HILBERT ALGEBRAS*)

By

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W. Ambrose[1] defined a *proper* H*-*algebra* $\mathfrak{H}$: $\mathfrak{H}$ is a Hilbert space and a ring subject to the conditions: 1) if $ax=0$ for all $x \in \mathfrak{H}$, then $a=0$, and 2) for any $a \in \mathfrak{H}$ there is $a^* \in \mathfrak{H}$ such that

$$(ax, y) = (x, a^*y), \ (xa, y) = (x, ya^*)$$

for all $x, y \in \mathfrak{H}$, and proved that if $\mathfrak{H}$ satisfies the condition

$$\text{(B)} \qquad \sup_{\|x\| = \|y\| = 1} \|xy\| < +\infty,$$

then $\mathfrak{H}$ is a direct sum of *simple* 2-sided ideals $\sum_{\lambda \in \Lambda} \mathfrak{H}_\lambda$ such that $\mathfrak{H}_\rho \perp \mathfrak{H}_\lambda$ for $\rho \neq \lambda$ and $\mathfrak{H}_\lambda$ is isometric to a full-matrix algebra: for some set Λ all complex valued functions $a(\lambda, \rho)$ $(\lambda, \rho \in \Lambda)$ with

$$\sum_{\lambda \, \rho} |a(\lambda, \rho)|^2 < +\infty$$

constitute a proper H*-algebra, being called a *full-matrix algebra*, if we put

$$ab(\lambda, \rho) = \sum_{\tau \in \Lambda} a(\lambda, \tau) b(\tau, \rho),$$

$$a^*(\lambda, \rho) = \overline{a(\rho, \lambda)},$$

$$(a, b) = \frac{1}{\alpha^2} \sum_{\lambda \, \rho} a(\lambda, \rho) \overline{b(\lambda, \rho)},$$

for some positive number α, wich we shall call the *order* of a full-matrix algebra. He used the condition (B) essentially in his proof, while it will be proved that any H*-algebra satisfies the condition (B) (cf. § 2). He remarked further that a group ring on a compact group is a proper H*-algebra: let $\mathfrak{G}$ be a compact group. All complex valued measurable functions $a(\sigma)$ $(\sigma \in \mathfrak{G})$ with

$$\int |a(\sigma)|^2 \, d\sigma < +\infty$$

for Haar measure constitute a proper H*-algebra if we put

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1) W. Ambrose: Structure theorems for a special class of Banach algebras, Trans. Amer. Math. Soc. **57** (1945) 364-386.

$$ab(\sigma)=\int a(\sigma\tau^{-1})\, b(\tau)\, d\tau,$$
$$a^*(\sigma)=\overline{a(\sigma^{-1})},$$
$$(a,\ b)=\int a(\sigma)\, \overline{b(\sigma)}\, d\sigma.$$

If a group $\mathfrak{G}$ is locally compact, then its group ring is not necessarily a Hilbert space as remarked by I. E. Segal.[2] But all complex valued measurable functions $a(\sigma)$ $(\sigma \varepsilon \mathfrak{G})$, such that $a(\sigma)=0$ except for some compact set Λ_a and

$$\int |a(\sigma)|^2 d\sigma < +\infty,$$

satisfy the condition of proper H* algebra, except completeness, and for Haar measure $m(\Lambda_a)$ we have

$$\|ab\| \leqq m(\Lambda_a)^{1/2} \|a\|\, \|b\|$$

for all such functions $b(\sigma)$, if $\mathfrak{G}$ has a 2-sided Haar measure. Thus we need to consider proper H*-algebra, which is not complete. This is the purpose of this paper.

§1. Fundamental definitions.

Let $\mathfrak{H}$ be a Hilbert space, which need not be separable.

DEFINITION. A linear manifold $\mathfrak{A}$ of $\mathfrak{H}$ is called a *Hilbert algebra*, if

(1) $\mathfrak{A}$ is dense in $\mathfrak{H}$;

(2) $\mathfrak{A}$ is a ring: for any $a, b \varepsilon \mathfrak{A}$ there is defined $ab \varepsilon \mathfrak{A}$ such that

$$(ab)c=a(bc),\ a(b+c)=ab+ac,\ (a+b)c=ac+bc$$

and $(\alpha a)b=a(\alpha b)=\alpha ab$ for any complex number α;

(3) for any $a \varepsilon \mathfrak{A}$ there exists an *adjoint element* $a^* \varepsilon \mathfrak{A}$ such that

$$(ab,\ c)=(b,\ a^*c),\ (bc,\ c)=(b,\ ca^*);$$

(4) for any $a \varepsilon \mathfrak{A}$ there exists a positive number α_a such that

$$\|ax\| \leqq \alpha_a \|x\| \qquad \text{for all } x \varepsilon \mathfrak{A};$$

(5) by (1) and (4), for every $a \varepsilon \mathfrak{A}$ we obtain uniquely a bounded linear operator T_a on $\mathfrak{H}$ such that

$$T_a x = ax \qquad \text{for all } x \varepsilon \mathfrak{A}.$$

For an element $f \varepsilon \mathfrak{H}$, if $T_x f=0$ for all $x \varepsilon \mathfrak{A}$, then we have $f=0$.

First we shall write fundamental properties of Hilbert algebra. Let $\mathfrak{A}$ be a

2) I. E. Segal: The group ring of a locally compact group, Proc. Nat. Acad. Sci. U.S.A. **27** (1941) 348-351.

Hilbert algebra in a Hilbert phace $\mathfrak{H}$ in the sequel.

THEOREM 1. 1. *For an element* $a \varepsilon \mathfrak{A}$, *if* $ax=0$ *for all* $x \varepsilon \mathfrak{A}$, *then we have* $a=0$.

PROOF. By (3), for any $x, y \varepsilon \mathfrak{A}$ we have

$$(T_x a, y) = (xa, y) = (a, x^*y) = (ay^*, x^*) = 0.$$

Since $\mathfrak{A}$ is dense in $\mathfrak{H}$ by (1), we have thus $T_x a = 0$ for all $x \varepsilon \mathfrak{A}$, and hence $a=0$ by (5).

By this theorem we have obviously:

THEOREM 1. 2. $T_a = T_b$ *if and only if* $a=b$.

THEOREM 1. 3. *For any* $a \varepsilon \mathfrak{A}$ *we have* $T_a^* = T_{a^*}$, *and hence the adjoint element* a^* *is detemined uniquely*.

PROOF. For any $x, y \varepsilon \mathfrak{A}$ we have by (3)

$$(T_a x, y) = (ax, y) = (x, a^*y) = (x, T_{a^*}y).$$

Since $\mathfrak{A}$ is dense in $\mathfrak{H}$ by (1), we have thus $T_{a^*} = T_a^*$.

By definition we see easily:

THEOREM 1. 4. $a^{**} = a$, $(\alpha a)^* = \bar{\alpha} a^*$,

$$(ab)^* = b^*a^*, \qquad (a+b)^* = a^* + b^*.$$

THEOREM 1. 5. $T_{\alpha a} = \alpha T_a$, $T_{ab} = T_a T_b$, $T_{a+b} = T_a + T_b$.

THEOREM 1. 6. $\mathfrak{A}\mathfrak{A}$ *is complete in* $\mathfrak{H}$: *there is no element except* 0 *in* $\mathfrak{H}$, *which is orthogonal to* xy *for all* $x, y \varepsilon \mathfrak{A}$.

PROOF. For an element $f \varepsilon \mathfrak{H}$, if $(xy, f) = 0$ for all $x, y \varepsilon \mathfrak{A}$, then we have by theorem 1. 3

$$(T_x f, y) = (f, T_x^* y) = (f, T_{x^*}y) = (f, x^*y) = 0$$

for all $x, y \varepsilon \mathfrak{A}$, and hence $f=0$ by (1) and (5).

THEOREM 1. 7. *For any* $a, b \varepsilon \mathfrak{A}$ *we have*

$$(a, b) = (b^*, a^*), \qquad \|a\| = \|a^*\|.$$

PROOF. For any $x, y, z \varepsilon \mathfrak{A}$ we have by (3) and theorem 1. 4

$$(x, yz) = (xz^*, y) = (z^*y^*, x^*) = ((yz)^*, x^*).$$

By the previous theorem, for any $a \varepsilon \mathfrak{A}$ there exist $a_\nu \varepsilon \mathfrak{A}$ $(\nu = 1, 2, \cdots)$, as linear forms from $\mathfrak{A}\mathfrak{A}$, such that $\lim_{\nu\to\infty} a_\nu = a$ and $(x, a_\nu) = (a_\nu^*, x^*)$ for all $x \varepsilon \mathfrak{A}$ and $\nu = 1, 2, \cdots$. As $x \varepsilon \mathfrak{A}$ may be arbitrary, we have then

$$\|c_\nu - a_\mu\|^2 = \|a_\nu^* - a_\mu^*\|^2 \qquad (\nu, \mu = 1, 2, \cdots),$$

and hence there exists $f \varepsilon \mathfrak{H}$ for which $\lim_{\nu\to\infty} a_\nu^* = f$. Then we have $(x, a) = (f, x^*)$ for all $x \varepsilon \mathfrak{H}$. Therefore we have

$$(f, xy) = ((xy)^*, a) = (a^*, xy)$$

for all $x, y \varepsilon \mathfrak{A}$, and hence $f = a^*$ by the previous theorem.

