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A REMARK ON A CLASS OF CONVOLUTION TRANSFORMS

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1. Introduction. In this paper the kernels of a class of convolution transforms will be treated. The convolution transform is defined (as usual) by

$$f(x)=\int_{-\infty}^{\infty} G(x-t)Y(t)\,dt\,, \tag{1.1}$$

where $G(t)$ is the kernel. In our case the bilateral Laplace transform of $G(t)$ exists in a certain vertical strip, and

$$[F(s)]^{-1}=\int_{-\infty}^{\infty} e^{-st}G(t)\,dt\,. \tag{1.2}$$

We shall limit $F(s)$ (which is of course limited also by (1.2)) as follows:

$$F(s)=e^{bs}\prod_{k=1}^{\infty}\{(1-sa_k^{-1})e^{s/a_k}/(1-sc_k^{-1})e^{s/c_k}\} \tag{1.3}$$

where b, $\{a_k\}_{k=1}^{\infty}$ and $\{c_k\}_{k=1}^{\infty}$ are real numbers, $0\leqq(a_k/c_k)<1$ and $\sum_{k=1}^{\infty}a_k^{-2}<\infty$. We shall permit $c_k=\infty$ when $a_k>0$ or $c_k=-\infty$ when $a_k<0$ by which we shall understand $(1-sc_k^{-1})e^{s/c_k}\equiv 1$.

Y. Tanno in some papers (see [3], [4] and [5] for example) treated these transforms which were first introduced by Hirschman and Widder [1].

We shall show that one of the main conditions imposed on the kernels and in fact on $F(s)$ in [5] (see formula (5) of [5, p. 157]) is satisfied automatically for a much greater class than that treated in [3] [4] and [5]. In the following papers we shall treat the behaviour and inversion of this class of transforms.

2. Properties of $F(s)$ defined in (1.3). The investigation of $F(s)$ will be useful for that of $G(t)$. For the following Theorem we have to define

$$\alpha_1=\max\{a_k,\ -\infty\,|\,a_k<0\},\ \alpha_2=\min\{a_k,\ \infty\,|\,a_k>0\}, \tag{2.1}$$

(2. 2) $$\gamma_1 = \max\{c_k, \ -\infty \mid c_k < 0\} \text{ and } \gamma_2 = \min\{c_k, \ \infty \mid c_k > 0\}.$$

THEOREM 2.1. *Let* $F(s)$ *be defined by* (1.3), $s=\sigma+i\tau$, $(\gamma_1+\alpha_1)/2 \leqq \sigma \leqq (\gamma_2+\alpha_2)/2$ *and* $-\infty < \tau < \infty$; *then*

(2. 3) $$|F(\sigma + i\tau)|^{-1} \leqq |F(\sigma)|^{-1}.$$

PROOF. By definition

$$|F(\sigma + i\tau)| = e^{b\sigma} \prod_{k=1}^{\infty} \left\{\left[\left(1 - \frac{\sigma}{a_k}\right)^2 + \frac{\tau^2}{a_k^2}\right] e^{\sigma/a_k} \Big/ \left[\left(1 - \frac{\sigma}{c_k}\right)^2 + \frac{\tau^2}{c_k^2}\right] e^{\sigma/c_k}\right\}^{\frac{1}{2}}.$$

A simple calculation will show that for $k=1, 2, \cdots$ and $(\alpha_1+\gamma_1)/2 \leqq \sigma \leqq (\alpha_2+\gamma_2)/2$ the function $\left\{\left[\left(1 - \frac{\sigma}{a_k}\right)^2 + \left(\frac{\tau}{a_k}\right)^2\right] \Big/ \left[\left(1 - \frac{\sigma}{c_k}\right)^2 + \left(\frac{\tau}{c_k}\right)^2\right]\right\}$ has a single minimum at $\tau = 0$ as a function of τ. Q.E.D.

For the next Theorem we shall need the following definitions.

DEFINITION 2.1. For a sequence $\{b_k\}$ of real numbers, the integer $N(\{b_k\}, x)$ is the number of elements of the sequence $\{b_k\}$ between 0 and x, (may also be equal to x).

