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SELF-INJECTIVE RINGS

TOYONORI KATO

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Throughout this paper each ring R will be a ring with identity element and each module will be unital. We denote the category of right (resp. left) R-modules by $\mathfrak{M}_R$ (resp. ${}_R\mathfrak{M}$).

If A is a module over a ring R, we denote the dual of A by A^*. We may form the double dual A^{**} of A and get the natural R-homomorphism

$$\delta_A : A \longrightarrow A^{**}.$$

Following H.Bass [2, p. 476], we call A torsionless if δ_A is a monomorphism and reflexive if δ_A is an isomorphism. Let A be a right R-module. If X is a subset of A, then we set

$$l(X) = \{b \in A^* \mid bX = 0\}.$$

Similarly, if X is a subset of A^*, then we set

$$r(X) = \{a \in A \mid Xa = 0\}.$$

Note that, if X is a subset of a ring R, $l(X)$ (resp. $r(X)$) is just the left (resp. right) annihilator of X in R since $(R_R)^* = {}_RR$. We call a ring R right PF if every faithful right R-module is a generator in $\mathfrak{M}_R$. A ring R is PF if R is both right and left PF. A ring which is both right and left self-injective is called self-injective.

In the present paper we shall show the following theorems:

THEOREM 7. *The following conditions on a ring R are equivalent*:
(1) *R is right* PF.
(2) *R is right self-injective and $l(I) \neq 0$ for each maximal right ideal I.*
(3) *R is an injective cogenerator in $\mathfrak{M}_R$.*

THEOREM 8. *The following conditions on a ring R are equivalent*:
(1) *Every submodule and every quotient module of a reflexive right R-module is reflexive and every submodule and every quotient module of a reflexive left R-module is also reflexive.*

(2) *Every finitely generated right, and every finitely generated left, R-module is reflexive.*

(3) *Every cyclic right, and every cyclic left, R-module is reflexive.*

(4) *R is* PF.

(5) *R is self-injective and $l(I)\neq 0$ for each maximal right ideal I.*

(6) *R is an injective cogenerator both in $\mathfrak{M}_R$ and in ${}_R\mathfrak{M}$.*

THEOREM 9. *Let R be a ring characterized in Theorem 8, and A a right R-module. Then*

(1) *$r(l(A_0))=A_0$ for every submodule A_0 of A.*

(2) *If A is finitely generated (or reflexive), then $l(r(B_0))=B_0$ for every submodule B_0 of A^*.*

THEOREM 10. *The following conditions on a ring R are equivalent:*

(1) *R is* PF *and the Jacobson radical of R is nilpotent.*

(2) *R is* QF.

1. Cogenerators. In this section we study rings R for which R is an injective cogenerator in $\mathfrak{M}_R$ and deduce the interesting characterizations of such rings. The following proposition paves the way for the definition of cogenrators in $\mathfrak{M}_R$.

PROPOSITION 1. *The following conditions on right R-modules A and C are equivalent:*

(1) *There is a monomorphism from A into a direct product of copies of C.*

(2) *For any nonzero map $B\to A$ ($B\in\mathfrak{M}_R$), there is a map $A\to C$ such that the composition map $B\to A\to C$ is nonzero.*

(3) *For any nonzero element a of A, there is a map f: $A\to C$ such that $f(a)\neq 0$.*

This result is clear and we omit the proof.

A right R-module C is called a cogenerator in $\mathfrak{M}_R$ if it satisfies the conditions in Proposition 1 for each right R-module A. Note that a ring R is a cogenerator in $\mathfrak{M}_R$ if and only if every right R-module is torsionless.

If A is a right R-module then $E(A)$ will denote the injective hull of A, and let $B\,'\!\supset A$ signify that a right R-module B is an essential extension of A.

The following proposition is a useful characterization of being a cogenerator in $\mathfrak{M}_R$.