By (4) and theorem 1. 7 we obtain immediately:

THEOREM 1. 8. For any $a \varepsilon \mathfrak{A}$ there exists a positive number β_a such that

$$\|xa\| \leqq \beta_a \|x\| \qquad \text{for all } x \varepsilon \mathfrak{A}.$$

By this theorem, for every $a \varepsilon \mathfrak{A}$ we obtain uniquely a bounded linear opetator S_a on $\mathfrak{H}$ such that

$$S_a x = xa \qquad \text{for all } x \varepsilon \mathfrak{A}.$$

By definition we see easily:

THEOREM 1. 9. $S_a^* = S_{a^*}$, $S_{ab} = S_b S_a$, $T_a S_b = S_b T_a$, $S_{\alpha a} = \alpha S_a$, $S_{a+b} = S_a + S_b$.

THEOREM 1. 10. $S_a = S_b$ if and only if $a = b$.

PROOF. If $S_a = 0$, then we have $T_x a = S_a x = 0$ for all $x \varepsilon \mathfrak{A}$, and hence $a = 0$ by (5).

THEOREM 1. 11. *For an element $f \varepsilon \mathfrak{H}$, if $S_x f = 0$ for all $x \varepsilon \mathfrak{A}$, then we have $f = 0$.*

PROOF. For any $x, y \varepsilon \mathfrak{A}$ we have by theorems 1.3, 1.9

$$(T_x f, y) = (f, T_x^* y) = (f, x^* y) = (f, S_y x^*) = (S_{y^*} f, x^*).$$

Therefore if $S_x f = 0$ for all $x \varepsilon \mathfrak{A}$, then we have $T_x f = 0$ for all $x \varepsilon \mathfrak{A}$, and hence $f = 0$ by (5).

By definition we have obviously

$$(T_a b)^* = S_{a^*} b^*, \qquad (S_a b)^* = T_{a^*} b^*$$

for all $a, b \varepsilon \mathfrak{A}$, and hence by theorem 1.7

$$\|T_a b\| = \|S_{a^*} b^*\|, \qquad \|b\| = \|b^*\|.$$

Therefore we obtain $\|T_a\| = \|S_{a^*}\| = \|S_a^*\| = \|S_a\|$ by theorem 1.9, that is, we have:

THEOREM 1. 12. $\|T_a\| = \|S_a\|$ *for all $a \varepsilon \mathfrak{A}$.*

§ 2. Closed algebras.

Let $\mathfrak{A}$ be a Hilbert algebra in a Hilbert space $\mathfrak{H}$. First we will prove:

THEOREM 2. 1. *If* $\lim_{\nu\to\infty} a_\nu = a_0$, $a_\nu \varepsilon \mathfrak{A}$ $(\nu = 0, 1, 2, \cdots)$ *and* $\|T_{a_\nu}\| \leqq \gamma$ $(\nu = 1, 2, \cdots)$ *for some* $\gamma > 0$, *then we have*

$$\lim_{\nu\to\infty} T_{a_\nu} = T_{a_0}, \quad \lim_{\nu\to\infty} T_{a_\nu^*} = T_{a_0^*},$$
$$\lim_{\nu\to\infty} S_{a_\nu} = S_{x_0}, \quad \lim_{\nu\to\infty} S_{a_\nu^*} = S_{a_0^*}.$$

PROOF. Since $T_{a_\nu}x = S_x a_\nu$ for all $x \varepsilon \mathfrak{A}$, we have by assumption

$$\lim_{\nu\to\infty} T_{a_\nu}x = S_x a_0 = T_{a_0}x \qquad \text{for all } x \varepsilon \mathfrak{A}.$$

For any $f \varepsilon \mathfrak{H}$ there exist $x_\nu \varepsilon$ $(\nu = 1, 2, \cdots)$ by § 1 (1), such that $\lim_{\nu\to 8} x_\nu = f$. We have then

$$\|T_{a_\nu}f - T_{a_0}f\| \leqq (\|T_{a_\nu}\| + \|T_{a_0}\|)\|f - x_\mu\| + \|T_{a_\nu}x_\mu - T_{a_0}x_\mu\|$$

for any ν, $\mu = 1, 2, \cdots$. Therefore we obtain

$$\overline{\lim_{\nu\to\infty}} \|T_{a_\nu}f - T_{a_0}f\| \leqq (\gamma + \|T_{a_0}\|)\|f - x_\mu\|,$$

and hence $\lim_{\nu\to\infty} T_{a_\nu}f = T_{a_0}f$ for all $f \varepsilon \mathfrak{H}$.

Since $\lim_{\nu\to\infty} a_\nu^* = a_0^*$ by theorem 1.7 and $\|T_{a_\nu^*}\| = \|T_{a_\nu}^*\| \leqq \gamma$ $(\nu = 1, 2, \cdots)$ by theorem 1.3, we have also $\lim_{\nu\to\infty} T_{a_\nu^*} = T_{a_0^*}$, as proved above. Furthermore, since $\|T_{a_\nu}\| = \|S_{a_\nu}\|$ by theorem 1.12, we can prove similarly also the other equations.

DEFINITION. A Hilbert algebra $\mathfrak{A}$ is said to be *closed*, if $\lim_{\nu\to\infty} a_\nu = f \varepsilon \mathfrak{H}$, $a_\nu \varepsilon \mathfrak{A}$ $(\nu = 1, 2, \cdots)$ and $\sup_{\nu\geqq 1} \|T_{a_\nu}\| < +\infty$ imply $f \varepsilon \mathfrak{A}$.

DEFINITION. A Hilbert algebra $\tilde{\mathfrak{A}}$ is called an *extension* of a Hilbert algebra $\mathfrak{A}$, if $\tilde{\mathfrak{A}}$ contains $\mathfrak{A}$ as a subalgebra.

DEFINITION. A Hilbert algebra $\mathfrak{A}$ is said to be *maximal*, if there is no extension of $\mathfrak{A}$ except itself.

By Zorn's lemma or transfinite induction we see easily:

THEOREM 2. 2. *Every Hilbert algebra has a max'mal extension.*

THEOREM 2. 3. *If a Hilbert algelra $\mathfrak{A}$ is maximal, then $\mathfrak{A}$ is closed.*

PROOF. Let $\mathfrak{A}$ be a maximal Hilbert algebra. If $\lim_{\nu\to\infty} a_\nu = f \varepsilon \mathfrak{H}$, $a_\nu \varepsilon \mathfrak{A}$ and $\|T_{a_\nu}\| < \gamma$ $(\nu = 1, 2, \cdots)$, then we have

$$\lim_{\nu,\mu\to\infty} \|a_\nu - a_\mu\| = 0, \quad \|T_{a_\nu - a_\mu}\| \leqq 2\gamma \ (\nu, \ \mu = 1, 2, \cdots),$$

and hence $\lim_{\nu,\mu\to\infty} (T_{a_\nu} - T_{a_\mu}) = 0$ by theorem 2.1. Consequently there exists a bounded linear operator $\tilde{T}_f$ on $\mathfrak{H}$ such that $\lim_{\nu\to\infty} T_{a_\nu} = \tilde{T}_f$. Such $\tilde{T}_f$ is determined uniquely corresponding to f, because if

$$\lim_{\nu\to\infty} a_\nu = \lim_{\nu\to\infty} b_\nu, \quad \|T_{a_\nu}\| \leqq \gamma, \ \|T_{b_\nu}\| \leqq \gamma,$$

then we have $\lim_{\nu\to\infty} (a_\nu - b_\nu) = 0$, $\|T_{a_\nu} - T_{b_\nu}\| \leqq 2\gamma$, and hence $\lim_{\nu\to\infty} T_{a_\nu} = \lim_{\nu\to\infty} T_{b_\nu}$ by theorem 2.1.

