NOTES ON FOURIER ANALYSIS (XXV): ON THE $|C|$-SUMMABILITY OF THE FOURIER SERIES*)

By

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§ 1. Introduction. On the absolute convergence of the Fourier series, Bernstein[1] has proved the following two theorems.

(1. 1) *If* $f(x) \in \mathrm{Lip}\,\alpha$ $(\alpha > 1/2)$, *then the Fourier series of* $f(x)$ *is absolutely convergent.*

(1. 2) *If* $f(x)$ *is continuous and the series*

$$\sum_{n=1}^{\infty} \omega(1/n)/\sqrt{n}$$

converges, where $\omega(\delta)$ *is the modulus of continuity of* $f(x)$, *then the Fourier series of* $f(x)$ *is abosolutely convergent.*

On the other hand Salem has proved the following theorem, which is the generalization of well known Zygmund's Theorem:

(1. 3) *If* $f(x)$ *is of bounded variation and the series*

$$\sum_{n=1}^{\infty} \omega(1/n)^{1/2}/n$$

converges, then the Fourier series of $f(x)$ *is absolutely convergent.*

As the generalization of theorems (1. 2) and (1. 3), Szász[2] has proved the following theorem:

(1. 4) *If* $f(x) \in L^p$ $(1 \leqq p \leqq 2)$ *and the series*

$$\sum_{n=1}^{\infty} \omega_p^{(m)}(1/n)/n^{1/p'}$$

converges, then the Fourier series of $f(x)$ *is absolutely convergent, where*

$$\omega_p^{(m)}(h) \equiv \sup_{0 \leqq t \leqq h} \left\{ \int_0^{2\pi} |\Delta_m f(x;t)|^p\, dx \right\}^{1/p},$$

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1) S. Bernstein, Comp. Rend. de Paris, 199 (1934).

2) O. Szász, Trans. Amer. Math. Soc., 42 (1936).

and $1/p + 1/p' = 1$.

Our object of this paper is to prove the following two theorems. The former is the analogue of Hyslop's[3] Theorem, that is,

(1.5) *If* $f(x) \varepsilon L^p$ $(1 < p \leqq 2)$ *and the series*

$$\sum_{n=1}^{\infty} \omega_p(1/n)/n^{\delta+1/p'} \tag{1}$$

converges, then the Fourier series of $f(x)$ *is* $|C, \delta|$*-summable, where*

$$\omega_p(h) \equiv \sup_{0 \leqq t \leqq h} \left(\int_0^{2\pi} |f(x+t) - f(x)|^p \, dx \right)^{1/p},$$

and $-1 < \delta < 1/p$.

(1.6) *If* $f(x) \varepsilon L^p$ $(p > 2)$ *and the series*

$$\sum_{n=1}^{\infty} \omega_p(1/n)/n^{\delta+1/2} \tag{2}$$

converges, then the Fourier series of $f(x)$ *is* $|C, \delta|$*-summable, where* $-1 < \delta < 1/2 - 1/p$.

§ 2. **Proof of (1.5).** Let $f(x) \varepsilon L^p$ $(1 < p \leqq 2)$, $0 < \delta < 1/p$ and

$$f(x) \sim \frac{1}{2} a_0 + \sum_{n=1}^{\infty} \left(a_n \cos nx + b_n \sin nx \right) \equiv \sum_0^{\infty} A_n(x). \tag{3}$$

If we denote by $\sigma_n^\delta(x)$ and $\tau_n^\delta(x)$ the (C, δ)-mean of (3) and the sequence $\{nA_n(x)\}$, respectively, then $\sigma_n^\delta(x) - \sigma_{n-1}^\delta(x) = \tau_n^\delta(x)/n$.

Consequently for the proof of (1.5) it is sufficient that

$$\sum_{n=1}^{\infty} |\tau_n^\delta(x)|/n$$

converges a.e., or

$$\sum_{n=1}^{\infty} \int_{-\pi}^{\pi} |\tau_n^\delta(x)| n^{-1} dx < \infty.$$

Since

$$A_n(x) = \frac{1}{\pi} \int_0^\pi \varphi(t) \cos nt \, dt,$$

$$\tau_n^\delta(x) = \frac{1}{\pi} \frac{1}{A_n^\delta} \int_0^\pi \varphi(t) \sum_{k=0}^n A_k^{\delta-1}(n-k) \cos(n-k) t \, dt$$

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3) J. M. Hyslop, Proc. Lond. Math. Soc., 41 (1936).

