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ON A CONFORMALLY FLAT RIEMANNIAN SPACE WITH POSITIVE RICCI CURVATURE

MARIKO TANI

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1. Introduction. A Riemannian space is called conformally flat when its Weyl conformal curvature tensor vanishes. It is well known that the p-th $(0 < p < n)$ Betti numbers of a compact orientable conformally flat Riemannian space with positive Ricci curvature are all zero.[1] In this paper we shall prove the following theorems in the case when its scalar curvature is constant.

THEOREM A. *Let M be a compact orientable conformally flat Riemannian space with constant scalar curvature and with positive Ricci curvature, then M is of constant curvature.*

In the proof of this theorem, we can find that the same conclusion is obtained by replacing the condition about the Ricci curvature by that of positive sectional curvature. Hence we have:

THEOREM B. *Let M be a compact orientable conformally flat Riemannian space with constant scalar curvature and with positive sectional curvature, then M is of constant curvature.*

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2. Definitions and notations. Let M be an n-dimensional Riemannian space and p be a point in M. $T_p(M)$ and $\{x^a\}$ denote the tangent space to M at p and a local coordinate system around p, respectively. As usual, g_{ab}, $R^a{}_{bcd}$ and $R_{bc} = R^a{}_{bca}$ denote the Riemannian metric, the curvature tensor and the Ricci tensor, respectively. Take an orthonormal basis $X_1, \cdots, X_n$ at p, then we have

$$g_{ab}\, X_i{}^a\, X_j{}^b = \delta_{ij}\,,$$

1) K. Yano and S. Bochner, [1], pp. 79–80.

where $X_i = X_i^a \partial/\partial x^a$. The sectional curvature of 2-plane spanned by X_i and X_j is given by

$$\rho(X_i, X_j) = -R_{abcd} X_i^a X_j^b X_i^c X_j^d ,$$

where $R_{abcd} = g_{ae} R^e{}_{bcd}$ and we have

$$\sum_j \rho(X_i, X_j) = R_{ab} X_i^a X_i^b .$$

We call $R_{ab} X_i^a X_i^b$ the Ricci curvature with respect to X_i. The scalar curvature R is given by

$$R = \sum_{i,j} \rho(X_i, X_j) .$$

The Weyl conformal curvature tensor is defined by

$$C^a{}_{bcd} = R^a{}_{bcd} - \frac{1}{n-2}(R_{bc}\delta^a_d - R_{bd}\delta^a_c + g_{bc}R^a{}_d - g_{bd}R^a{}_c)$$
$$+ \frac{R}{(n-1)(n-2)}(g_{bc}\delta^a_d - g_{bd}\delta^a_c) .$$

A Riemannian space is called conformally flat if $C^a{}_{bcd} = 0$ for $n > 3$ and if $C_{bcd} = 0$ for $n = 3$, where the tensor C_{bcd} is defined by

$$C_{bcd} = \frac{1}{n-2}(\nabla_d R_{bc} - \nabla_c R_{bd}) - \frac{1}{2(n-1)(n-2)}(g_{bc}\nabla_d R - g_{bd}\nabla_c R),$$

$\nabla_d R_{bc}$ and $\nabla_d R$ being the covariant derivatives of R_{bc} and R.

Generally it is known that the following relation holds between the two tensors $C^a{}_{bcd}$ and C_{bcd}:

$$\nabla_a C^a{}_{bcd} = (n-3)\, C_{bcd} .$$

Therefore, if the space is conformally flat ($n \geqq 3$), then the relation $C_{bcd} = 0$ is always true.

In our case, R being constant, we have

$$\nabla_d R_{bc} - \nabla_c R_{bd} = 0 . \tag{2.1}$$

Now we consider about Lichnérowicz formula.[2] Denoting the volume

2) A. Lichnérowicz, [2], p. 10.

element of M by dV, we have the following integral formula:

$$\int_M \{(\nabla^d R^{bc} - \nabla^c R^{bd})(\nabla_d R_{bc} - \nabla_c R_{bd}) - K(p)\}\, dV$$

$$= (1/2) \int_M (\nabla^a R^{bcde} \nabla_a R_{bcde})\, dV\,,$$

where K is a real valued function defined on M as follows:

$$K(p) = -R_{ac}{}^{bc} R^{ad}{}_{ef} R_{bd}{}^{ef} + (1/2) R_{ab}{}^{cd} R_{cd}{}^{ef} R_{ef}{}^{ab} + 2R_{ac}{}^{bd} R^{ae}{}_{bf} R^{c}{}_{ed}{}^{f}\,.$$

By (2.1) we have

$$\int_M K(p)\, dV \leqq 0\,. \tag{2.2}$$

On the other hand, as shown later, we obtain $K(p) \geqq 0$ under our assumption, and arrive at the conclusion of Theorem A.

3. Calculation of $K(p)$. In a conformally flat space, the curvature tensor is given by

$$R^a{}_{bcd} = \frac{1}{n-2}(R_{bc}\delta^a_d - R_{bd}\delta^a_c + g_{bc}R^a{}_d - g_{bd}R^a{}_c)$$

$$- \frac{R}{(n-1)(n-2)}(g_{bc}\delta^a_d - g_{bd}\delta^a_c)\,.$$

With respect to an orthonormal basis $X_1, \cdots, X_n$, we have

$$\begin{aligned} &R_{ijkh} = 0, \qquad \text{if } i, j, k \text{ and } h \text{ are all different,} \\ &R_{ijih} = -\frac{1}{n-2} R_{jh}\,, \qquad \text{if } j \neq h\,, \\ &R_{ijij} = -\frac{1}{n-2}\Big(R_{ii} + R_{jj} - \frac{R}{n-1}\Big), \qquad \text{if } i \neq j\,. \end{aligned} \tag{3.1}$$

Now we choose an orthonormal basis such that $R_{jh} = 0$, $(j \neq h)$, then non-vanishing components of the curvature tensor are only of the type R_{ijij}.