We denote by $\tilde{\mathfrak{A}}$ the set of all such elements $f \varepsilon \mathfrak{H}$. Then we see easily that $\tilde{\mathfrak{A}}$ is a linear manifold of $\mathfrak{H}$, $\tilde{\mathfrak{A}} \supset \mathfrak{A}$, and

$$T_a = \tilde{T}_a \qquad \text{for all } a \varepsilon \mathfrak{A}.$$

For any $f, g \varepsilon \tilde{\mathfrak{A}}$ there exist $a_\nu \varepsilon \mathfrak{A}$ and $b_\nu \varepsilon \mathfrak{A}$ $(\nu = 1, 2, \cdots)$ such that

$$\lim_{\nu\to\infty} a_\nu = f, \ \lim_{\nu\to\infty} b_\nu = g, \ \|T_{a_\nu}\| \leqq \gamma, \ \|T_{b_\nu}\| \leqq \gamma$$

for some positive number γ. Since

$$\|a_\nu b_\nu - \tilde{T}_f g\| \leqq \|T_{a_\nu}\| \, \|b_\nu - g\| + \|T_{a_\nu} g - \tilde{T}_f g\|,$$

we have then $\lim_{\nu\to\infty} a_\nu b_\nu = \tilde{T}_f g$ and further $\|T_{a_\nu b_\nu}\| \leqq \gamma^2$, and hence Since $\lim_{\mu,\nu\to\infty} \|a_\nu^* - a_\mu^*\| = 0$ by theorem 1.7 and $\|T_{a_\nu^*}\| = \|T_{a_\nu}^*\| \leqq \gamma$ by .... 1.2, there exists $f^* \varepsilon \tilde{\mathfrak{A}}$ for which $\lim_{\nu\to\infty} a_\nu^* = f^*$, and we have $\tilde{T}_{f^*} = \tilde{T}_f^*$, because for all $x, y \varepsilon \mathfrak{A}$ we have

$$\begin{aligned}(\tilde{T}_f^* x, y) &= \lim_{\nu\to\infty} (T_{a_\nu^*} x, y) = \lim_{\nu\to\infty} (a_\nu^* x, y) \\ &= \lim_{\nu\to\infty} (x, T_{a_\nu} y) = (x, \tilde{T}_f y).\end{aligned}$$

Furthermore if $\lim_{\nu\to\infty} c_\nu = h \varepsilon \tilde{\mathfrak{A}}$, $c_\nu \varepsilon \mathfrak{A}$, and $\|T_{c_\nu}\| \leqq \gamma$ $(\nu = 1, 2, \cdots)$ for some positive number γ, then we have

$$(\tilde{T}_g f, h) = \lim_{\nu\to\infty} (b_\nu a_\nu, c_\nu) = \lim_{\nu\to\infty} (b_\nu, c_\nu a_\nu^*) = (g, \tilde{T}_h f^*).$$

Therefore, putting $fg = \tilde{T}_f g$ for $f, g \varepsilon \tilde{\mathfrak{A}}$, we obtain a Hilbert algebra $\tilde{\mathfrak{A}}$, which is an extension of $\mathfrak{A}$. If $\mathfrak{A}$ is not closed, then $\tilde{\mathfrak{A}}$ dose not coincide with $\mathfrak{A}$ by its construction, contradicting the assumption that $\mathfrak{A}$ is maximal. Thus $\mathfrak{A}$ is closed.

Let $\tilde{\mathfrak{A}}$ and $\hat{\mathfrak{A}}$ be two extensions of $\mathfrak{A}$. For any common element f of $\tilde{\mathfrak{A}}$ and $\hat{\mathfrak{A}}$, puttig $\tilde{T}_f \tilde{g} = f\tilde{g}$ for $\tilde{g} \varepsilon \tilde{\mathfrak{A}}$ and $\hat{T}_f \hat{g} = f\hat{g}$ for $\hat{g} \varepsilon \hat{\mathfrak{A}}$, we have for all $x, y \varepsilon \mathfrak{A}$

$$(\tilde{T}_f x, y) = (f, yx^*) = (\hat{T}_f x, y),$$

and hence $T_f = T_f$ by §1 (1). Therefore the intersection of all closed extensions of $\mathfrak{A}$ is also a closed extension of $\mathfrak{A}$, which is called the *closure* of $\mathfrak{A}$.

DEFINITION. A Hilbert algebra $\mathfrak{A}$ is said to be *bounded*, if $\mathfrak{A}$ satisfies the condition

$$\text{(B)} \qquad \sup_{\|x\| = \|y\| = 1} \|xy\| < +\infty,$$

and this value is called its *order*.

THEOREM 2. 4. *If a Hilbert algebra $\mathfrak{A}$ is bounded, then the closure of $\mathfrak{A}$ coincides with the whole space $\mathfrak{H}$.*

PROOF. For any $f \varepsilon \mathfrak{H}$ there exist $a_\nu \varepsilon \mathfrak{A}$ $(\nu = 1, 2, \cdots\cdots\cdots)$ by §1 (1) such that $\lim_{\nu\to\infty} a_\nu = f$. If $\mathfrak{A}$ is bounded, then we have $\sup_{\nu\geqq 1} \|T_{a_\nu}\| < +\infty$ and hence f belongs to the closure of $\mathfrak{A}$.

THEOREM 2. 5. *If the whole space $\mathfrak{H}$ is an extension of a Hilbert algebra $\mathfrak{A}$, then $\mathfrak{A}$ is bounded.*

PROOF. Let $\mathfrak{H}$ be a Hilbert algebra. If $\mathfrak{H}$ is not bounded, then there exist x_ν and $y_\nu \varepsilon \mathfrak{H}$ $(\nu = 1, 2, \cdots)$ such that

$$\|x_\nu\| = \|y_\nu\| = 1, \qquad \|x_\nu y_\nu\| \geqq \nu^2,$$

and we have for any $z \varepsilon \mathfrak{H}$

$$\left|\left(\frac{1}{\nu} x_\nu y_\nu, z\right)\right| = \left|\left(\frac{1}{\nu} x_\nu, z y_\nu^*\right)\right| \leqq \frac{1}{\nu} \|T_z\|.$$

Thus $\frac{1}{\nu} x_\nu y_\nu$ $(\nu = 1, 2, \cdots)$ is weakly convergent, contradicting

$$\lim_{\nu\to\infty} \left\|\frac{1}{\nu} x_\nu y_\nu\right\| = +\infty.$$

REMARK. Every proper H*-algebra defined by W. Ambrose is a Hilbert algebra. Indeed a proper H*-algebra $\mathfrak{H}$ satisfies obviously the condition of Hilbert algebra except (4). $\mathfrak{H}$ satisfies further (4), because if $\mathfrak{H}$ dose not satisfy (4), then there exist $a \varepsilon \mathfrak{H}$ and $x_\nu \varepsilon \mathfrak{H}$ such that $\|x_\nu\| = 1$, $\|ax_\nu\| \geqq \nu^2$ $(\nu = 1, 2, \cdots)$, and we have

$$\lim_{\nu\to\infty} \left(\frac{1}{\nu} ax_\nu, y\right) = \lim_{\nu\to\infty} \left(\frac{1}{\nu} x_\nu, a^* y\right) = 0$$

for all $y \varepsilon \mathfrak{H}$, that is, $\frac{1}{\nu} ax_\nu$ $(\nu = 1,2,\cdots)$ is weakly convergent, contradicting $\lim_{\nu\to\infty} \left\|\frac{1}{\nu} ax_\nu\right\| = +\infty$. Furthermore $\mathfrak{H}$ is bounded by theorem 2. 5.

§ 3. Associative operators.

Let $\mathfrak{A}$ be a closed Hilbert algebra in a Hilbert space $\mathfrak{H}$.

DEFINITION. A bounded linear operator A on $\mathfrak{H}$ is said to be *associative* with $\mathfrak{A}$, if $A\mathfrak{A} \subset \mathfrak{A}$ and we have

$$(Ax)\,y = Axy \qquad \text{for all } x, y \in \mathfrak{A},$$

that is, $AS_x = S_x A$ for all $x \in \mathfrak{A}$.

By definition we have obviously:

THEOREM 3. 1. *The set of all associative operators constitutes a ring of operators: if A and B are both assoiative with $\mathfrak{A}$, then $\alpha A + \beta B$, AB are all associative with $\mathfrak{A}$.*

THEOREM 3. 2. *T_a is associative with $\mathfrak{A}$ for all $a \in \mathfrak{A}$.*

THEOREM 3. 3. *If A is associative with $\mathfrak{A}$, then we have $AT_a = T_{Aa}$ for every $a \in \mathfrak{A}$.*

LEMMA 3. 1. *For a sequence of bounded linear operators A_ν $(\nu = 1, 2, \cdots)$ on $\mathfrak{H}$, if $\lim\limits_{\nu\to\infty} (A_\nu x, y)$ exists for any $x, y \in \mathfrak{H}$, then we have $\sup\limits_{\nu\geqq 1} \| A_\nu \| < +\infty$.*

PROOF. If $\sup\limits_{\nu\geqq 1} \| A_\nu \| = +\infty$, then there exist $x_\nu \in \mathfrak{H}$, μ_ν $(\nu = 1, 2, \cdots)$ such that $\| x_\nu \| = 1$ and $\| A_{\mu_\nu} x_\nu \| \geqq \nu^2$, and we have for any $y \in \mathfrak{H}$

$$\lim_{\nu\to\infty} \left(\frac{1}{\nu} A_{\mu_\nu} x_\nu, y \right) = \lim_{\nu\to\infty} \left(\frac{1}{\nu} x_\nu, A^*_{\mu_\nu} y \right) = 0,$$

that is, $\frac{1}{\nu} A_{\mu_\nu} x_\nu$ $(\nu = 1, 2, \cdots)$ is weakly convergent, contradicting

$$\lim_{\nu\to\infty} \left\| \frac{1}{\nu} A_{\mu_\nu} x_\nu \right\| = +\infty.$$