PROPOSITION 2. *A right R-module C is a cogenerator in $\mathfrak{M}_R$ if and only if C contains a copy of the injective hull $E(U)$ of each simple right R-module U.*

PROOF. Let C be a cogenerator in $\mathfrak{M}_R$, and U a simple right R-module. By Proposition 1 we have a map $g: E(U) \to C$ such that $U \to E(U) \to C$ is nonzero where $U \to E(U)$ is inclusion. The map g is a monomorphism since $\mathrm{Ker}(g) \cap U = 0$ and $E(U)' \supset U$. Conversely let C contain a copy of $E(U)$ for each simple right R-module U. Let A be any right R-module, and $a \neq 0 \in A$. Then there is a map $f: aR \to E(U)$ such that $f(a) \neq 0$ for some simple right R-module U. The map f can be extended to a map f' of A into $E(U)$ since $E(U)$ is injective. The map f' is considered as a map of A into C by assumption and $f'(a) \neq 0$, which completes the proof.

The following theorem will be of some interest.

THEOREM 1. *The following conditions on a right self-injective ring R are equivalent*:

(1) $r(l(I)) = I$ *for each right ideal* I.

(2) $l(I) \neq 0$ *for each maximal right ideal* I.

(3) R *is a cogenerator in* $\mathfrak{M}_R$.

PROOF. The implication (1)⇒(2) is trivial.

(2) implies (3). Let U be a simple right R-module. Then $U \approx R/I$ for some maximal right ideal I. Then we have

$$U^* \approx (R/I)^* \approx l(I) \neq 0 .$$

Hence R contains a copy of U. Thus R is a cogenerator in $\mathfrak{M}_R$ since R is right self-injective.

(3) implies (1). Since R is a cogenerator in $\mathfrak{M}_R$, every cyclic right R-module is torsionless. Thus $r(l(I)) = I$ for each right ideal I by [6, Theorem 1].

We denote by J the Jacobson radical of a ring R.

On the rings characterized in the preceding theorem, B.L.Osofsky [8] has proved the following surprising result.

THEOREM 2. *Let R be an injective cogenerator in* $\mathfrak{M}_R$. *Then* R/J *is Artinian.*

REMARK. It is easy to see that, if R is an injective cogenerator in $\mathfrak{M}_R$ with zero Jacobson radical, then R is Artinian. In fact, R does not contain any proper essential right ideal since $r(l(I)) = I$ for each right ideal I and $J = \{x \in R \mid R' \supset r(x)\} = 0$ (see [3, Theorem 3.1] or [9, Lemma 4.1]).

Note that, if R is right self-injective, R/J is Artinian if and only if every simple right R-module is isomorphic to eR/eJ for some idempotent $e \in R$. The 'only if' part is evident since idempotents can be lifted modulo J (see [7, Proposition 4] or [9, Corollary 3.2]). The 'if' part follows since R/J

is an injective cogenerator in $\mathfrak{M}_{R/J}$ (see [9, Theorem 4.8]).

Recall the definition of generators in $\mathfrak{M}_R$. A right R-module M is a generator in $\mathfrak{M}_R$ if each right R-module is an epimorph of a direct sum of copies of M. Note that M is a generator in $\mathfrak{M}_R$ if and only if R is a direct summand of a finite direct sum of copies of M.

PROPOSITION 3. *A projective right R-module M is a generator in* $\mathfrak{M}_R$ *if and only if each simple right R-module is an epimorph of M.*

Proof. *Let M* be a generator in $\mathfrak{M}_R$, U a simple right R-module. Then we can find easily a nonzero map $M \to U$, which is necessarily an epimorphism since U is simple. Conversely, let us suppose that each simple right R-module is an epimorph of M. Let I be a maximal right ideal of R. Then there exists a map $f: M \to R$ such that the composition map $M \to R \to R/I$ is nonzero, or equivalently, $f(M) \not\subset I$ since M is projective. Hence R is a sum of those right ideals which are epimorph of M, which completes the proof.

Now, let $\{U_i\}$ be the family of all non-isomorphic simple right R-modules, and C the direct sum of the family $\{E(U_i)\}$.

THEOREM 3. *The following conditions on a ring R are equivalent*:
(1) *R is an injective cogenerator in* $\mathfrak{M}_R$.
(2) C is a generator in $\mathfrak{M}_R$.

