{ab}.

Let $u^{\lambda\mu}$ be a skew symmetric tensor field in M^{n+1} and assume that it is tangent to M^n at any point of M^n, i.e., we have on M^n

(3.3) $$u^{\lambda\mu}=B_a^{\lambda}B_b^{\mu}v^{ab},$$

where v^{ab} is a skew symmetric tensor field on M^n.

Defining $B^a{}_\lambda$ by $B^a{}_\lambda=g^{ab}g_{\lambda\mu}B_b^{\mu}$, (3.3) reduces to

$$u_{\lambda\mu}=B^a{}_\lambda B^b{}_\mu v_{ab}$$

in terms of covariant components of the tensor.

If we differentiate the last equation covariantly along M^n, we obtain

$$\nabla_c u_{\lambda\mu}\equiv B_c^{\nu}\nabla_\nu u_{\lambda\mu}$$
$$=(H_c{}^a{}_\lambda B^b{}_\mu+B^a{}_\lambda H_c{}^b{}_\mu)v_{ab}+B^a{}_\lambda B^b{}_\mu\nabla_c v_{ab}.$$

Transvecting this with $B_d{}^\lambda$ we can get

(3.4) $$B_c^{\nu}B_d{}^{\lambda}\nabla_\nu u_{\lambda\mu}=H_c{}^b{}_\mu v_{db}+B^b{}_\mu\nabla_c v_{db}.$$

Now we assume that our M^n is totally umbilic. Then there exists a vector field C_λ on M^n locally such as $H_c{}^b{}_\mu=C_\mu\delta_c{}^b$ and hence (3.4) reduces to

$$B_c^{\nu}B_d{}^{\lambda}\nabla_\nu u_{\lambda\mu}=C_\mu v_{dc}+B^b{}_\mu\nabla_c v_{db}.$$

This equation shows that v_{db} is a Killing tensor on M^n provided that $u_{\lambda\mu}$ is Killing.

Now let $M^{n+1}=E^{n+1}$ and apply the above argument to the sphere $S^n:\Sigma(y^\lambda)^2=1$. The condition in order that a skew symmetric tensor $u_{\lambda\mu}$ to be tangent to S^n everywhere is $y^\alpha u_{\alpha\mu}=0$. As a Killing tensor in E^{n+1} is of the form (3.2), we know that

$$u_{\lambda\mu}=y^\alpha a_{\alpha\lambda\mu}$$

is a Killing tensor defined globally on S^n, where $a_{\alpha\lambda\mu}$ is a skew symmetric constant tensor in E^{n+1}.

BIBLIOGRAPHY

[1] L. P. EISENHART, Continuous groups of transformations, Princeton Univ. Press, 1933.
[2] A. FRÖHLICHER, Zur Differentialgeometrie der komplexen Strukturen, Math. Ann., 129 (1955), 50-95.
[3] T. FUKAMI AND S. ISHIHARA, Almost Hermitian structure on S^6, Tôhoku Math. Journ., 7(1955), 151-156.
[4] Y. OGAWA, On C-harmonic forms in a compact Sasakian space, Tôhoku Math. Journ., 19(1967), 267-296.

[5] K. YANO, On harmonic and Killing vector fields, Ann. of Math., 55(1952), 38–45.
[6] K. YANO, Some remarks on tensor fields and curvature, Ann. of Math., 55(1952), 328–347.
[7] K. YANO, AND S. BOCHNER, Curvature and Betti numbers, Ann. of Math., Studies, N. 32, 1953.

DEPARTMENT OF MATHEMATICS
OCHANOMIZU UNIVERSITY
TOKYO, JAPAN