THEOREM 3. 4. *If A_ν $(\nu = 1, 2, \cdots)$ are all associative with $\mathfrak{A}$ and $\lim\limits_{\nu\to\infty} A_\nu = A$, then A is also associative with $\mathfrak{A}$.*

PROOF. By the previous lemma A is obviously a bouded linear operator on $\mathfrak{H}$. For any $a \in \mathfrak{A}$ we have by theorem 3. 3

$$\| T_{A_\nu a} \| = \| A_\nu T_a \| \leqq \| A_\nu \| \, \| T_a \| \quad \text{and} \quad \lim_{\nu\to\infty} A_\nu a = Aa,$$

and hence $Aa \in \mathfrak{A}$, since $\mathfrak{A}$ is closed by assumption. For any $x, y \in \mathfrak{A}$ we have furthermore

$$(Ax)\,y = \lim_{\nu\to\infty} S_y A_\nu x = \lim_{\nu\to\infty} A_\nu S_y x = Axy.$$

For a system of projection operators P_λ $(\lambda \in \Lambda)$ on $\mathfrak{H}$, $\bigcap\limits_\lambda P_\lambda$ means the projection operator of the intersection of all $P_\lambda \mathfrak{H}$ $(\lambda \in \Lambda)$, and $\bigcup\limits_\lambda P_\lambda$ the projection operator of the closed linear manifold spanned by all $P_\lambda \mathfrak{H}$ $(\lambda \in \Lambda)$.

LEMMA 3. 2. *For any projection operators P and Q, we have*

$$\lim_{\nu\to\infty} (PQ)^\nu = P \cap Q, \qquad P \cup Q = 1-((1-P)\cap(1-Q)).$$

PROOF. Since PQP is a positive definite self-adjoint operator and $\|PQPx\| \leqq \|x\|$ for all $x \in \mathfrak{H}$, by spectral theory we obtain a projection operator P_0 as

$$P_0 = \lim_{\nu\to\infty} (PQP)^\nu .$$

As $PQP\,(P\cap Q) = P\cap Q$, we have then obviously $P_0 \geqq P\cap Q$. On the other hand, since for any $x \in \mathfrak{H}$

$$\|x\| \geqq \|Px\| \geqq \|QPx\| \geqq \|PQPx\| \geqq \|P_0x\|, \quad P_0P_0x = P_0x,$$

we have $\|P_0x\| = \|PP_0x\| = \|QPP_0x\| = \|P_0x\|$ for all $x \in \mathfrak{H}$. Therefore we obtain $P_0x = PP_0x = QPP_0x$, and hence $P_0 \leqq P\cap Q$. Thus we have

$$P \cap Q = \lim_{\nu\to\infty} (PQP)^\nu = \lim_{\nu\to\infty} (PQ)^\nu,$$

since $(P\cap Q)\,Q = P\cap Q$. By definition we have obviously the other equation.

THEOREM 3. 5. *For a system of projection operators P_λ $(\lambda \in \Lambda)$, if P_λ $(\lambda \in \Lambda)$ are all associative with $\mathfrak{A}$, then $\bigcap_\lambda P_\lambda$ and $\bigcup_\lambda P_\lambda$ are both associative with $\mathfrak{A}$.*

PROOF. By theorem 3. 4 and lemma 3. 2 we can assume that for any $\lambda_1, \lambda_2 \in \Lambda$ there exists $\lambda \in \Lambda$ such that $P_\lambda \leqq P_{\lambda_1} \cap P_{\lambda_2}$. Putting $P = \bigcap P_\lambda$, we have then

$$\|Px\| = \inf_{\lambda\in\Lambda} \|P_\lambda x\| \qquad \text{for all } x \in \mathfrak{A}.^{3)}$$

For any $a, b \in \mathfrak{A}$ there exists $\lambda_\nu \in \Lambda$ $(\nu = 1, 2, \cdots\cdots\cdots)$ such that $P_{\lambda_1} \geqq P_{\lambda_2} \geqq \cdots\cdots$ and

$$\lim_{\nu\to\infty} \|P_{\lambda_\nu} a\| = \inf_{\lambda\in\Lambda} \|P_\lambda a\|, \quad \lim_{\nu\to\infty} \|P_{\lambda_\nu} ab\| = \inf_{\lambda\in\Lambda} \|P_\lambda ab\|.$$

For $P_0 = \lim_{\nu\to\infty} P_{\lambda_\nu}$, we have thus $\|P_0a\| = \|Pa\|$, $\|P_0ab\| = \|Pab\|$. As obvionsly $P_0 \geqq P$, we have hence $P_0a = Pa$, $P_0ab = Pab$. Since P_0 is associative with $\mathfrak{A}$ by theorem 3. 4, we obtain therefore

$$Pa \in \mathfrak{A}, \; (Pa)\,b = (P_0a)\,b = P_0ab = Pab.$$

Similarly we can prove the other relation.

THEOREM 3. 6. *If a bounded self-adjoint operator H is associative with $\mathfrak{A}$,*

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3) H.Nakano: Funktionen mehrerer hypermaximaler normaler Operatoren, Proc. Phis.-Math. Soc. Japan, **21** (1939) 713-728, Satz 2.

then for its spectral system $E_\lambda\ (-\infty < \lambda < +\infty)$:

$$H = \int_{-\infty}^{\infty} \lambda\, dE_\lambda, \quad \lim_{\lambda \to \lambda_0 + 0} E_\lambda = E_{\lambda_0},$$

E_λ *is associative with* $\mathfrak{A}$ *for any* λ.

Proof. As H is bounded, we assume $\|Hx\| \leqq \gamma \|x\|$ for all $x \in \mathfrak{H}$. Since

$$|\lambda| = \sum_{\nu=1}^{\infty} \alpha_\nu \lambda^2 (1-\lambda^2)^\nu, \quad \alpha_0 = 1, \quad \alpha_\nu = \frac{2\nu - 1}{2\nu}\, \alpha_{\nu-1},$$

is uniformly convergent for $|\lambda| \leqq 1$, putting

$$H_1 = \frac{1}{\gamma + |\lambda_0|} (H - \lambda_0), \quad -\gamma \leqq \lambda_0 \leqq \gamma$$

we see easily by spectral theory[4])

$$(1 - E_{\lambda_0})\, H_1 = \frac{1}{2}\, H_1 + \frac{1}{2} \sum_{\nu=1}^{\infty} \alpha_\nu H_1^2 (1 - H_1^2)^\nu,$$

$$E_{\lambda_0} = \lim_{\nu \to \infty} \{1 - (1 - E_{\lambda_0}) H_1\}^\nu.$$

Therefore we obtain by theorems 3.1, 3.4 that E_{λ_0} is assciative with $\mathfrak{A}$.

§ 4. Units.

Let $\mathfrak{A}$ be a closed Hilbert algebra in a Hilbert space $\mathfrak{H}$.

Definition. An element $h \in \mathfrak{A}$ is said to be *self-adjoint*, if $h = h^*$.

By theorems 1.3 and 1.9 we have then obviously:

Theorem 4. 1. *An element* $a \in \mathfrak{A}$ *is self-adjoint if and only if* T_a *is self-adjoint, or* S_a *is self-adjoint.*

Definition. An element $u \in \mathfrak{A}$ is called a *unit*, if u is self-adjoint and idempotent: $u = u^*$ and $uu = u$.

By theorems 1.5 and 1.9 we see at once:

Theorem 4. 2. *An element* $a \in \mathfrak{A}$ *is a unit if and only if* T_a *is a projection operator, or* S_a *is a projection operator.*

Definition. For units $u_1, u_2 \in \mathfrak{A}$ we shall write $u_1 \geqq u_2$ if $T_{u_1} \geqq T_{u_2}$ as projection operators.

By theorem 1.5 and calculus of projection operators we see easily that

4) Cf. J. von Neumann: Allgemeine Eigenwerttheorie Hermitescher Funktionaloperatoren, Math. Ann. **102** (1930) 49-131, M. H. Stone: Linear transformations in Hilbert space, New York (1932).

for units we have:

THEOREM 4. 3. *If* $u_1 \geqq u_2$ *and* $u_2 \geqq u_3$, *then* $u_1 \geqq u_3$; $u_1 \geqq u_2$ *implies* $u_1 u_2 = u_2 u_1 = u_2$; *for* $u_1 \geqq u_2$ *we obtain a unit* $u_1 - u_2 \leqq u_1$; *and if* $u_1 u_2 = 0$ *and* $u_2 \geqq u_3$, *then we have* $u_1 u_3 = u_3 u_1 = 0$.

THEOREM 4. 4. *For units,* $u_1 u_2 = 0$ *implies* $(u_1, u_2) = 0$.

PROOF. $(u_1, u_2) = (u_1, u_2 u_2) = (u_1 u_2, u_2) = 0$.

THEOREM 4. 5. *For a unit u and an associative projection operator* $P \leqq T_u$ *we obtain a unit Pu for which we have* $T_{Pu} = P$.

PROOF. By theorem 3.3 we have $T_{Pu} = PT_u = P$, and hence Pu is a unit by theorem 4.2.