$$= \frac{1}{\pi} \frac{n}{A_n^\delta} \int_0^\pi \varphi(t) \sum_{k=0}^\infty A_k^{\delta-1} \cos(n-k)t\,dt$$

$$- \frac{1}{\pi} \frac{1}{A_n^\delta} \int_0^\pi \varphi(t) \left\{ \sum_{k=n+1}^\infty n A_k^{\delta-1} \cos(n-k)t + \sum_{k=0}^n k A_k^{\delta-1} \cos(n-k)t \right\} dt$$

$$= \frac{1}{\pi} \frac{n}{A_n^\delta} \int_0^\pi \varphi(t) p(t) \cos nt\,dt + \frac{1}{\pi} \frac{n}{A_n^\delta} \int_0^\pi \varphi(t) q(t) \sin nt\,dt$$

$$- \frac{1}{\pi} \frac{1}{A_n^\delta} \int_0^\pi \varphi(t) \left\{ \sum_{k=n+1}^\infty n A_k^{\delta-1} \cos(n-k)t + \sum_{k=0}^n k A_k^{\delta-1} \cos(n-k)t \right\} dt$$

$$\equiv \frac{1}{\pi} \frac{n}{A_n^\delta} \left[P_n(x) + Q_n(x) \right] - \frac{1}{\pi} \frac{1}{A_n^\delta} R_n(x),$$

say, where

$$p(t) = \sum_{k=0}^\infty A_k^{\delta-1} \cos kt,$$

and

$$q(t) = \sum_{k=0}^\infty A_k^{\delta-1} \sin kt.$$

$\varphi(t)\,p(t)$ is an even function of t and $1/_\pi P_n(x)$ is its n-th Fourier coefficient. Consequently for any $h > 0$,

$$\left(\sum_{n=1}^\infty |P_n(x) \sin nh|^{p'} \right)^{p/p'} \leqq A \int_0^\pi |\varphi(t+h)\,p(t+h) - \varphi(t-h)\,p(t-h)|^p\,dt$$

$$\leqq A_p \int_0^\pi |\varphi(t+h) - \varphi(t-h)|^p\,|p(t+h)|^p\,dt$$

$$+ A_p \int_0^\pi |p(t+h) - p(t-h)|^p\,|\varphi(t-h)|^p\,dt$$

$$\equiv \alpha(x) + \beta(x),$$

say. Then

$$\int_0^\pi \alpha(x)\,dt \leqq A_p \int_0^\pi dx \int_0^\pi |f(x+t+h) - f(x+t-h)|^p\,|p(t+h)|^p\,dt$$

$$+ A_p \int_0^\pi dx \int_0^\pi |f(x-t-h) - f(x-t+h)|^p\,|p(t+h)|^p\,dt$$

$$\leqq A_p \int_0^\pi \omega_p^p(h)\,|p(t+h)|^p\,dt \leqq A_p\,\omega_p^p(h) \int_0^\pi (t+h)^{-\delta p}\,dt \leqq A_p \omega_p^p(h),$$

$$\int_0^\pi \beta(x)\,dx \leqq A_p \int_0^\pi dx \left(\int_{-h}^h + \int_h^\pi \right) |p(t+2h) - p(t)|^p\,|\varphi(t)|^p\,dt \equiv \beta_1 + \beta_2,$$

say.

$$\beta_1 \leqq A_p \int_0^\pi dx \int_{-h}^{h} \Big(|p(t+2h)|^p + |p(t)|^p \Big) |\varphi(t)|^p \, dt$$
$$\leqq A_p \int_{-h}^{h} \omega_p^p(t/2)\, t^{-p\delta}\, dt \leqq A_p\, \omega_p^p(h),$$

and

$$\beta_2 \leqq A_p \int_0^\pi dx \int_h^\pi h^p\, t^{-p(1+\delta)} |\varphi(t)|^p \, dt \leqq A_p\, h^p \int_h^\pi \omega_p^p(t)\, t^{-p(1+\delta)}\, dt.$$

Consequently,

$$\int_0^\pi \Big(\sum_{n=1}^\infty |P_n(x) \sin nh|^{p'} \Big)^{p/p'} dx \leqq \int_0^\pi \alpha(x)\, dx + \int_0^\pi \beta(x)\, dx$$
$$\leqq A_p\, \omega_p^p(h) + A_p\, h^p \int_h^\pi \omega_p^p(t)\, t^{-p(1+\delta)}\, dt.$$