Proof. (1) implies (2). $E(U_i)$ is projective, or equivalently, C is projective since R is a cogenerator in $\mathfrak{M}_R$. Each simple right R-module is isomorphic to $E(U_i)/E(U_i)J$ for some simple right R-module U_i since R/J is Artinian (see [4, p. 214]). Thus each simple right R-module is an epimorph of C and hence C is a generator in $\mathfrak{M}_R$ by Proposition 3.

(2) implies (1). Since C is a generator in $\mathfrak{M}_R$, R is right self-injective and a finite direct sum of indecomposable right ideals each of which contains a minimal right ideal of R (see [1, p. 702]). Then R/J is Artinian. Hence R contains a copy of each simple right R-module, thus completing the proof.

2. Simple modules. In this section we shall consider simple modules over rings characterized in Theorem 1 and obtain a precise information on the structure of these rings.

Let R be a right self-injective ring, L a finitely generated left ideal of R. Then $l(r(L))=L$ (see [5] or [6, Theorem 13]). We shall use this fact to show the following.

LEMMA 1. *Let R be a right self-injective ring satisfying the conditions in Theorem* 1. *Let I be a maximal right ideal of R. Then l(I) is a minimal left ideal of R.*

PROOF. Let $a \neq 0 \in l(I)$; then we have

$$I \subset r(l(I)) \subset r(a) \neq R.$$

Then $I = r(a)$ since I is maximal. Hence we have

$$l(I) = l(r(a)) = Ra,$$

which completes the proof.

THEOREM 4. *Let R be a right self-injective ring satisfying the conditions in Theorem* 1. *Then the mapping*

$$U \longrightarrow U^*$$

gives a one-to-one correspondence between isomorphism classes of simple right R-modules and isomorphism classes of simple left R-modules.

PROOF. Let U be a simple right R-module. Then $U \approx R/I$ for some maximal right ideal I. $U^* \approx l(I)$ is simple by Lemma 1. Now, we may assume that U is a minimal right ideal of R since R is a cogenerator in $\mathfrak{M}_R$. Then U is reflexive since R is right self-injective and $r(l(U)) = U$ (see [6, Proposition 5]). This implies that our correspondence is one-to-one. Finally, our correspondence is 'onto' since R/J is Artinian.

Let R be a right self-injective ring characterized in Theorem 1. In the proof of Theorem 4 we have just seen that each simple right R-module is reflexive. Hence each simple left R-module is also reflexive by Theorem 4.

Y. Utumi [10, Theorem 1.3] has proved the following interesting result on the Jacobson radical J of a right self-injective ring.

THEOREM 5. *Let R be a right self-injective ring, and I a right (or left) ideal of R. Then $I \subset J$ if and only if I does not contain any nonzero idempotent.*

THEOREM 6. *Let R be a right self-injective ring characterized in Theorem* 1.

(1) *$r(L)$ is a minimal right ideal for each maximal left ideal L.*

(2) *$er(J)$ (resp. $l(J)e$) is a minimal right (resp. left) ideal for every primitive idempotent e, and each simple right (resp. left) R-module is isomorphic to $er(J)$ (resp. $l(J)e$) for some primitive idempotent e.*

(3) *$l(J) = r(J)$ and $l(r(J)) = J$.*

(4) *Every nonzero right (resp. left) ideal contains a minimal right (resp. left) ideal.*

PROOF. (1) Let L be a maximal left ideal of R. Then $r(L)\approx(R/L)^*$ is simple by Theorem 4.

(2) Let e be a primitive idempotent. Then $(1-e)R+eJ$ (resp. $R(1-e)+Je$) is a maximal right ideal (resp. left ideal). Hence $er(J)=r(R(1-e)+Je)$ (resp. $l(J)e=l((1-e)R+eJ)$) is minimal by (1) (resp. by Lemma 1). Note that each simple right (resp. left) R-module is isomorphic to eR/eJ (resp. Re/Je) for some primitive idempotent e. Since $(eR/eJ)^*\approx l(J)e$, $(Re/Je)^*\approx er(J)$, each simple right (resp. left) R-module is isomorphic to $er(J)$ (resp. $l(J)e$) by Theorem 4.