THEOREM 4. 6. *Let* $h \varepsilon \mathfrak{A}$ *be self-adjoint and* E_λ $(-\infty < \lambda < +\infty)$ *the spectial system of* T_h. *For any positive number* ε *there exists an associative self-adjornt operator H such that Hh is a unit and* $T_{Hh} = 1 - E_\varepsilon$.

PROOF. By spectral theory we have for any $\varepsilon > 0$

$$\| T_h x \| \geqq \varepsilon \| x \| \qquad \text{for } x \varepsilon (1 - E_\varepsilon) \mathfrak{H}.$$

As T_h is bounded, we assume $\| T_h x \| \leqq \gamma \| x \|$ for $x \varepsilon \mathfrak{H}$. Then we obtain the inverse A of $\frac{1}{\gamma} T_h$ in $(1 - E_\varepsilon) \mathfrak{H}$, as

$$Ax = \sum_{\nu=1}^{\infty} \left(1 - \frac{1}{\gamma} T_h\right)^\nu x \qquad \text{for } x \varepsilon (1 - E_\varepsilon) \mathfrak{H},$$

and $\frac{1}{\gamma} AT_h x = x$ for $x \varepsilon (1 - E_\varepsilon) \mathfrak{H}$. Putting $H = \frac{1}{\gamma} A (1 - E_\varepsilon)$, we obtain a bounded self-adjoint operator H on $\mathfrak{H}$ and

$$HT_h x = \frac{1}{\gamma} AT_h (1 - E_\varepsilon) x = (1 - E_\varepsilon) x \qquad \text{for } x \varepsilon \mathfrak{H}.$$

Sinec $H = \sum_{\nu=1}^{\infty} \left(1 - \frac{1}{\gamma} T_h\right)^\nu (1 - E_\varepsilon)$, H is associative with $\mathfrak{A}$ by theorems 3.1 and 3.4, and hence $T_{Hh} = HT_h = 1 - E_\varepsilon$ by theorem 3.3. Consequently Hh is a unit by theorem 4.2.

THEOREM 4. 7. *Every closed Hilbert algebra* $\mathfrak{A}$ *has a unit* $u \neq 0$.

PROOF. For any $a \varepsilon \mathfrak{A}$, $a + a^*$ and $ia \cdot ia^*$ are both self-adjoint. Therefore there exists a self-adjoint element $h \neq 0$. For a self-adjoint element $h \neq 0$, since one of T_h and T_{-h} is not negative definite, we can assume that T_h is not negative definite. Then, for the spectral system E_λ of T_h, there exists a positive number ε such that $1 - E_\varepsilon \neq 0$, and by the previous theorem there exists a unit u for which we have $T_u = 1 - E_\varepsilon \neq 0$, and hence $u \neq 0$ by theorem 1.2.

Definition. A unit $u \neq 0$ is said to be *minimal*, if there is no unit $v \leqq u$ except itself and 0.

Theorem 4. 8. *In order that a unit* $u \in \mathfrak{A}$ *be minimal, it is necessary and sufficient that* $u\mathfrak{A}u$ *is one-dimensional.*

Proof Let $u\mathfrak{A}u$ be not one-dimensional. Then there exists $x \in u\mathfrak{A}u$ such that $x \neq 0$ and $(u, x) = 0$. Since

$$x^* = (uxu)^* = ux^*u \in u\mathfrak{A}u, \quad (u, x^*) = (x, u) = 0,$$

there exists then a self-adjoint element $h \in u\mathfrak{A}u$ such that

$$(u, h) = 0, \quad \|u\| = \|h\| \neq 0.$$

T_h is self-adioint by theorem 4.1. Let E_λ $(-\infty < \lambda < +\infty)$ be the spectral system of T_h. As $uh = hu = h$, we have $T_u T_h = T_h T_u = T_h$ by thnorem 1.5, and hence T_u is commutative with E_λ for all λ by spectral theory. We will now prove that there exists λ_0 for which

$$T_u E_{\lambda_0} \neq 0, \quad T_u(1 - E_{\lambda_0}) \neq 0.$$

If there is no such λ_0, then we have

$$T_h = T_h T_u = \int_{-\infty}^{\infty} \lambda \, dE_\lambda T_u = \lambda T_u = T_{\lambda u}$$

for some λ, and hence $h = \lambda u$ by theorem 1.2, contradicting

$$\|h\|^2 = (\lambda u, h) = \lambda(u, h) = 0.$$

If $T_u E_{\lambda_0} \neq 0$ and $T_u(1 - E_{\lambda_0}) \neq 0$, then, since E_λ is associative with $\mathfrak{A}$ by theorem 3.6, we obtain a unit $v = T_u E_{\lambda_0} u \leqq u$ by theorem 4.5, and we have $T_v = T_u E_{\lambda_0} \neq 0$, $T_u - T_v \neq 0$, that is, $v \neq 0$, $u \neq v$ by theorem 1.2. Therefore u is not minimal.

Conversely if a unit $u \neq 0$ is not minimal, then there exists a unit $v \leqq u$ such that $v \neq 0$ and $u - v \neq 0$. Then, since $v(u - v) = 0$, we have $(v, u - v) = 0$ by theorem 4.4, and

$$v = uvu \in u\mathfrak{A}u, \quad u - v = u(u - v)u \in u\mathfrak{A}u.$$

Therefore $u\mathfrak{A}u$ is not one-dimensional.

§ 5. Discrete algebras.

Let $\mathfrak{A}$ be a closed Hilbert algebra in a Hilbert space $\mathfrak{H}$.

Definition. A linear manifold $\mathfrak{p} \subset \mathfrak{A}$ is calld an *ideal*, if $\mathfrak{p}$ satisfies

(1) $\quad x\mathfrak{p}y \subset \mathfrak{p}$ for all $x, y \in \mathfrak{A}$;

(2) $\quad x \in \mathfrak{p}$ implies $x^* \in \mathfrak{p}$;

(3) $\quad a_\nu \in \mathfrak{p}, \lim_{\nu\to\infty} a_\nu = a \in \mathfrak{A}$ implies $a \in \mathfrak{p}$.

THEOREM 5.1. *If a projection operator P is associative with $\mathfrak{A}$ and commutative with T_x for all $x \in \mathfrak{A}$, then $P\mathfrak{A}$ is an ideal and closed as a subalgebra.*

PROOF. Since for any $x, y \in \mathfrak{A}$ we have

$$(Px)y = Pxy, \quad x(Py) = T_x Py = PT_x y = Pxy,$$

$P\mathfrak{A}$ satisfies (1). Since for any $x, y, z \in \mathfrak{A}$ we have,

$$((Px)y, z) = (x(Py), z) = (Py, x^* z) = (y, (Px^*)z),$$

we obtain by definition,

$$(Px)^* = Px^*$$

and hence $P\mathfrak{A}$ satisfies (2). Furthermore for $\lim_{\nu\to\infty} Pa_\nu = a \in \mathfrak{A}$ we have $\lim_{\nu\to\infty} Pa_\nu = Pa = a$, and hence $a \in P\mathfrak{A}$, that is, $P\mathfrak{A}$ satisfies (3).

If $\lim_{\nu\to\infty} a_\nu = a$, $a_\nu \in P\mathfrak{A}$, and $\| T_{a_\nu} x \| \leqq \gamma \| x \|$ for $x \in P\mathfrak{A}$ $(\nu = 1, 2, \cdots)$, then we have for any $x \in \mathfrak{A}$

$$\| T_{a_\nu} x \| = \| T_{Pa_\nu} x \| = \| PT_{a_\nu} x \| = \| T_{a_\nu} Px \| \leqq \gamma \| Px \| \leqq \gamma \| x \|.$$

Since $\mathfrak{A}$ is closed by assumption, we have then $a \in \mathfrak{A}$, and obviously $Pa = a$. Therefore $P\mathfrak{A}$ is closed as a subalgebra.

DEFINITION. An ideal $\mathfrak{p}$ is said to be *simple*, if there is no ideal contained in $\mathfrak{p}$ except itself and $\{0\}$.

