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ON THE TOPOLOGY OF COMPACT CONTACT MANIFOLDS

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1. Introduction. In an earlier paper [3] the author proved that the second betti number of a compact normal regular contact manifold of strictly positive curvature vanishes. This was subsequently strengthened by E.M.Moskal [5] by removing the regularity condition and another proof[2] was recently given by S.Tanno [9].

PROPOSITION 1. *The second betti number of a compact normal contact manifold of positive curvature is zero.*

In this paper we prove

THEOREM 1. *The second betti number of a compact normal regular contact manifold with non-negative sectional curvature is zero.*

The proof is based upon Proposition 2 below as well as the technique used to obtain Theorem 2 of [3].

Proposition 2 has other interesting consequences. Indeed an application of a result due to B.Kostant [4] yields

THEOREM 2. *A compact simply connected*[3] *(normal) contact symmetric space is isometric with a sphere.*

This also follows from a statement due to M.Okumura ([6], Theorem 3.2). Employing Theorem 1 and Proposition 3 below, we obtain

THEOREM 3. *A compact torsion free 5-dimensional normal regular contact manifold with non-negative sectional curvature is homeomorphic with a sphere.*

1) Research supported by National Science Foundation Grant GP-5477.
2) An error in the proof originally due to Moskal led to the proof given by Tanno which is substantially the same.
3) D. Blair and the auther have shown that the fundamental group of a compact symmetric normal contact manifold is finite.

For, M is an integral homology sphere, so by the Hurewicz isomorphism theorem, the Whitehead homotopy type theorem and the generalized Poincaré conjecture [7], M is homeomorphic with a sphere.

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2. Contact manifolds. *An almost contact structure* (J, X_0, ω) on a $2n+1$ dimensional C^∞ manifold M is given by a linear transformation field J, a vector field X_0 and a 1-form ω on M with the properties

(1) $$\omega(X_0) = 1\,,$$

(2) $$JX_0 = 0\,,$$

(3) $$\omega \circ J = 0\,,$$

(4) $$J^2 = -I + \omega(\cdot)X_0$$

where I is the identity transformation field. If M has a (J, X_0, ω)-structure, a Riemannian metric $<,>$ can be found such that

(5) $$\omega = <X_0, \cdot>,$$

(6) $$<JX, JY> = <X, Y> - \omega(X)\omega(Y)$$

and M is then said to carry a $(J, X_0, \omega, <,>)$-structure. Formula (6) along with (1)−(5) says that J is skew-symmetric with respect to $<,>$, that is

(6') $$<JX, Y> = -<X, JY>$$

for all vector fields X and Y.

The almost contact structure is called *normal* if for all vector fields X and Y on M

$$J^2[X, Y] + [JX, JY] - J[X, JY] - J[JX, Y] + d\omega(X, Y)X_0 = 0.$$

A $2n+1$ dimensional C^∞ manifold M is said to have a *contact structure* and M is called a *contact manifold* if it carries global 1-form ω with the property

(7) $$\omega \wedge (d\omega)^n \neq 0\,.$$

In this case, there exists an associated $(J, X_0, \omega, <,>)$-structure with ω the 1-form defining the contact structure and

(8) $$<X, JY> = d\omega(X, Y).$$

This structure is called a *contact metric structure.*

It can easily be shown that the vector field X_0 generates a one-parameter group of isometries with respect to $<,>$.

Observe that (7) implies that a contact manifold is orientable.

A normal contact metric manifold is occasionally referred to as a *Sasakian manifold.*

3. Harmonic forms. Let a be a harmonic p-form on a compact Sasakian manifold. Then, for $1\leqq p\leqq n$, a is 'orthogonal' to X_0, that is $\iota(X_0)a$ vanishes [8].

Let ∇ denote the operator of covariant differentiation with respect to the Riemannian connexion. Then, since X_0 is a Killing field with respect to the metric $<,>$ we obtain

LEMMA 1. *On a Sasakian manifold*

$$2\nabla_Y X_0+JY=0, \ \forall Y.$$

For, $(\nabla_X\omega)(Y)+(\nabla_Y\omega)(X)=0$. So $<X, JY>=d\omega(X, Y)=(\nabla_X\omega)(Y)-(\nabla_Y\omega)(X)$ $=-2(\nabla_Y\omega)(X)=-2<X, \nabla_Y X_0>$. The vector field X being arbitrary and $<,>$ being nondegenerate, Lemma 1 follows.

PROPOSITION 2. *There are no covariant constant p-forms on a compact Sasakian manifold M for $1\leqq p\leqq 2n$.*

PROOF. Let a be a covariant constant p-form, $1\leqq p\leqq n$ and $X_1, X_2, \cdots, X_p$ any p vector fields on M. Then, since $\iota(X_0)a=0$, Lemma 1 implies

$$\begin{aligned}(\nabla_{JX_1}\iota(X_0)a)(X_2, \cdots, X_p)&=\iota(\nabla_{JX_1}X_0)a(X_2, \cdots, X_p)\\ &=-\frac{1}{2}a(J^2X_1, X_2, \cdots, X_p)\\ &=\frac{1}{2}a(X_1, X_2, \cdots, X_p).\end{aligned}$$

Hence, $a=0$. Denote by $*a$ the $(2n+1-p)$-form corresponding to a under the Hodge star operator. Since $*a$ is covariant constant whenever a is, and $*$ is an isomorphism, the same is true for forms of complementary degree.