(3) $er(J)\subset l(J)$ (resp. $l(J)e\subset r(J)$) for every primitive idempotent e by (2). Thus $l(J)=r(J)$, since the identity element 1 is a sum of primitive idempotents by the proof of Theorem 3. By (1), we have

$$er(J)=r(R(1-e)+Je)\neq 0$$

for each nonzero idempotent e. Thus $l(r(J))$ does not contain any nonzero idempotent and $l(r(J))=J$ by Theorem 5.

(4) Let L be a nonzero left ideal. Take a finitely generated left subideal $L'\neq 0$. $r(L')\subset I$ for some maximal right ideal I. Then

$$L\supset L'=l(r(L'))\supset l(I)\,.$$

Thus L contains a minimal left ideal $l(I)$. Similarly, each nonzero right ideal contains a minimal right ideal (or by the proof of Theorem 3).

3. PF-rings. *A* ring R is called right PF if every faithful right R-module is a generator in $\mathfrak{M}_R$. In this section we give characterizations of right PF-rings.

Let C be defined as in Theorem 3.

THEOREM 7. *The following conditions on a ring R are equivalent*:

(1) *R is right* PF.

(2) *R is right self-injective and $l(I)\neq 0$ for each maximal right ideal I.*

(3) *R is an injective cogenerator in $\mathfrak{M}_R$.*

(4) *R/J is Artinian and $E(U)$ is projective for each simple right R-module U.*

PROOF. (2)$\Leftrightarrow$(3) by Theorem 1.

(3) implies (4). R/J is Artinian by Theorem 2. $E(U)$ is projective for each simple right R-module U since R is a cogenerator in $\mathfrak{M}_R$.

(4) implies (3). Let U be a simple right R-module. Since $E(U)$ is indecomposable injective and projective, R contains a copy of $E(U)$ by [4, Corollary 2.5]. Hence R is a cogenerator in $\mathfrak{M}_R$. R is right self-injective since R is a

cogenerator in $\mathfrak{M}_R$ and R/J is Artinian by [4, Corollary 4.2].

(1) implies (3). C is faithful since C is a cogenerator in $\mathfrak{M}_R$. By assumption, C is a generator in $\mathfrak{M}_R$, Hence R is an injective cogenerator in $\mathfrak{M}_R$ by Theorem 3.

(3) implies (1). Let M be a faithful right R-module. Then M is a cogenerator in $\mathfrak{M}_R$ by the next lemma. Hence M contains a copy of C, which is injective since R/J is Artinian. Hence we have an epimorphism

$$M \longrightarrow C \longrightarrow 0 .$$

Thus M is a generator in $\mathfrak{M}_R$, since C is a generator in $\mathfrak{M}_R$ by Theorem 3.

LEMMA 2. *Let R be a cogenerator in $\mathfrak{M}_R$. Then the following conditions on a right R-module M are equivalent*:

(1) *M is faithful.*

(2) *M is a cogenerator in $\mathfrak{M}_R$.*

PROOF. (2) trivially implies (1).

(1) implies (2). Let U be a simple right R-module. Note that every nonzero submodule of $E(U)$ contains U since $E(U)\,'\supset U$. Now, we may assume that $E(U)\subset R$ since R is a cogenerator in $\mathfrak{M}_R$. Then $MU\neq 0$ since M is faithful. Take $m\in M$ such that $mU\neq 0$. Then the map

$$E(U) \longrightarrow mE(U)\subset M$$

must be a monomorphism since $mU\neq 0$. Thus M contains a copy of $E(U)$, thus completing the proof.

It may be interesting to compare our Theorem 7 with the following theorem, which was proved in G.Azumaya [1] or in Y.Utumi [10].