Let $u \in \mathfrak{A}$ be a minimal unit, and $e_\lambda \in \mathfrak{A}u$ $(\lambda \in \Lambda)$ a maximal orthogonal system contained in $\mathfrak{A}u$ such that

$$\| e_\lambda \| = \| u \|, \quad e_{\lambda_0} = u \quad \text{fos some } \lambda_0 \in \Lambda$$

that is, $\mathfrak{A}u$ contains no element except 0, which is orthogonal to all e_λ $(\lambda \in \Lambda$

As $e_\lambda = e_\lambda u$, we have then

$$e_\lambda^* e_\rho = u e_\lambda^* e_\rho u \in u\mathfrak{A}u.$$

Since $u\mathfrak{A}u$ is one-dimensional by theorem 4.8, there exists a complex numb $\delta_{\lambda,\rho}$ such that

$$(\qquad e_\lambda^* e_\rho = \delta_{\lambda,\rho} u,$$

and $\delta_{\lambda,\rho} \| u \|^2 = (\delta_{\lambda,\rho} u, u) = (e_\lambda^* e_\rho, u) = (e_\rho, e_\lambda)$. Therefore we have by assumpti

$$\delta_{\lambda,\rho} = \begin{cases} 1 & \text{for } \lambda = \rho, \\ 0 & \text{for } \lambda \neq \rho. \end{cases}$$

By (*) we see easily:

$$(**) \qquad \begin{aligned} &(e_\rho e_\lambda^*)(e_\mu e_\kappa^*) = \delta_{\lambda,\mu} e_\rho e_\kappa^*, \\ &(e_\rho e_\lambda^*, e_\mu e_\kappa^*) = (e_\mu^* e_\rho, e_\kappa^* e_\lambda) = \delta_{\rho,\mu}\delta_{\lambda,\kappa}\|u\|^2. \end{aligned}$$

Thus $e_\rho e_\lambda^*$ $(\rho, \lambda \in \Lambda)$ constitute an orthogonal system. Since $e_\lambda e_\lambda^*$ is a unit by (**), we obtain a projection operator P as

$$P = \bigcup_{\lambda \in \Lambda} T_{e_\lambda e_\lambda^*}$$

which is associative with $\mathfrak{A}$ by theorem 3.5. For any $\lambda \in \Lambda$ we have by (*)

$$Pe_\lambda = Pe_\lambda e_\lambda^* e_\lambda = PT_{e_\lambda e_\lambda^*} e_\lambda = T_{e_\lambda e_\lambda^*} e_\lambda = e_\lambda,$$

and hence $(1-P)\,\mathfrak{A}u = \{0\}$, because e_λ $(\lambda \in \Lambda)$ is a maximal orthogonal system in $\mathfrak{A}u$. Since $\mathfrak{A}$ is dense in $\mathfrak{H}$, we have $(1-P)S_u = 0$, namely $P \geqq S_u$, and e_λ $(\lambda \in \Lambda)$ is a complete orthogonal system of $S_u\mathfrak{H}$. For any $\rho \in \Lambda$, since by (*) we have $S_{e_\rho^*} S_u = S_{e_\rho^*}$, and

$$(S_{e_\rho^*} x, e_\lambda e_\rho^*) = (x, e_\lambda e_\rho^* e_\rho) = (S_u x, e_\lambda)$$

for all $x \in \mathfrak{H}$ and $\lambda \in \Lambda$, we see easily that $e_\lambda e_\rho^*$ $(\lambda \in \Lambda)$ is a complete orthogonal system of $S_{e_\rho^*}\mathfrak{H}$. Furthermore we have by (**)

$$T_{e_\lambda e_\lambda^*} e_\lambda e_\rho^* = e_\lambda e_\rho^*,$$

and hence we obtain $P \geqq \bigcup_{\lambda \in \Lambda} S_{e_\lambda e_\lambda^*}$. On the other hand, e_λ^* $(\lambda \in \Lambda)$ is a maximal orthogonal system of $u\mathfrak{A}$, because $(x, y) = (y^*, x^*)$ for any $x, y \in \mathfrak{A}$ by theorem 1.7. Therefore we can prove similarly that for any $\rho \in \Lambda$, $e_\rho e_\lambda^*$ $(\lambda \in \Lambda)$ is a complete orthogonal system of $T_{e_\rho}\mathfrak{H}$ and $\bigcup_\lambda T_{e_\lambda e_\lambda^*} \leqq \bigcup S_{e_\lambda e_\lambda^*}$. Consequently we have

$$P = \bigcup_\lambda T_{e_\lambda e_\lambda^*} = \bigcup_\lambda S_{e_\lambda e_\lambda^*}$$

and $e_\rho e_\lambda^*(\rho, \lambda \in \Lambda)$ is a complete orthogonal system of $P\mathfrak{H}$. Since $S_{e_\lambda e_\lambda^*}$ is commutative with T_a for all $a \in \mathfrak{A}$, P is commutative with T_a for all $a \in \mathfrak{A}$.[5] Therefore $P\mathfrak{A}$ is an ideal by theorem 5.1.

Since $e_\rho e_\lambda^*$ $(\rho, \lambda \in \Lambda)$ is a complete orthogonal system of $P\mathfrak{A}$ and $\|e_\rho e_\lambda^*\| = \|u\|$ by (**), every element $a \in P\mathfrak{A}$ is represented uniquely as

$$a = \sum_{\rho,\lambda} \alpha_{\rho,\lambda} e_\rho e_\lambda^*, \quad \sum_{\rho,\lambda} |\alpha_{\rho,\lambda}|^2 = \frac{\|a\|^2}{\|u\|^2},$$

5) H. Nakano: Funktionen ..., Satz 3.

and for $b = \sum_{\rho,\lambda} \beta_{\rho,\lambda} e_\rho e_\lambda^* \varepsilon P\mathfrak{A}$ we have by (**)

$$ab = \sum_{\rho,\lambda} \{\alpha_{\rho,\lambda} e_\rho e_\lambda^* \sum_{\mu,\kappa} \beta_{\mu,\kappa} e_\mu e_\kappa^*\}$$

$$= \sum_{\rho,\mu} \sum_{\kappa} \alpha_{\rho,\mu} \beta_{\mu,\kappa} e_\rho e_\kappa^* = \sum_{\rho,\kappa} (\sum_{\mu} \alpha_{\rho,\mu} \beta_{\mu,\kappa}) e_\rho e_\kappa^*,$$

$$(a, b) = \| u \|^2 \sum_{\rho,\kappa} \alpha_{\rho,\kappa} \overline{\beta}_{\rho,\kappa}.$$

Therefore we see easily by Schwarz's inequality that we have

$$(***) \qquad \|xy\| \leqq \frac{1}{\|u\|} \|x\| \, \|y\| \qquad \text{for } x, y \varepsilon P\mathfrak{A}$$

and hence $P\mathfrak{A}$ is bounded as a subalgebra. Since $P\mathfrak{A}$ is closed as a subalgebra by theorem 5.1, we have hence $P\mathfrak{A} = P\mathfrak{H}$ by theorem 2.3.

We can prove easily that $P\mathfrak{A}$ is a simple ideal, in customary way: for an ideal $\mathfrak{p} \subset P\mathfrak{A}$, if $0 \neq a \varepsilon \mathfrak{p}$, then there exists ρ_0, $\lambda_0 \varepsilon \Lambda$ such that

$$a = \sum_{\rho,\lambda} \alpha_{\rho,\lambda} e_\rho e_\lambda^*, \quad \alpha_{\rho_0,\lambda_0} \neq 0,$$

and then $\mathfrak{p}$ contains $(1/\alpha_{\rho_0,\lambda_0}) e_{\rho_0} e_{\lambda_0}^* a e_{\rho_0} e_{\lambda_0}^* = e_{\rho_0} e_{\lambda_0}^*$, and hence all $e_\rho e_\lambda^*$, since we have by (**)

$$e_\rho e_\lambda^* = (e_\rho e_{\rho_0}^*) (e_{\rho_0} e_{\lambda_0}^*) (e_{\lambda_0} e_\lambda^*)$$

for any ρ, $\lambda \varepsilon \Lambda$. Consequently we have $\mathfrak{p} \supset P\mathfrak{A}$.

For any minimal unit $v \varepsilon P\mathfrak{A}$ we obtain similary a projection operator Q, such that $Q\mathfrak{A}$ is a simple ideal and $Q\mathfrak{A} \ni v$. Then the intersection of $P\mathfrak{A}$ and $Q\mathfrak{A}$ is also an ideal containing v, and hence $P\mathfrak{A} = Q\mathfrak{A}$. Therefore we have $\| u \| = \| v \|$ by (***).

Since $(1-P)\mathfrak{A}$ is also an ideal and closed as a subalgebra by theorem 5.1, we obtain by Zorn's lemma or transfinite induction:

TTHEOREM 5. 2. *For any closed Hilbert algebra* $\mathfrak{A}$ *there exists uniquely a system of associative projection operators* P_λ $(\lambda \varepsilon \Lambda)$ *such that* P_λ *is commutative with* T_a *for all* $a \varepsilon \mathfrak{A}$; $P_\lambda P_\rho = 0$ *for* $\lambda \neq \rho$; $P_\lambda \mathfrak{A}$ *is a simple ideal and isometric to a full-matrix algebra with the order* $1/\|u\|$ *for any minimal unit* u *of* $P_\lambda \mathfrak{A}$; $P_\lambda \mathfrak{A} = P_\lambda \mathfrak{H}$; *and* $(1 - \bigcup_\lambda P_\lambda)\mathfrak{A}$ *has no minimal unit.*

DEFINITION. A Hilbrt algebra $\mathfrak{A}$ is said to be *discrete*, if for any unit $u \neq 0$ there exists a minimal unit $v \leqq u$.