DEFINITION 2.2. For a pair of sequence $\{a_k\}$ and $\{c_k\}$ of real numbers we define

(2. 4) $$N_+ \equiv N_+(\{a_k\}, \{c_k\}) \equiv \liminf_{x \to \infty} (N(\{a_k\}, x) - N(\{c_k\}, x)),$$

(2. 5) $$N_- \equiv N_-(\{a_k\}, \{c_k\}) \equiv \liminf_{x \to -\infty} (N(\{a_k\}, x) - N(\{c_k\}, x)).$$

and

(2. 6) $$N \equiv N(\{a_k\}, \{c_k\}) \equiv N_+ + N_-.$$

REMARK. N_+, N_- and N may be $0, 1, 2, \cdots$ or $+\infty$.

THEOREM 2.2. *Let* $F(s)$ *be defined by* (1.3), *then for all integers* n *satisfying* $n \leqq N$,

(2. 7) $$|F(\sigma + i\tau)|^{-1} = O(|\tau|^{-n}) \qquad |\tau| \to \infty$$

uniformly in the strip $|\sigma| \leqq R$.

PROOF. Define $F_+(s)$ and $F_-(s)$ by

$$F_+(s)=\prod_{a_k>0}\{(1-sa_k^{-1})/(1-sc_k^{-1})\}\ \exp(s(a_k^{-1}-c_k^{-1})) \tag{2.8}$$

and

$$F_-(s)=e^{-bs}F(s)/F_+(s). \tag{2.9}$$

It is enough to prove that for every n_1 and n_2 satisfying $n_1\leqq N_+$ and $n_2\leqq N_-$ we have

$$|F_+(\sigma+i\tau)|^{-1}=O(|\tau|^{-n_1})\qquad |\tau|\to\infty \tag{2.10}$$

and

$$|F_-(\sigma+i\tau)|^{-1}=O(|\tau|^{-n_2})\qquad |\tau|\to\infty. \tag{2.11}$$

Let $\{a_{k(i)}\}$ and $\{c_{k(i)}\}$ be a finite (and in this case of the same number of elements) or infinite subsequences of $\{a_k\}$ and $\{c_k\}$ respectively such that $a_{k(i)}>0$ and if $a_{k_0}>0$ then $k_0\in\{k(i)\}$. Obviously

$$F_+(s)=\prod(1-sa_{k(i)}^{-1})\ \exp sa_{k(i)}^{-1}\Big/\prod(1-sc_{k(i)}^{-1})\ \exp\ sc_{k(i)}^{-1}$$

and the two finite or infinite products do converge. Define the subsequence of $\{k(i)\}$ for which $c_{k(i)}=\infty$ by $\{k(i,1)\}$ and $a_{k_j}, c_{k_j}^*$ be a rearrangement of $a_{k(i)}$, and $c_{k(i)}$ for which $c_{k(i)}<\infty$ such that $a_{k_j}<a_{k_{j+1}}$, $c_{k_j}^*<c_{k_{j+1}}^*$.

One has now $F_+(s)\equiv F_{+,1}(s)\,F_{+,2}(s)$, where

$$F_{+,1}(s)=\prod\left(1-\frac{s}{a_{k(i,1)}}\right)\exp\frac{s}{a_{k(i,1)}}\quad ,$$

$$F_{+,1}(s)=\prod\left[\left(1-\frac{s}{a_{k_j}}\right)\Big/\left(1-\frac{s}{c_{k_j}^*}\right)\right]\exp\left(\frac{s}{a_{k_j}^{-1}}-\frac{s}{c_{k_j}^*}\right)\ .$$

If the product $F_{+,1}(s)$ is infinite or its number of elements is equal to N_+ then the proof of (2.10) follows immediately. If $F_{+,1}(s)$ is a finite product of p elements (p may be zero), then: $N^*=N_+(\{a_{k_i}\},\{c_{k_i}^*\})=N_+-p$. To conclude the proof we should show for all finite n, $n\leqq N^*$ that

$$|F_{+,2}(\sigma+i\tau)|^{-1}=O(|\tau|^{-n})\qquad |\tau|\to\infty$$

uniformly in the strip $|\sigma| \leqq R$ (R arbitrary).

The range of values of $N(\{a_{k_i}\}, x) - N(\{c^*_{k_i}\}, x)$ as a function of x is included in $\{0, 1, 2, \cdots\}$. Therefore for any finite n, $n \leqq N^* \equiv \liminf_{x\to\infty} (N(\{a_{k_i}\}, x) - N(\{c^*_{k_i}\}, x))$ there exists an x_0 such that for $x \geqq x_0$ $N(\{a_{k_i}\}, x) - N(\{c^*_{k_i}\}, x) \geqq n$.