Let us put $h = \pi 2^{-(\lambda+1)}$, then

$$\int_0^\pi \Big(\sum_{n=2^{\lambda-1}+1}^{2^\lambda} |P_n(x)|^{p'} \Big)^{p/p'} dx$$
$$\leqq A_p\, \omega_p^p(\pi 2^{-(\lambda+1)}) + A_p\, 2^{-p(\lambda+1)} \int_h^\pi 2^{-(\lambda+1)}\, \omega_p^p(t)\, t^{-p(1+\delta)}\, dt.$$

$$\sum_{n=1}^\infty \int_0^\pi |P_n(x)|\, n^{-\delta}\, dx = \sum_{\lambda=1}^\infty \Big(\sum_{n=2^{\lambda-1}+}^{2^\lambda} \int_0^\pi |P_n(x)|\, n^{-\delta}\, dx \Big)$$

$$\leqq \sum_{\lambda=1}^\infty \int_0^\pi \Big(\sum_{n=2^{\lambda-1}+1}^{2^\lambda} |P_n(x)|^{p'} \Big)^{1/p'} \Big(\sum_{n=2^{\lambda-1}+1}^{2^\lambda} n^{-\delta p} \Big)^{1/p} dx$$

$$\leqq A_p \sum_{\lambda=1}^\infty \Big[\int_0^\pi \Big(\sum_{n=2^{\lambda-1}+1}^{2^\lambda} |P_n(x)|^{p'} \Big)^{p/p'} \Big]^{1/p} \cdot 2^{-\lambda(\delta p-1)/p}\, dx$$

$$= A_p \sum_{\lambda=1}^\infty 2^{-\lambda(\delta-1/p)} \Big\{ A_p\, \omega_p^p(\pi 2^{-(\lambda+1)}) + A_p 2^{-p(\lambda+1)} \int_h^\pi 2^{-(\lambda+1)}\, \omega_p^p(t)\, t^{-p(1+\delta)}\, dt \Big\}^{1/p}$$

$$\leqq A_p \sum_{\lambda=1}^\infty \omega_p(\pi 2^{-(\lambda+1)})\, 2^{-\lambda(\delta-1/p)} + A_p \sum_{\lambda=1}^\infty 2^{-\lambda(\delta-1/p+1)} \Big(\int_h^\pi 2^{-(\lambda+1)} \omega_p^p(t)\, t^{-p(1+\delta)}\, dt \Big)^{1/p}$$

$$\equiv A_p(K+L), \text{ say.}$$

Since, by the hypothesis, $n^{-(\delta+1/p')}\, \omega_p(1/n) \downarrow 0$, the convergence of $\Sigma\, n^{-(\delta+1/p')}$ $\omega_p(1/n)$ is equivalent to that of $\Sigma 2^\lambda 2^{-\lambda(\delta+1/p')}\, \omega_p(\pi 2^{-\lambda})$. That is, the series K converges. On the other hand, by $n^{-(\delta+1/p')}\, \omega_p(1/n) = o(1/n)$,

$$\int_h^\pi 2^{-(\lambda+1)} t^{-p(1+\delta)}\, \omega_p^p(t)\, dt \leqq A \sum_{n=1}^{2^{\lambda+1}} n^{p(1+\delta)-2}\, \omega_p^p(1/n) \leqq A \sum_{n=1}^{2^{\lambda+1}} n^{p(1+2\delta)-3} \leqq A 2^{\lambda p(+2p\delta-2)}.$$

$$L \leqq A \sum_{\lambda=1}^{\infty} 2^{-\lambda(\delta+1/p')} 2^{\lambda(1+\delta-2/p)}$$

$$= A \sum_{\lambda=1}^{\infty} 2^{\lambda(1+\delta-\delta-2/p-1/p')} \leqq A \sum_{\lambda=1}^{\infty} 2^{\lambda(\delta-1/p)} < \infty .$$

From above estimations $\Sigma |P_n(x)| n^{-\delta}$ converges a.e., we can also prove that $\Sigma |Q_n(x)| n^{-\delta}$ converges a.e., by the same way.

Now

$$|R_n(x)| n^{-\delta} \leqq A_p n^{-\delta} \left| \int_0^{1/n} \right| + A_p n^{-\delta} \left| \int_{1/n}^{\pi} \right| \equiv \gamma_n(x) + \delta_n(x), \text{ say.}$$

$$\int_0^{\pi} \gamma_n(x)\, dx \leqq A_p n^{-\delta} \int_0^{1/n} n^{\delta} t^{-1}\, dt \int_0^{\pi} |\varphi(t)|\, dx$$

$$\leqq A_p \int_0^{1/n} t^{-1}\, dt \left(\int_0^{\pi} |\varphi(t)|^p\, dx \right)^{1/p}$$

$$\leqq A_p \int_0^{1/n} t^{-1} \omega_p(t) dt \leqq A_p \sum_{m=n}^{\infty} \omega_p(1/m)\, m^{-1}.$$

$$\sum_{n=1}^{\infty} \int_0^{\pi} \gamma_n(x)\, n^{-1}\, dx \leqq A_p \sum_{n=1}^{\infty} n^{-1} \sum_{m=n}^{\infty} \omega_p(1/m)\, m^{-1}$$

$$\leqq A_p \sum_{n=1}^{\infty} n^{-1+\delta-1/p} \sum_{m=n}^{\infty} \omega_p(1/m)\, m^{-(\delta+1/p)} < \infty .$$