THEOREM 5. 3. *In order that a closed Hilbert algebra* $\mathfrak{A}$ *be discrete, it is necessary and sufficient that for every self-adjoint element* $h \varepsilon \mathfrak{A}$, T_h *has no* *continuous spectrum.*

PROOF. If $\mathfrak{A}$ is not discrete, then we see easily by theorems 4.3, 4.4, and 4.5 that there exists a system of units u_λ $(0 \leqq \lambda \leqq 1)$ such that $u_\lambda \leqq u_\mu$ for $\lambda < \mu$ and $\| u_\lambda \|$ is a continuous function of λ for $0 \leqq \lambda \leqq 1$. By spectral theory as

$$H = \int_0^1 \lambda \, d_\lambda \, T_{u_\lambda},$$

we obtain then a bounded self-adjoint operator H with continuous spectrum. Furthermore H is associative with $\mathfrak{A}$ by theorem 3.6. Putting $h = Hu_1$, we have then $T_h = HT_{u_1} = H$ by theorem 3.3.

Conversely if there is a self-adjoint elemenet $h \in \mathfrak{A}$ for which T_h has a continuous spectrum, then for the spectral system E_λ $(-\infty < \lambda < +\infty)$ of T_h, putting

$$P = \bigcup_\lambda (E_\lambda - E_{\lambda-0}),$$

we obtain an associative projection operator P by theorems 3.4 and 3.6, and putting $k = (1-P)h$, we obtain a self-adjoint element $k \in \mathfrak{A}$, for which the spectral system of T_k consists only of continuous spectrum, since $T_k = (1-P)T_h$ by theorem 3.3. Then we see easily by theorem 4.6 that there exists a system of units u_λ $(0 \leqq \lambda \leqq 1)$ such that $u_\lambda \leqq u_\mu$ for $\lambda < \mu$, $u_0 \neq v_1$, and $\| u_\lambda \|$ is a continuous function of λ. Putting $u = u_1 - u_0$, we obtain then a unit u by theorem 4.3, and for any positive number ε there exists a finite number of units $v_1, v_2, \cdots, v_\kappa$ such that

$$u = v_1 + \cdots + v_\kappa, \quad v_\nu v_\mu = 0 \text{ for } \nu \neq \mu, \quad \| v_\nu \| \leqq \varepsilon.$$

If there exists a minimal unit $v \leqq u$, then we obtain an associative projection operator P by theorem 5.2 such that P is commutative with T_a for all $a \in \mathfrak{A}$, $P\mathfrak{A} \ni v$, and

$$\| xy \| \leqq \frac{1}{\| v \|} \| x \| \| y \| \qquad \text{for all } x, y \in P\mathfrak{A}.$$

Since $Pu = Pv_1 + \cdots + Pv_k$, we have $Pv_\nu \neq 0$ for some ν.

For such ν, since $T_{Pv_\nu} = PT_{v_\nu}$ and PT_{v_ν} is also a projection operator P_{v_ν} is a unit by theorem 4.2, and $\| P_{u_\nu} \| \leqq \| v_\nu \| \leqq \varepsilon$, cotradicting that $\| P_{v_\nu} \| \geqq \| v \|$ and $\varepsilon > 0$ may be arbitary.

THEOREM 5.4. *In order that a closed Hilbert algebra $\mathfrak{A}$ be bounded, it is necessary and sufficient that for any unit $u \neq 0$ there exists a finite number of minimal units $u_1, u_2, \cdots, u_\kappa$ such that*

$$u = u_1 + \cdots + u_\kappa, \quad u_\nu u_\mu = 0 \quad for\ \nu \neq \mu.$$

PROOF. If $\|xy\| \leqq \gamma \|x\| \|y\|$ for all $x, y \in \mathfrak{A}$, then we have $\|u\| \geqq 1/\gamma$ for any unit $u \neq 0$. Since for two units $u \geqq v$, $u - v$ is also a unit by theorem 4.3 and

$$\|u\|^2 = \|v\|^2 + \|u - v\|^2$$

by theorem 4.4. Therefore we see easily that the condition of theorem is satisfied.

Conversely if the codition in theorem is satisfied, then $\mathfrak{A}$ is obviously discrete by definition, and there exists a system of associative projection operators $P_\lambda\ (\lambda \in \Lambda)$ indicated in theorem 5.2 with $\bigcup_\lambda P_\lambda = 1$. Let u_λ be a minimal unit in $P_\lambda \mathfrak{A}$. Then we have by theorem 5.2

$$\|xy\| \leqq \frac{1}{\|u_\lambda\|} \|x\| \|y\| \qquad \text{for all } x, y \in P_\lambda \mathfrak{A}.$$

If $\inf_{\lambda \in \Lambda} \|u_\lambda\| = 0$, then there exists $\lambda_\nu \in \Lambda$ $(\nu = 1, 2, \cdots)$ for which we have

$$\sum_{\nu=1}^{\infty} \|u_{\lambda_\nu}\|^2 < +\infty.$$

Since $u_{\lambda_\nu} u_{\lambda_\mu} = (P_{\lambda_\nu} u_{\lambda_\nu})(P_{\lambda_\mu} u_{\lambda_\mu}) = P_{\lambda_\nu} P_{\lambda_\mu} u_{\lambda_\nu} u_{\lambda_\mu} = 0$ for $\nu \neq \mu$, $u_{\lambda_1} + \cdots + u_\lambda$ is a unit and $\|T_{u_{\lambda_1} + \cdots + u_{\lambda_\nu}}\| \leqq 1$. Therefore we obtain a unit $u = u_{\lambda_1} + u_{\lambda_2} + \cdots$. For such unit u, by assumption there exists a finite number of minimal units $v_1, v_2, \cdots, v_k$ such tbat

$$u = v_1 + \cdots + v_\kappa, \qquad v_\nu v_\mu = 0 \text{ for } \nu \neq \mu.$$

We have then $u_{\lambda_\nu} = P_{\lambda_\nu} v_1 + \cdots + P_{\lambda_\nu} v_\kappa$ $(\nu = 1, 2, \cdots)$, and, since $P_{\lambda_\nu} v_\mu = P_{\lambda_\nu} T_{v_\mu} v_\mu$, $P_{\lambda_\nu} v_\mu$ is also a unit by theorem 4.5. Thus $v_1, v_2, \cdots, v_\kappa$ must coincide with a finite number of $u_{\lambda_1}, u_{\lambda_2}, \cdots$, contradicting $u_{\lambda_\nu} \neq 0$ $(\nu = 1, 2, \cdots)$. Therefore there exists a positive number ε such that $\|u_\lambda\| \geqq \varepsilon$ for all $\lambda \in \Lambda$, and we have then for any $x, y \in \mathfrak{A}$

$$\begin{aligned} \|xy\|^2 &= \|\sum_{\lambda \in \Lambda} (P_\lambda x)(P_\lambda y)\|^2 = \sum_{\lambda \in \Lambda} \|(P_\lambda x)(P_\lambda y)\|^2 \\ &\leqq \frac{1}{\varepsilon^2} \sum_{\lambda \in \Lambda} \|P_\lambda x\|^2 \|P_\lambda y\|^2 \leqq \frac{1}{\varepsilon^2} (\sum_\lambda \|P_\lambda x\|^2)(\sum_\lambda \|P_\lambda y\|^2) \\ &= \frac{1}{\varepsilon^2} \|x\|^2 \|y\|^2. \end{aligned}$$

THEOREM 5.5. *In order that a closed Hilbert algebra $\mathfrak{A}$ be bounded ana every simple ideal be finite-dimensional, it is necessary and sufficient that T_a is completely continuous for all $a \in \mathfrak{A}$.*

PROOF First we assume that T_a is completely continuous for every $a \varepsilon \mathfrak{A}$. Since $\mathfrak{A}$ is discrete by theorem 5.3, for any unit $u \neq 0$ there exists a sequence of minimal units u_ν $(\nu = 1, 2, \cdots)$ by theorem 4.3 such that

$$u = u_1 + u_2 + \cdots, \quad u_\nu u_\mu = 0 \text{ for } \nu \neq \mu.$$

As T_u is completely continuous by assumption, $T_u \mathfrak{H}$ must be finite-dimensional, and

$$T_u \mathfrak{H} \ni T_u u_\nu = u_\nu \quad (\nu = 1, 2, \quad).$$

Thus $u_\nu = 0$ except for finite ν, and hence $\mathfrak{A}$ is bounded by the previous theorem. Let P_λ $(\lambda \varepsilon \Lambda)$ be a system of projection operators indicated in theorem 5.2. For any minimal unit $u \varepsilon P_\lambda \mathfrak{A}$, since $T_u \mathfrak{H}$ is finite-dimensional, $P_\lambda \mathfrak{A}$ is also finite-dimensinal by its construction. Therefore every simple ideal is finite-dimensional.

