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ON A CLASS OF OPERATORS

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1. We consider bounded linear operators on a Hilbert space H. Denote by $\sigma(T)$ the spectrum, by $\sigma_p(T)$ the point spectrum and by $\pi(T)$ the approximate point spectrum of an operator T. As in [3], an operator T is said to be of class (N) in case $\|T^2x\| \geqq \|Tx\|^2$ for all unit vectors $x \in H$. A. Wintner [8] calls an operator T normaloid if $\|T\| = \sup\{|(Tx, x)| : x \in H, \|x\|=1\}$. It is known that T is normaloid if and only if $\|T\| = \sup\{|\lambda| : \lambda \in \sigma(T)\}$ or equivalently, $\|T^n\| = \|T\|^n$ for $n = 1, 2, \cdots$. If T is a hyponormal operator, that is $\|Tx\| \geqq \|T^*x\|$ for all $x \in H$, then T is of class (N). In fact, if T is a hyponormal operator, we have

$$\|Tx\|^2 = (T^*Tx, x) \leqq \|T^*(Tx)\| \leqq \|T^2x\|,$$

for any unit vector $x \in H$.

2. In this section we prove some theorems on an operator of class (N). The following theorem is suggested by [6] and [7].

THEOREM 1. *For an operator T of class (N),*

(i) *T is normaloid,*

and

(ii) *T^{-1} is also of class (N) if T is invertible.*

PROOF. To prove (i), it is sufficient to show that $\|T^nx\| \geqq \|Tx\|^n$ for each unit vector $x \in H$ and $n=1, 2, \cdots$. If $n \leqq 2$, the inequality is obvious by the definition of class (N). Suppose that $\|T^kx\| \geqq \|Tx\|^k$ for $k=1, 2, \cdots, n$ and $x \in H$, $\|x\|=1$. Then

$$\|T^{n+1}x\| = \|Tx\| \left\| T^n \frac{Tx}{\|Tx\|} \right\| \geqq \|Tx\| \left\| T \frac{Tx}{\|Tx\|} \right\|^n$$

$$= \|Tx\|^{1-n}\|T^2x\|^n \geqq \|Tx\|^{1-n}\|Tx\|^{2n} = \|Tx\|^{n+1}$$

for $x \in H$, $\|x\|=1$ and the induction is completed.

To prove (ii), let $y \in H$ be an arbitrary unit vector. Then there is an $x \in H$ such as $y=T^2x$. As T is of class (N), we have

$$\|T^{-1}y\|^2 = \|Tx\|^2 = \|x\|^2 \left\| T\frac{x}{\|x\|} \right\|^2 \leqq \|x\|^2 \left\| T^2 \frac{x}{\|x\|} \right\|$$

$$= \|x\| \|T^2x\| = \|x\| \|y\| = \|x\| = \|T^{-2}y\| .$$

and T^{-1} is of class (N).

As an immediate consequence of Theorem 1 we have the following corollary.

COROLLARY. *If T is an operator of class (N) and $\sigma(T)$ lies on the unit circle, T is a unitary operator.*

In the case of hyponormal operator this is nothing but a reshlt of [6] and [7].

PROOF. If $\sigma(T)$ lies on the unit circle, then $\|T\|=\|T^{-1}\|=1$ by Theorem 1. Hence we have

$$\|x\| \geqq \|Tx\| = \|T^{-1}x\| \left\| T^2 \frac{T^{-1}x}{\|T^{-1}x\|} \right\| \geqq \|T^{-1}x\| \left\| T \frac{T^{-1}x}{\|T^{-1}x\|} \right\|^2$$

$$= \frac{\|x\|^2}{\|T^{-1}x\|} \geqq \|x\| ,$$

and $\|Tx\|=\|x\|$ for $x \in H$ and T is a unitary operator.

In [1] T. Andô has proved that every completely continuous hyponormal operator is necessarily normal. The following theorem is a slight generalization of it.

THEOREM 2. *Let T be an operator of class (N) such that $T^{*p_1}T^{q_1} \cdots T^{*p_m}T^{q_m}$ is completely continuous for some non-negative integers $p_1, q_1, \cdots, p_m, q_m$. Then T is necessarily a normal operator.*

To prove the theorem, we shall need some preliminary lemmas. The following lemma is well-known (see [5]), but we cite here for convenience.