And

$$\int_0^{\pi} \delta_n(x)\, n^{-1}\, dx \leqq A_p \int_0^{\pi} n^{-1}\, dx \int_{1/n}^{\pi} |\varphi(t)|\, dt$$

$$+ A_p \int_0^{\pi} n^{-\delta}\, dx \int_{1/n}^{\pi} |\varphi(t)|\, t^{-(1+\delta)}\, dt + A_p \int_0^{\pi} n^{-2}\, dx \int_{1/n}^{\pi} |\varphi(t)|\, t^{-2}\, dt$$

$$\leqq A_p n^{-2} \int_{1/n}^{\pi} \omega_p(t)\, dt + A_p n^{-(1+\delta)} \int_{1/n}^{\pi} \omega_p(t)\, t^{-(1+\delta)}\, dt$$

$$+ A^p n^{-2} \int_{1/n}^{\pi} \omega_p(t)\, t^{-2}\, dt.$$

$$\sum_{n=1}^{\infty} \int_0^{\pi} |\delta_n(x)|\, n^{-1}\, dx \leqq A_p \sum_{n=1}^{\infty} n^{-2} \int_{1/n}^{\pi} \omega_p(t)\, dt$$

$$+ A_p \sum_{n=1}^{\infty} n^{-(1+\delta)} \int_{1/n}^{\pi} \omega_p(t)\, t^{-(1+\delta)}\, dt + A_p \sum_{n=1}^{\infty} n^{-2} \int_{1/n}^{\pi} \omega_p(t)\, t^{-2}\, dt.$$

$$\leqq A_p + A_p \sum_{n=1}^{\infty} n^{-(1+\delta)} \left(\sum_{m=1}^{n} \omega_p(1/m)\, m^{\delta-1} \right) + A_p \sum_{n=1}^{\infty} n^{-2} \left(\sum_{m=1}^{n} \omega_p(1/m) \right)$$

$$\leqq A_p + A_p \sum_{m=1}^{\infty} \omega_p(1/m)\, m^{\delta-1} \sum_{n=m}^{\infty} n^{-(1+\delta)} + A_p \sum_{m=1}^{\infty} \omega_p(1/m) \sum_{n=m}^{\infty} n^{-2}$$

$$\leqq A_p + A_p \sum_{m=1}^{\infty} \omega_p (1/m)\, m^{-1} \leqq A_p + A_p \sum_{m=1}^{\infty} \omega_p (1/m)\, m^{-(\delta+1/p')} < \infty .$$

Consequently

$$\sum_{n=1}^{\infty} \int_0^{\pi} | R_n (x) |\, n^{-(\delta+1)}\, dx < \infty$$

and $\Sigma | R_n (x) | / n^{\delta+1}$ converges a.e.

§ 3. **The case** $-1 < \delta \leqq 0$. Let

$$f(x) \sim \sum_{-\infty}^{\infty} c_\nu e^{i\nu x},$$

and

$$\sum_{n=1}^{\infty} n^{\delta-1/p'} \omega_p (1/n) < \infty \quad (0 \leqq \delta < 1).$$

Following the Szász argument, we have successively

$$\sum_{\nu=n+1}^{2n} | c_\nu |^{p'} = O ((\omega_p (1/n))^{p'}),$$

$$\sum_{\nu=n+1}^{2n} | c_\nu |\, \nu^{\delta} \leqq \Big(\sum_{\nu=n+1}^{2n} \nu^{\delta p} \Big)^{1/p} \Big(\sum_{\nu=n+1}^{2n} | c_\nu |^{p'} \Big)^{1/p'}$$

$$= O\Big(n^{\delta+1/p} \omega_p (1/n) \Big) = O\Big(\sum_{\nu=n+1}^{2n} \nu^{\delta-1+1/p} \omega_p (1/\nu) \Big).$$

$$\sum_{n=1}^{\infty} | c_n |\, n^{\delta} = O\Big(\sum_{n=1}^{\infty} n^{\delta-1/p'} \omega_p (1/n) \Big) = O (1).$$

Consequently the Fourier series of $f(x)$ is $|C, -\delta|$-summable by a theorem due to Sunouchi[4].

§ 4. **Proof of** (1. 6). If we use the Perseval's relation, in stead of the Hausdorff-Young inequality in the proof of (1.5), we can quite similarly prove (1. 6).

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4) G. Sunouchi, Journ. Math. Soc. Jap., 1; 2 (1949).