Let us choose m to be the biggest integer satisfying $a_{k_{m+n}} \leqq x_0 + R$ and therefore $c^*_{k_{m+1}} > x_0 + R$.

One may write $F_{+,2}(s) = F_3(s) \cdot F_4(s) \cdot F_5(s)$, where $s = \sigma + i\tau$ and

$$F_3(s) = \prod_{i=1}^{m} \left[\left(1 - \frac{s}{a_{k_i}}\right) \exp \frac{s}{a_{k_i}} \Big/ \left(1 - \frac{s}{c^*_{k_i}}\right) \exp \frac{s}{c^*_{k_i}}\right],$$

$$F_4(s) = \prod_{j=1}^{n} \left(1 - \frac{s}{a_{k_{m+j}}}\right) \exp \frac{s}{a_{k_{m+j}}},$$

$$F_5(s) = \prod_{i=m+1}^{\infty} \left[\left(1 - \frac{s}{a_{k_{i+n}}}\right) \exp \frac{s}{a_{k_{i+n}}} \Big/ \left(1 - \frac{s}{c^*_{i+n}}\right) \exp \frac{s}{c^*_{k_{i+n}}}\right].$$

By the arguments which were used by I. I. Hirschman and D.V.Widder in [2] $|F_4(\sigma + i\tau)|^{-1} = O(|\tau|^{-n})$ $|\tau| \to \infty$ uniformly in $|\sigma| \leqq R$. Since for $x \geqq x_0$ $N(\{a_{k_i}\}, x) - N(\{c^*_{k_i}\}, x) \geqq n$, and therefore $N(\{a_{k_{i+n}}\}, x) - N(c^*_{k_i}, x) \geqq 0$ we have $a_{k_{i+n}} \leqq c^*_{k_i}$ for $i \geqq m$, therefore we have by Theorem 2.1 $|F_5(s)|^{-1} \leqq |F_5(\sigma)|^{-1}$. For $|\tau| \geqq 1$ and for $|\sigma| \leqq R$ (and since $a_{k_i} \leqq c^*_{k_i}$)

$$|F_3(s)| \geqq \exp\left(-R\left(\sum_{i=1}^{m} a^{-1}_{k_i} - \sum_{i=1}^{m} c^{-1}_{k_i}\right)\right)$$
$$\cdot \prod_{i=1}^{m} \left\{\left[(1-\sigma/a_{k_i})^2 + \tau^2/a^2_{k_i}\right] \Big/ \left[(1-\sigma/c^*_{k_i})^2 + \tau^2/c^*_{k_i}\right]\right\} \geqq k. \qquad \text{Q.E.D.}$$

REMARK 2.3. One can note that for the class treated by Y.Tanno in [5] if there are infinitely many positive a_k's we have $N_- = \infty$; if there are infinitely many negative a_k's we have $N_+ = \infty$; since $\{a_k\}$ was an infinite sequence $N_+ + N_- = \infty$. For the class defined by Y.Tanno in [3] and [4],

i.e., $$\Omega = \lim_{k\to\infty} a_k/k, \qquad \lim_{k\to\infty} c_k/k = \Omega_1 \qquad \text{and}$$
$$F(s) = \prod_{k=1}^{\infty} (1 - s^2/a^2_k) / \prod_{k=1}^{\infty} (1 - s^2/c^2_k)$$

both $N_+ = \infty$ and $N_- = \infty$. Theorem 2.2 implies that for the class defined in [3], [4] and [5] $|F(s)|^{-1} = O(|\tau|^{-n})$, $|\tau| \to \infty$, for any n and therefore the restriction (5) of [5, §1] is unnecessary. Moreover it is mentioned in [5, p.162] that "if $G(t)$ satisfies (5) of §1 for any positive α then $G(t) \in C^\infty(-\infty, \infty)$" and since it does for the class defined in [5] the kernels treated there are infinitely differentiable.

REMARK 2.4. Though the goal of this paper was an estimate of $F(s)$ to show $G(t) \in C^{\infty}(-\infty, \infty)$ for the convolution transforms of [5], we, in fact, introduced a rather wide class of convolution transforms that will satisfy $N_{+}+N_{-} \geqq n$. In forthcoming papers we shall treat the behaviour of the kernels of such transforms, convergence properties and for $N_{+}+N_{-} \geqq 2$ an inversion theory similar to that given by Y.Tanno.

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