Let $b_p=b_p(M)$ denote the p^{th} betti number of M. We shall require the following well-known fact [1].

LEMMA 2. *In a compact and orientable Riemannian manifold there are no harmonic p-forms $a=a_{i_1\cdots i_p}dx^{i_1}\wedge\cdots\wedge dx^{i_p}$ satisfying the quadratic inequality*

$$F_p(a)=R_{ij}a^{ii_2\cdots i_p}a^j{}_{i_2\cdots i_p}+\frac{p-1}{2}R_{ijkl}a^{iji_3\cdots i_p}a_{kli_3\cdots i_p}\geqq 0$$

unless ∇a *vanishes where* R_{ijkl} *and* R_{ij} *are the components of the Riemann curvature and Ricci tensors, respectively.*

For $p=1$, we have

$$F_1(a) = R_{ij}a^i a^j,$$

so if the Ricci curvature is positive semi-definite ∇a vanishes. Proposition 2 then gives

PROPOSITION 3. *If the Ricci curvature of a compact Sasakian manifold is positive semi-definite,* $b_1 = 0$.

4. **The main results**. A contact manifold M is *homogeneous* if there is a connected Lie group which acts transitively and effectively on M as a group of diffeomorphisms and leaves the contact form invariant. If M is also compact and simply connected, then it is regular, and is, in fact, a principal circle bundle over a homogeneous Hodge manifold (see [2], Theorem 6). It can be shown that an odd dimensional sphere possesses a homogeneous contact structure as a principal circle bundle over a homogeneous Hodge manifold. Making use of these facts, it was shown in a previous paper [3] that a simply connected homogeneous contact manifold of positive curvature in the invariant metric is globally isometric with a sphere.

PROOF OF THEOREM 1. Since the contact structure is normal and regular and the manifold M is compact, it may be considered as a principal circle bundle over a Hodge manifold B. Since the sectional curvatures of M are non-negative, the sectional curvatures of B are positive (see [3, §4]). The Gysin sequence of the circle bundle M over B is

$$\longrightarrow H^i(B,R) \xrightarrow{p^*} H^i(M,R) \longrightarrow H^{i-1}(B,R) \xrightarrow{L} H^{i+1}(B,R) \longrightarrow$$

where R denotes the reals, p^* is the map induced by the bundle projection and L is multiplication by the characteristic class of the bundle. By Proposition 3, $b_1(M)$ vanishes, so since $b_2(B)=1$, we conclude that $b_2(M)$ also vanishes.

A *contact symmetric space* is a homogeneous contact manifold which is Riemannian symmetric with respect to the contact metric structure.

PROOF OF THEOREM 2. Since a harmonic form on a compact symmetric space has vanishing covariant derivative with respect to the connexion of the invariant metric, then by Proposition 2, since the manifold is normal, it is a rational homology sphere. But a compact simply connected symmetric space which is a rational homology sphere is isometric with a sphere except for $SU(3)/SO(3)$. To see this, we first note that the holonomy group of a compact simply connected Riemannian manifold M which is topologically a cohomology sphere, is the special orthogonal group $SO(d)$, $d = \dim M$, with the exception of $SU(3)/SO(3)$. This is a consequence of Corollary 2.2 of [4] and the fact that simple connectedness implies that the holonomy group is contained in $SO(d)$. Since M is in addition a symmetric space G/H, the holonomy group is H.

The above facts imply that the isotropy group is $SO(d)$, unless $d=5$. In particular, the isometry group is transitive on tangents, M is of rank 1, and M is not complex or quaternionic projective space, so $M = S^d$, unless $d=5$. The exceptional case does not occur as one sees from the classification of homogeneous contact manifolds given by Boothby and Wang [2].

REMARKS. (a) $SU(3)/SO(3)$ may yet carry a contact structure, but it will not be invariant.

(b) $SU(3)/SO(3)$ has 2-torsion. This is a conseque nceof the exact sequence

$$\longrightarrow \pi_2(SU(3)) \longrightarrow \pi_2(SU(3)/SO(3)) \longrightarrow \pi_1(SO(3)) \longrightarrow \pi_1(SU(3)) \longrightarrow 0 .$$

For, $\pi_2(SU(3)) = \pi_1(SU(3)) = 0$, $\pi_1(SO(3)) \approx Z_2$, and since $SU(3)/SO(3)$ is simply connected, $\pi_2(SU(3)/SO(3)) \approx H_2(SU(3)/SO(3), Z)$. Hence, $H_2(SU(3)/SO(3)) \approx Z_2$.

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