If we set

$$A = -\sum_{\substack{i,j,k,h\\ l,m}} R_{ikjk} R_{ihlm} R_{jhlm}\,, \qquad B = (1/2) \sum_{\substack{i,j,k\\ h,l,m}} R_{ijkh} R_{khlm} R_{lmij}\,,$$

$$C=2\sum_{\substack{i,j,k,h\\l,m}} R_{ikjh}R_{iljm}R_{klhm},$$

in accordance with our choice of the basis, we have

$$A=-2\sum_{i,k,h} R_{ikik}R_{ihih}{}^2, \qquad B=2\sum_{i,h} R_{ihih}{}^3,$$

$$C=2\sum_{i,k,h} R_{ikik}R_{ihih}R_{khkh}.$$

As a result, it remains to calculate the following formula:

$$(1/2)\,K(p)=-\sum_{i,k,h} R_{ikik}R_{ihih}{}^2+\sum_{i,k,h} R_{ikik}R_{ihih}R_{khkh}.$$

Substitution of (3. 1) into this formula gives

$$\frac{(n-2)^3}{2}K(p)=\sum_{i,k,h\neq}\left\{\left(R_{ii}+R_{kk}-\frac{R}{n-1}\right)\left(R_{ii}+R_{hh}-\frac{R}{n-1}\right)^2\right.$$

$$\left.-\left(R_{ii}+R_{kk}-\frac{R}{n-1}\right)\left(R_{ii}+R_{kk}-\frac{R}{n-1}\right)\left(R_{kk}+R_{hh}-\frac{R}{n-1}\right)\right\}$$

$$=\sum_{i,k,h\neq}\left\{(R_{ii}{}^3+R_{ii}{}^2R_{hh})+\frac{R}{n-1}(-R_{ii}{}^2+R_{ii}R_{hh})\right\}$$

$$=(n-1)(n-2)\sum_i R_{ii}{}^3-(n-2)\sum_i R_{ii}{}^2(R-R_{ii})$$

$$+\frac{R}{n-1}\left\{-(n-1)(n-2)\sum_i R_{ii}{}^2+(n-2)\sum_i R_{ii}(R-R_{ii})\right\},$$

$$\frac{(n-2)^2}{2}K(p)=(n-1)\sum_i R_{ii}{}^3-2\sum_i R_{ii}{}^2(R-R_{ii})+\frac{R^3}{n-1}-\frac{R}{n-1}\sum_i R_{ii}{}^2.$$

Since R^3 and $R\sum_i R_{ii}{}^2$ can be expressed in the forms

$$R^3=\sum_i R_{ii}{}^3+3\sum_i R_{ii}{}^2(R-R_{ii})+6\sum_{i<j<k} R_{ii}R_{jj}R_{kk},$$

$$R\sum_i R_{ii}{}^2=\sum_i R_{ii}{}^3+\sum_i R_{ii}{}^2(R-R_{ii}),$$

we have

$$(3.2)\qquad \begin{aligned} \frac{(n-1)(n-2)^2}{4} K(p) &= \frac{(n-1)(n-2)}{2} \sum_i R_{ii}{}^3 - (n-2) \sum_i R_{ii}{}^2(R-R_{ii}) \\ &\quad + 3 \sum_{i<j<k} R_{ii} R_{jj} R_{kk} \\ &= \sum_{\substack{i,j,k \\ j<k}} R_{ii}(R_{ii} - R_{jj})(R_{ii} - R_{kk}) . \end{aligned}$$

Next we shall prove that the right hand side of (3. 2) is non-negative. By the assumption we have $R_{ii} > 0$, $i = 1, \cdots, n$. Without loss of generality, we can assume that $R_{11} \leqq R_{22} \leqq \cdots \leqq R_{nn}$. For the term for which $i<j<k$ or $j < k < i$, we have

$$R_{ii}(R_{ii} - R_{jj})(R_{ii} - R_{kk}) \geqq 0 .$$

On the other hand for the term for which $j < i < k$, we can find a suitable term such that the sum of these terms is non-negative, that is,

$$\begin{aligned} &R_{ii}(R_{ii} - R_{jj})(R_{ii} - R_{kk}) + R_{kk}(R_{kk} - R_{jj})(R_{kk} - R_{ii}) \\ &\quad = (R_{ii} - R_{kk})^2(R_{kk} + R_{ii} - R_{jj}) \geqq 0 . \end{aligned}$$

Thus the right hand side of (3. 2) can be expressed as the sum of only non-negative terms. Consequently, we get $K(p) \geqq 0$ for each point of M. By (2. 2), as $K(p)$ must be identically zero, every non-negative term is zero. Therefore from (3. 2), we have $R_{11} = R_{22} = \cdots = R_{nn}$, and by (3. 1) we conclude that M is of constant curvature.

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DEPARTMENT OF MATHEMATICS,
OCHANOMIZU UNIVERSITY,
TOKYO, JAPAN.