LEMMA 1. *For any operator T, $\sigma(T) \cap \{\lambda : |\lambda| = \|T\|\} \subset \pi(T)$, and if $\mu \in \sigma(T) \cap \{\lambda : |\lambda| = \|T\|\}$, $Tx_n - \mu x_n \to 0$ $(n \to \infty)$ is equivalent to $T^* x_n - \bar{\mu} x_n \to 0$ $(n \to \infty)$ for any sequence $\{x_n\}$ of unit vectors in H.*

The essential part of our proof is the following lemma.

LEMMA 2. *Let T be an operator such that $T^{*p_1}T^{q_1}\cdots T^{*p_m}T^{q_m}$ is completety continuous for some non-negative integers $p_1, q_1, \cdots, p_m, q_m$. Then the condition $\mu \in \sigma(T) \cap \{\lambda : |\lambda| = \|T\|\}$ implies $\mu \in \sigma_p(T)$ and $\bar{\mu} \in \sigma_p(T^*)$.*

PROOF. To simplify the notations, we shall treat the case where $T^{*p}T^q$ is completely continuous for some non-negative integers p and q. Since $\mu \in \sigma(T) \cap \{\lambda : |\lambda| = \|T\|\}$, there is a sequence $\{x_n\}$ of unit vectors in H such as $\|Tx_n - \mu x_n\| \to 0$ and $\|T^{*p}T^q x_n - \bar{\mu}^p \mu^q x_n\| \to 0$ $(n \to \infty)$ by Lemma 1. As $T^{*p}T^q$ is completely continuous, we may assume that (if necessary, by choosing a suitable subsequence) the sequence $\{T^{*p}T^q x_n\}$ converges to a certain vector $x \in H$. Let x_0 be $x/\bar{\mu}^p\mu^q$, then $\|x_n - x_0\| \to 0$ $(n \to \infty)$. Therefore $Tx_0 = \mu x_0$ and so $T^* x_0 = \bar{\mu} x_0$ by Lemma 1, i.e., $\mu \in \sigma_p(T)$ and $\bar{\mu} \in \sigma_p(T^*)$.

PROOF OF THEOREM 2. Throughout the proof, $\mathfrak{N}_T(\lambda)$ means the λ-th proper subspace of an operator T, that is $\mathfrak{N}_T(\lambda) = \{x \in H : Tx = \lambda x\}$. At first, we notice that there is at least one $\lambda \in \sigma_p(T)$ such as $\mathfrak{N}_T(\lambda) \cap \mathfrak{N}_{T^*}(\bar{\lambda}) \neq (0)$. In fact, since T is normaloid by Theorem 1, there is a $\lambda_0 \in \sigma(T)$ such as $|\lambda_0| = \|T\|$. Thus $\lambda_0 \in \sigma_p(T)$ and $\bar{\lambda}_0 \in \sigma_p(T^*)$ by Lemma 2 and $\mathfrak{N}_T(\lambda_0) \cap \mathfrak{N}_{T^*}(\bar{\lambda}_0) \neq (0)$ by the proof of Lemma 2. Now it is easy to see that $\{\mathfrak{N}_T(\lambda) \cap \mathfrak{N}_{T^*}(\bar{\lambda}) : \lambda \in \sigma_p(T)\}$ is a mutually orthogonal family. Let H_0 be $\sum_{\lambda \in \sigma_p(T)} \oplus (\mathfrak{N}_T(\lambda) \cap \mathfrak{N}_{T^*}(\bar{\lambda}))$, then H_0 reduces T and the restriction of T onto H_0 is normal. To complete the proof of the theorem, we have only to prove that the restriction T_1 of T onto $H_1 = H_0^{\perp}$ is 0. Suppose the contrary. Then T_1 is a non-zero operator of class (N) and $T_1^{*p}T_1^q$ is also completely continuous. By Theorem 1, T_1 is normaloid and there exists a $\mu \in \sigma(T_1)$ such as $|\mu| = \|T_1\|$. Hence $T_1 x = \mu x$ for some non-zero vector $x \in H_1$ and then $T_1^* x = \bar{\mu} x$ by the proof of Lemma 2. Therefore $\mathfrak{N}_T(\mu) \cap \mathfrak{N}_{T^*}(\bar{\mu}) \neq (0)$ and this is orthogonal to H_0. This is a contradiction.

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