ON THE KLEIN'S GEOMETRY*)

BY

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Introduction

Klein's definition of geometry can be stated as follows. Suppose that a point-set M and a group G of 1-1 transformations, each of which maps M onto itself, are given. An arbitrary subset A of M is said to be a figure and a figure A is said to be congruent to a figure B when B is a image of A under a transformation of G. We denote this relation by $A \equiv B$. Then, as G is a group, the concept of congruence satisfies the following three conditions (equivalence relations):

1) Every figure is congruent to itself.

2) If a figure A is congruent to a figure B, then B is congruent to A.

3) If a figure A is congruent to a figure B and B to C, then A is congruent to C.

It therefore enables us to divide the figures in M into mutually disjoint congruence classes. It is the Klein's geometry to study the common properties of all figures of the same classes, namely the properties which are invariant under all transformations of G.

But a kind of geometry will be constructed, if the set of transformations Λ satisfies the above conditions, even if it does not form a group.

Thus there arises a question that whether Λ forms a group or not as an implication of the fact that the concept of congruence satisfies the above three conditions.

This question was pointed out by Professor S. Sasaki, but there exists a very simple example such that Λ is not a group.

On the other hand, the developments of the theories of projective, affine, Euclidean and Möbius' geometries are based on the fact that G's are groups of projective,affine, Euclidean, and Möbius' transformations respectively. And it will be hard to expect fine theories, if the concept of congruence satisfies only the above three conditions.

Therefore it will be necessary for the foundation of Klein's geometry to find conditions other than the above three which express the conditions of congruence, in order that Λ forms a group.

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The purpose of the present paper is to explain the relations between the concept of congruence and that of group, and to give a sufficient condition for Λ in order that it forms a group.

§1. We here give a simple example of a set of transformations of M which satisfies the conditions 1), 2), and 3), but is not a group.

EXAMPLE 1. Let M be a set consisting only three points and G be the set of all 1-1 transformations of M onto itself. And let Λ be the subset of G which exclude only the identity transformation. Then Λ is not a group, but the pair (M, Λ) satisfies the conditions 1), 2) and 3).

To prove this we remark first that when a set M and a set of transformations of M are given, then in general the conditions 2) and 3) imply the condition 1), if Λ is not empty.

Hence it is only sufficient to verify that the conditions 2) and 3) are satisfied, but it is easy.

As we have seen from this example, the fact that figure A is congruent to itself merely implies the fact that there exists a transformation which maps A onto A, but each point of A may be different from its image and by this freedom Λ is not always a group.

In order to restrict somewhat this freedom, we shall extend the concepts of figures and of congruence as follows:

A figure is an ordered finite system $A = \{A_i\}$ $(i = 1, \ldots, m)$ such that each A_i is an arbitrary subset of M. And we define that a figure $A = \{A_i\}$ $(i = 1, \ldots, m)$ is congruent to a figure $B = \{B_j\}$ $(j = 1, \ldots, n)$, if $m = n$ and there exists a transformation f of Λ such $f(A_i) = B_i$ for $i = 1, \ldots, m$. We denote this relation by $A \equiv B$ or $B = f(A)$. We call A_i components of the figure A.

Hereafter we always consider the figures and congruences in the above sense. Thus an arbitrary finite subset of M can be considered as such a figure that each point of this set is a component, when an order is given.

Now, we assume that the pair (M, Λ) satisfies the conditions 1), 2) and 3) for every generalized figures. Then, if M is a finite set, so Λ must be a group.

In fact, if we consider M as the figure each component of which is one point, then conditions 1), 2) and 3) are equivalent to the following conditions 1), 2) and 3) respectively:

1*) Λ includes the identity transformation.

2*) If $f \in \Lambda$, then $f^{-1} \in \Lambda$.

3*) If $f, g \in \Lambda$, then $f \cdot g \in \Lambda$.

§2. Our problem was solved when M is finite, but we can not conclude that the set of transformations in consideration forms a group when M is infinite.

We shall give such an example.

EXAMPLE 2. Let M be an infinite point-set. We consider only 1-1 transformations of M onto itself, each of which operates as identity except a certain finite point-set. Such transformations form a group G. Let Λ be the subset of G which exclude only the identity transformation. If we consider the pair (M, Λ), we can prove that (M, Λ) satisfies the conditions 1), 2) and 3) for generalized figures.

By the same remark stated in § 1, it is sufficient to prove that conditions 2) and 3) are satisfied. But if $f \in \Lambda$, then $f^{-1} \in \Lambda$, so condition 2) is satisfied. Hence there remains only to prove that the condition 3) is satisfied.

Let $f, g \in \Lambda$ be transformations which define the congruences $A \equiv B$ and $B \equiv C$ respectively.

CASE i). All components of the figure A are finite.

Let $A = \{A_i\}$, then $M - \bigcup_i A_i$ is an infinite set, so it contains two different points a_1, a_2. We define the transformation $h: M \to M$ by

$$h \mid M - (a_1, a_2) = gf \mid M - a_1, a_2),$$

and

$$h(a_1) = a_2, \ h(a_2) = a_1.$$

Then $h \in \Lambda$ and $h(A_i) = gf(A_i)$, so we have $A \equiv C$.

CASE ii). At least one of the components of the figure A is infinite.

Let $A = \{A_i, A_j'\}$ be a figure whose components A_i are all infinite, while A_j' are all finite. Then $\bigcup_i A_i - \bigcup_j A_j'$ is a infinite set, so every point of this set is fixed under transformations f and g except a finite number of points. Hence there are two points a_1, a_2 in A_1 which are fixed both under transformations f and g. We now define the transformation $h: M \to M$ by

$$h \mid M - (a_1, a_2) = g \cdot f \mid M - (a_1, a_2),$$

and

$$h(a_1) = a_2, \ h(a_2) = a_1.$$

Then $h \in \Lambda$ and h defines the congruence $A \equiv C$.