THEOREM. *A ring R is right* PF *if and only if R is right self-injective and a finite direct sum of indecomposable right ideals each of which contains a minimal right ideal of R.*

It is to be noted that our characterizations of (right) PF-rings are far-reaching (see the next section). Note also that if R is a right PF-ring then $r(l(I))=I$ for each right ideal I of R.

4. Duality. Let us consider a problem to find out possible types of rings for which every finitely generated module is reflexive. In the following, we shall give a complete solution for this problem.

THEOREM 8. *The following conditions on a ring R are equivalent*:

(1) *Every submodule and every quotient module of a reflexive right R-module is reflexive and every submodule and every quotient module of a reflexive left R-module is reflexive.*

(2) *Every finitely generated right, and every finitely generated left, R-module is reflexive.*

(3) *Every cyclic right, and every cyclic left, R-module is reflexive.*

(4) *R is* PF.

(5) *R is self-injective and $l(I)\neq 0$ for each maximal right ideal I.*

(6) *R is an injective cogenerator both in $\mathfrak{M}_R$ and in ${}_R\mathfrak{M}$.*

PROOF. The implications (1)⇒(2)⇒(3) are trivial.

(3) implies (4). R is self-injective, each right ideal is a right annulet and each left ideal is a left annulet, by [6, Theorem 12]. Hence R is PF by Theorem 7.

(4) implies (5). This implication is trivial by Theorem 7.

(5) implies (6). Note $r(L)\neq 0$ for each maximal left ideal L by Theorem 6. Hence R is an injective cogenerator both in $\mathfrak{M}_R$ and in ${}_R\mathfrak{M}$.

(6) implies (1). Let

$$0 \to A \to B \to B/A \to 0$$

be an exact sequence of right (or left) R-modules with B reflexive. Then we have the commutative exact diagram

$$\begin{array}{ccccccccc} 0 & \to & A^{**} & \to & B^{**} & \to & (B/A)^{**} & \to & 0 \\ & & \delta_A \uparrow & & \wr & & \uparrow \delta_{B/A} & & \\ 0 & \to & A & \to & B & \to & B/A & \to & 0 \end{array}$$

since R is self-injective.
Now, $\delta_{B/A}$ is a monomorphism since R is a cogenerator both in $\mathfrak{M}_R$ and in ${}_R\mathfrak{M}$. Hence $\delta_{B/A}$ is an isomorphism. This, in turn, forces δ_A to be an isomorphism. This completes the proof of the theorem.

Let R be a ring, and A a right R-module. If X is a subset of A (resp. A^*) we denote its annihilator in A^* (resp. A) by $l(X)$ (resp. $r(X)$). Note that, if A_0 is a submodule of A, $r(l(A_0))=A_0$ if and only if A/A_0 is torsionless.

Now, recall the definition of $\delta_A : A \to A^{**}$. δ_A is defined by the rule

$$b\delta_A(a) = ba$$

for all $b \in A^*$, $a \in A$.

THEOREM 9. *Let R be a ring characterized in Theorem* 8, *and A a* *right R-module. Then*

(1) $r(l(A_0))=A_0$ *for every submodule* A_0 *of* A.

(2) *If* A *is finitely* generated (or reflexive), then $l(r(B_0))=B_0$ *for every submodule* B_0 *of* A^*.

PROOF. R is a cogenerator both in $\mathfrak{M}_R$ and in ${}_R\mathfrak{M}$. We use this fact to show (1) and (2).

(1) A/A_0 is torsionless since R is a cogenerator in $\mathfrak{M}_R$. Thus $r(l(A_0))=A_0$.

(2) The inclusion $l(r(B_0))\supset B_0$ is obvious. Since A is finitely generated, A is reflexive by Theorem 8. Hence each element of A^{**} is of the form $\delta_A(a)$ for some $a\in A$. Moreover, A^*/B_0 is torsionless since R is a cogenerator in ${}_R\mathfrak{M}$. Now, let $b\in A^*$, $b\notin B_0$. Then there exists an element $\delta_A(a)$ of A^{**} such that

$$B_0\delta_A(a)=0,\ \ b\delta_A(a)\neq 0$$

since A^*/B_0 is torsionless (see Proposition 1, (3)). Hence $B_0a=0$, $ba\neq 0$, or equivalently, $a\in r(B_0)$, $ba\neq 0$. Hence $b\notin l(r(B_0))$, which completes the proof of the theorem.