Conversely if T_a is not completely continuous for some $a \varepsilon \mathfrak{A}$, then $T_{a^* a}$ is also not completely continuous, because if $T_{a^* a}$ is completely continuous, then $\lim_{\nu\to\infty} (a_\nu, x) = 0$ for all $x \varepsilon \mathfrak{H}$ implies $\lim_{\nu\to\infty} T_{a^* a} a_\nu = 0$, and hence

$$\lim_{\nu\to\infty} \| T_a a_\nu \|^2 = \lim_{\nu\to\infty} (T_{a^* a} a_\nu, a_\nu) = 0,$$

contradicting that T_a is not completely continuous. Since $T_{a^* a}$ is self-adjoint and positive definite, by theorem 4.6 there exists a unit u for which $T_u \mathfrak{H}$ is infinite-dimensional. If $\mathfrak{A}$ is further bounded, then by the previous theorem there exists a finite number of minimal units $u_1, u_2, \cdots, u_k$ such that

$$u = u_1 + \cdots + v_k, \quad u_\nu u_\mu = 0 \text{ for } \nu \neq \mu.$$

Then, as $T_u = T_{u_1} + \cdots + T_{u_k}$, $T_{u_\nu} \mathfrak{H}$ is infinite-dimensional for some ν. Let P_λ $(\lambda \varepsilon \Lambda)$ be a projection operators indicated in theorem 5.2. There exists $\lambda \varepsilon \Lambda$ for which $P_\lambda \mathfrak{A} \ni u_\nu$, namely $P_\lambda u_\nu = u_\nu$, and hence $P_\lambda T_{u_\nu} = T_{u_\nu}$ by theorem 3.3, that is, $P_\lambda \mathfrak{A} \supset T_{u_\nu} \mathfrak{A}$. Therefore $P_\lambda \mathfrak{A}$ is a simple ideal but not finite-dimensional.

§ 6. **Maximal algebras.**

In the sequel we consider only maximal Hilbert algebra. Let $\mathfrak{A}$ be a maximal Hilbert algebra in a Hibert space $\mathfrak{H}$. $\mathfrak{A}$ is naturally closed by theorem 2.3.

THEOREM 6. 1. *If a projection operator P is commutative with T_a and S_a for all $a \varepsilon \mathfrak{A}$, then P is associative with $\mathfrak{A}$, $P\mathfrak{A}$ is an ideal and $P\mathfrak{A}$ is also maximal as a subalgebra.*

Proof. First we will prove that $P\mathfrak{A}\subset\mathfrak{A}$. If a, b, Pa, $Pb\in\mathfrak{A}$, then, since P is commutative with T_a and S_b by assumption, we see easily that

$$(Pa)\,b = a\,(Pb) = Pab, \qquad ((1-P)\,a)\,b = a\,((1-P)\,b) = (1-P)\,ab.$$

Therefore, denoting by $\tilde{\mathfrak{A}}$ the set of $Pa+(1-P)\,b$ for all a, $b\in\mathfrak{A}$, we obtain a Hilbert algebra, if we define

$$(Pa_1+(1-P)\,b_1)\,(Pa_2+(1-P)\,b_2) = Pa_1\,a_2+(1-P)\,b_1\,b_2,$$
$$(Pa+(1-P)\,b)^* = Pa^*+(1-P)\,b^*,$$

and $\tilde{\mathfrak{A}}$ contains obviously $\mathfrak{A}$ as a subalgebra. Since $\mathfrak{A}$ is maximal by assumption, we have $\tilde{\mathfrak{A}}=\mathfrak{A}$, and hence $P\mathfrak{A}\subset\mathfrak{A}$. Thus P is associative with $\mathfrak{A}$, and consequently $P\mathfrak{A}$ is an ideal by theorem 5.1.

For any extension $\hat{\mathfrak{A}}$ of $P\mathfrak{A}$ in the Hilbert space $P\mathfrak{H}$, we see also similarly that $\hat{\mathfrak{A}}+(1-P)\,\mathfrak{A}$ is a Hilbert algebra, if we define,

$$(x+a)\,(y+b) = xy+ab, \qquad (x+a)^* = x^*+a^*$$

for x, $y\in\tilde{\mathfrak{A}}$ and a, $b\in(1-P)\,\mathfrak{A}$, and $\hat{\mathfrak{A}}+(1-P)\,\mathfrak{A}$ contains $\mathfrak{A}$ as a subalgebra. Since $\mathfrak{A}$ is maximal by assumption, we have hence $\hat{\mathfrak{A}}=P\mathfrak{A}$, that is, $P\mathfrak{A}$ is maximal as a subalgebra.

Theorem 6.2. *For any ideal $\mathfrak{p}$ of $\mathfrak{A}$ there exists a projection operator P such that P is commutative with T_a and S_a for all $a\in\mathfrak{A}$ and $P\mathfrak{A}=\mathfrak{p}$.*

Proof. Let P be the projection operator of the closed linear manifold spanned by $\mathfrak{p}$. Since $\mathfrak{p}$ is an ideal by assumption, for any $a\in\mathfrak{A}$ we have

$$T_a\,Px = ax = PT_a\,Px \qquad \text{for all } x\in\mathfrak{p}.$$

As $\mathfrak{p}$ is dense in $P\mathfrak{H}$, we have hence $T_aP=PT_aP$ for all $a\in\mathfrak{A}$, and furthermore

$$PT_a = (T_{a^*}P)^* = (PT_{a^*}P)^* = PT_a\,P.$$

Thus P is commutative with T_a for all $a\in\mathfrak{A}$. Similarly we can prove that P is commutative with S_a for all $a\in\mathfrak{A}$. Therefore we have $P\mathfrak{A}\subset\mathfrak{A}$ by the previous theorem. Since $\mathfrak{p}$ is dense in $P\mathfrak{A}$, we have hence $\mathfrak{p}=P\mathfrak{A}$ by definition of ideals.

Theorem 6.3. *If two projection operators P and Q are both commutative with T_a and S_a for all $a\in\mathfrak{A}$, then we have $PQ=QP$.*

Proor. By theorem 6.1 $P\mathfrak{A}$ and $Q\mathfrak{A}$ are both ideals of $\mathfrak{A}$. First we assume that $P\mathfrak{A}$ and $Q\mathfrak{A}$ have no common element except 0. Then we have obviously

$$(P\mathfrak{A})\ (Q\mathfrak{A}) = \{0\}.$$

Therefore for any $a,\ b \in P\mathfrak{A}$ and $x \in Q\mathfrak{A}$ we have

$$(ab,\ x) = (b,\ a^*x) = 0.$$

Since $(P\mathfrak{A})\ (P\mathfrak{A})$ is complete in $P\mathfrak{H}$ by theorem 1.6, we obtain hence $PQ\mathfrak{A} = \{0\}$. As $\mathfrak{A}$ is dense in $\mathfrak{H}$, we have thus $PQ = 0$, and consequently $QP = 0$.

In general, the intersection $\mathfrak{p}$ of $P\mathfrak{A}$ and $Q\mathfrak{A}$ is obviously also an ideal of $\mathfrak{A}$. Therefore there exists a projection operator R such that $\mathfrak{p} = \mathrm{R}\mathfrak{A}$ and R is commutative with T_a and S_a for all $a \in \mathfrak{A}$. Then, since $P\mathfrak{A} \supset \mathrm{R}\mathfrak{A}$ and $\mathfrak{A}$ is dense in $\mathfrak{H}$, we have $P \geqq \mathrm{R}$. Similary we can prove that $Q \geqq \mathrm{R}$. Furtheremore $(P-\mathrm{R})\mathfrak{A}$ and $(Q-\mathrm{R})\mathfrak{A}$ have no common element except 0. Thus we have

$$(P-\mathrm{R})\ (Q-\mathrm{R}) = (Q-\mathrm{R})\ (P-\mathrm{R}) = 0,$$

and consequently $PQ = QP$.

Let $\mathfrak{P}$ be the set of all projection operators, which are commutative with T_a and S_a for all $a \in \mathfrak{A}$. By the theorems proved above, we see that $\mathfrak{P}$ is a Boolean algebra of projetion operators and $\mathfrak{P} \ni P_\lambda\ (\lambda \in \Lambda)$ implies $\mathfrak{P} \ni \bigcup_\lambda P_\lambda$, $\bigcap_\lambda P_\lambda$.[6]

For any atomic element $P \in \mathfrak{P}$, $P\mathfrak{A}$ is a simple ideal by theorem 6.2. Let $P_\lambda\ (\lambda \in \Lambda)$ be the system of all atomic elements of $\mathfrak{P}$. Then we have obviously $P_\lambda P_\rho = 0$ for $\lambda \neq \rho$. Putting $Q = 1 - \bigcup_\lambda P_\lambda$, we obtain $Q \in \mathfrak{P}$, and $Q\mathfrak{P}$ has no atomic element, and hence $Q\mathfrak{A}$ contains no simple ideal by theorem 6.2. Therefore we have:

THMORMM 6.4. *For a maximal Hilbert algebra $\mathfrak{A}$ there exists a system of projection operators $P_\lambda\ (\lambda \in \Lambda)$ such that $P_\lambda P_\rho = 0$ for $\lambda \neq \rho$, P_λ is commutative with T_a and S_a for all $a \in \mathfrak{A}$. $P_\lambda \mathfrak{A}$ is a simple ideal, and $(1 - \bigcup_\lambda P_\lambda)\mathfrak{A}$ contains no simple* [illegible]

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6) H. Nakano: Funktionen …. Satz. 3.