§3. The important cases in Klein's geometries are those in which M is a topological space, Λ is a set of homeomorphisms which operate continuously on M.

So we introduce topologies in the following way.

Let M be a topological space, and Λ be a set of homeomorphisms of M onto itself which is topologized so as continuously operates on M. In other words, there is a homeomorphism T_g of M onto itself which corresponds one to one to a point g of Hausdorff space Λ, and the transformation $T: \Delta \times M \to M$ defined by

$$T(g, m) = T_g(m) \qquad (g \in \Lambda, m \in M)$$

is continuous.

When M and Λ are locally bicompact, we expect by the examples of projective and affine geometries that by the conditions 1), 2) and 3) Λ forms a group, namely a topological group.

But the Example 2 shows that in this case we can not conclude that Λ forms a group in general.

In order to obtain a sufficient condition for Λ to form a group, we shall now introduce the concept of the fundamental figure of (M, Λ).

Fundamental figure F is a figure every component of which is one point and has the following property:

Let F' be an arbitrary figure congruent to F, and $\Lambda_{F'}$ be the subset of Λ such that every transformation corresponding to an element of $\Lambda_{F'}$ defines the congruence $F' \equiv F$, i. e.

$$\Lambda_{F'} = \{g \in \Lambda \mid T_g(F') = F\}.$$

Then $\Lambda_{F'}$ is always bicompact.

We can prove the following theorem.

THEOREM 1. *If a topologized pair* (M, Λ) *satisfies the conditions* 1), 2) and 3) *for generalized figures, and furthermore admits a fundamental figure* F, *then* Λ *is a topological group.*

PROOF. It is sufficient to show that $\{T_g \mid g \in \Lambda\}$ forms a group. In order to do it, we have to prove the following two facts:

I) For an arbitrary element $f \in \Lambda$, there exists an element f^{-1} in Λ such that $T_{f^{-1}} = (T_f)^{-1}$.

II) For arbitrary two elements $f, g \in \Lambda$, there exists an element h in Λ such that $T_h = T_f \cdot T_g$.

Set $F' = f(F)$, $F_m = \{F, m\}$, and $F'_m = \{F', f(m)\}$, where m is a point of M. Then the figure F_m is congruent to the figure F'_m, hence by 2) F'_m is congruent to F_m. Let Λ_n be the subset of such that every transformation corresponding to an element of Λ_m defines the congruence $F'_m \equiv F_m$, i. e.

$$\Lambda_m = \{g \in \Lambda \mid T_g(F'_m) = F_m\}.$$

Then Λ_m is a closed subset of the bicompact set $\Lambda_{F'}$. And the system $\{\Lambda_m \mid m \in M\}$ has the finite intersection property.

In fact, when Λ_{m_i} ($m_i \in M$, $i = 1, \ldots, n$) are given, the figure $(F, m_1 \ldots, m_n)$ is congruent to the figure $(F', m_1', \ldots, m'_n)$, where $m'_i = f(m_i)$. Hence $(F', m_1', \ldots, m'_n)$ is congruent to $(F, m_1, \ldots, m_n)$. Let T_{g_0} ($g_0 \in \Lambda$) be a transformation which defines this congiuence. Then we have $g_0 \in \Lambda_{mi}$ for $i = 1, \ldots, n$. Hence $\bigcap_i \Lambda_{m_i} \neq 0$. Therefore we have

$$\bigcap_{m \in M} \Lambda_m \neq 0.$$

Let f^{-1} be an element of this set. Then we have

$$T_{f^{-1}} = (T_f)^{-1}.$$

Thus it was proved that condition I) is satisfied.

Next, let f, $g \in \Lambda$ and $F' = T_f \cdot T_g(F)$. By the condition 3), the figure $F_m = \{F, m\}$ is congruent to the figure $F'_m = \{F', T_f \cdot T_g(m)\}$, so F'_m is congruent to F_m. Let Λ_n be the subset of Λ such that every transformation corresponding to an element of Λ_m defines the congrunce $F'_m \equiv F_m$, i. e.

$$\Lambda_m = \{g \in \Lambda \mid T_g(F'_m) = F_m\}.$$

Then Λ_m is a closed subset of the bicompact set $\Lambda_{F'}$. And the system $\{\Lambda_m \mid m \in M\}$ has the finite intersection property as well as the proof of I). Hence we have $\bigcap_{m \in M} \Lambda_m \neq 0$. Let k be an element of this set. Then we have $T_k = (T_f \cdot T_g)^{-1}$. Hence by I) there exists an element h in Λ such that $T_h = T_f \cdot T_g$.

Thus Theorem 1 is proved.

In particular, if Λ is bicompact, then (M, Λ) admits a fundamental figure. Hence we heve:

COROLLARY. *If a topologized pair* (M, Λ) *satisfies the conditions* 1), 2) *and* 3) *for generalized figures, and* Λ *is bicompact, then* Λ *is a topological group.*

Moreover, from Theorem 1 we can easily obtain the following results:

THEOREM 2. *In the projective (affine, Enclidean, and Möbius') space, if a set* *of projective (affine, Euclidean, and Möbius') tranformations satisfies the conditions* 1), 2) *and* 3) *for generalized figures, then the set is either the group of projective (affine, Euclidean, and Möbius') transformations or a subgroup of the group.*

Let n be the dimensions of this space. Then the fundamental figure consists of $n+1$ points which are linearly independent. So we have Theorem 2 from Theorem 1.

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