Theorem 9 is a generalization of a theorem of M.Hall, who considered the case: R is a QF-ring and A is a finite free R-module.

In connection with the preceding theorem, we have the following

LEMMA 3. *Let* R *be a ring characterized in Theorem* 8, *and* A *a infinite direct sum of right R-modules* A_i. *Then* $A^*=\Pi A_i^*\supset\sum A_i^*$, $l(r(\sum A_i^*))\neq\sum A_i^*$ *and* A *is not reflexive.*

PROOF. $r\left(\sum A_i^*\right)=0$ since each A_i is torsionless. Hence

$$l\left(r\left(\sum A_i^*\right)\right)=A^*=\Pi A_i^*\neq\sum A_i^*.$$

Therefore A is not reflexive by Theorem 9.

We can now prove the following

THEOREM 10. *The following conditions on a ring* R *are equivalent*:

(1) R *is* PF *and the Jacobson radical* J *of* R *is nilpotent.*

(2) R *is* QF.

PROOF. (1) implies (2). $(J^i/J^{i+1})_R$ is reflexive by Theorem 8. Since R/J

is Artinian, J^i/J^{i+1} is a direct sum of simple submodules. By Lemma 3, J^i/J^{i+1} must be a finite direct sum of simple submodules and hence J^i/J^{i+1} has a composition series for each i. Thus R_R has a composition series since J is nilpotent.

(2) trivially implies (1) by Theorem 8.

On duality over right PF-rings, we have the following

PROPOSITION 4. *The following conditions on a ring R are equivalent*:

(1) *If B is a left R-module such that δ_B is an epimorphism, and if A is any quotient module of B, then δ_A is an epimorphism. Moreover, every right R-module is torsionless.*

(2) *$\delta_{R/L}$ is an epimorphism for every left ideal L of R, and R/I is torsionless for every right ideal I of R.*

(3) *R is an injective cogenerator in $\mathfrak{M}_R$.*

(4) *R is right* PF.

PROOF. (1) trivially implies (2).

(2) implies (3). Let I be a right ideal of R. Since R/I is torsionless, $I=r(L)$ for some left ideal L of R by [6, Theorem 1]. By [6, Proposition 7], $\mathrm{Ext}^1_R(R/I, R)=0$ since $\delta_{R/L}$ is an epimorphism. Thus R is right self-injective. Moreover $r(l(I))=I$ for each right ideal I since R/I is torsionless. Hence R is an injective cogenerator in $\mathfrak{M}_R$ by Theorem 1.

(3)$\Leftrightarrow$(4) by Theorem 7.

(3) implies (1). Every right R-module is torsionless since R is a cogenerator in $\mathfrak{M}_R$. Next, let A be a quotient module of a left R-module B for which δ_B is an epimorphism. Then we have an exact sequence of right R-modules $0 \to A^* \to B^*$. From this we get the following commutative exact diagram

$$\begin{array}{ccccc} 0 & & & & \\ \uparrow & & & & \\ B^{**} & \longrightarrow & A^{**} & \longrightarrow & 0 \\ \delta_B \uparrow & & \uparrow \delta_A & & \\ B & \longrightarrow & A & \longrightarrow & 0 \end{array}$$

since R is right self-injective. Thus δ_A is an epimorphism.

5. Some open problems. It is an open question whether a right PF-ring is left PF or not. If a right PF-ring is necessarily left self-injective, then the above question is answered affirmatively from what we have seen.

We have seen that, in a right PF-ring, every right ideal is a right annulet. Is every left ideal a left annulet in a right PF-ring? In this connection, we have seen that finitely generated left ideals, maximal left ideals, and J are left annulets in a right PF-ring.

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MATHEMATICAL INSTITUTE
TÔHOKU UNIVERSITY
SENDAI, JAPAN.