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ISOMETRY BETWEEN $H^p(dm)$ AND THE HARDY CLASS H^p

NOZOMU MOCHIZUKI

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1. Let X be a compact Hausdorff space and A a logmodular algebra on X with the maximal ideal space M. Every $\theta \in M$ is represented by a uniquely determined multiplicative probablity measure m_θ on X, so that $\theta(f)=\int_X f\,dm_\theta$ for $f \in A$. We identify θ and m_θ. For $\theta_1, \theta_2 \in M$, the relation $\theta_1 \sim \theta_2$ defined by $\|\theta_1-\theta_2\| < 2$ is an equivalence relation and each equivalence class is called a part. We call a part P non-trivial if P does not reduce to a single point. P has the analytic structure in the sense that there exists a continuous one-to-one mapping τ of the open unit disk D onto P such that $\hat{f}\circ\tau$ is an analytic function on D. The purpose of this paper is to prove that if m is a representing measure belonging to a non-trivial part then $H^p(dm)$, $1 \leqq p \leqq \infty$, is isometrically isomorphic to the classical Hardy class H^p on the unit circle.

2. Let P be a non-trivial part of M. We fix $m \in P$. A is embedded in $L^\infty(dm)$ in a homomorphic and norm-decreasing manner. $H^p(dm)$, $1 \leqq p < \infty$, is defined as the $L^p(dm)$-completion of A; $H^\infty(dm)$ is defined by $H^\infty(dm)=L^\infty(dm) \cap H^2(dm)$, or equivalently, $H^\infty(dm)=\{f \mid f \in L^\infty(dm), \int_X fg\,dm=0 \text{ for } g \in A_m\}$, where A_m means the maximal ideal corresponding to m, so $H^\infty(dm)$ is w^*-closed in $L^\infty(dm)$. $L^\infty(dm)$ is represented as $C(Y)$; $H^\infty(dm)$ is a logmodular algebra on Y as a subalgebra of $C(Y)$ and Y is the Silov boundary. We denote by $\mathfrak{M}$ the maximal ideal space of $H^\infty(dm)$. Let $\theta \in P$. Then $d\theta=k\,dm$ for a bounded derivative k. Hence, θ is a bounded linear functional on A with respect to the $L^2(dm)$-norm, so θ is uniquely extended to a bounded linear functional on $H^2(dm)$. Especially, the latter is multiplicative on $H^\infty(dm)$. We denote by $j(\theta)$ the extended homomorphism. As a measure, $j(\theta)$ is supported in Y. Clearly, $j(\theta)(f)=\int_X f\,d\theta$ for $f \in H^\infty(dm)$ and j defines a mapping of P into $\mathfrak{M}$. Let $\mathfrak{P}=j(P)$. The following is the basic relation between θ and $j(\theta)$.

LEMMA. *Let $\theta \in P$, then $\int_X g\,d\theta=\int \hat{g}\,dj(\theta)$ for $g \in L^\infty(dm)$, in which $\hat{g}$*

means the represented function of g on Y.

PROOF. By the Riesz representation theorem, there is a positive measure $\tilde{\theta}$ on Y such that $\int_X g d\theta = \int_Y \hat{g} d\tilde{\theta}$.

Especially for $f \in H^\infty(dm)$, we have

$$\int_Y \hat{f} d\tilde{\theta} = \int_X f d\theta = j(\theta)(f) = \int_Y \hat{f} dj(\theta).$$

Since $H^\infty(dm)$ is logmodular on Y, we have $j(\theta) = \tilde{\theta}$.

The situation described above applies directly to the case where A is the algebra of all continuous functions on the unit circle C which are analytically extended on D. Let $H^\infty(D)$ denote the algebra of all bounded analytic functions $\hat{f}$ on D. $H^\infty(D)$ is isometrically isomorphic to the algebra of boundary functions $\tilde{f}$. Let $\mathfrak{R}$ be the maximal ideal space of $H^\infty(D)$. The structure of $\mathfrak{R}$ is extensively studied in [3]. Let φ_λ be the evaluation at $\lambda, \lambda \in D$. Then $\Delta = \{\varphi_\lambda | \lambda \in D\}$ is an open subset of $\mathfrak{R}$. The affirmative answer to the corona problem assures that Δ is dense in $\mathfrak{R}$. There is a natural projection π of $\mathfrak{R}$ onto $\bar{D}$, that is, $\pi = \hat{z}$ by definition and $\pi(\varphi_\lambda) = \lambda$ for $\lambda \in D$. Δ and D are homeomorphic. Let μ denote normalized Lebesgue measure on C. $L^\infty(d\mu)$ is represented as $C(\Gamma)$, where Γ becomes the Silov boundary of the logmodular algebra $H^\infty(D)$. Γ is contained in $\mathfrak{R} - \Delta$. It is easily seen that D and Δ correspond to P and $\mathfrak{P}$, respectively, that is, $\Delta = j(D)$. μ also corresponds to m. By Lemma and the Poisson integral formula, we have

$$f(0) = \int_C \tilde{f} d\mu = \int_\Gamma \hat{f} dj(\mu), \qquad f \in H^\infty(D).$$

THEOREM 1. *$H^p(dm)$, $1 \leqq p \leqq \infty$, is isometrically isomorphic to the Hardy class $H^p(D)$.*

PROOF. First, we discuss the case in which $p = \infty$. For this, we intend to reconstruct the analytic mapping of D onto $\mathfrak{P}$ instead of P as follows. We fix $Z \in H^2(dm)$ as in Theorem 7.4 in [1]. Since $|Z(x)| = 1$ a.e.dm, $Z \in H^\infty(dm)$. In the proof of that theorem, we can replace $f \in A$ by $f \in H^\infty(dm)$, having the analogous result; that is, $\hat{Z}$ is one-to-one from $\mathfrak{P}$ onto D and, since $Z \in H^\infty(dm)$, continuous. For every $f \in H^\infty(dm)$ and $\theta \in P$, we have

$$j(\theta)(f) = \sum_{n=0}^{\infty} a_n (\hat{Z}(j(\theta)))^n, \qquad \text{where } a_n = \int_X \bar{Z}^n f dm.$$

Put $\tau=\hat{Z}^{-1}$, then we have

$$(\hat{f}\circ\tau)(\lambda)=\sum_{n=0}^{\infty}a_n\lambda^n, \qquad \lambda\in D,$$

so $\hat{f}\circ\tau$ is an analytic function on D. Since $\int_X Zdm=0$, we have $\tau(0)=j(m)$. Now, for $f\in H^\infty(dm)$, we define the mapping τ^* by $\tau^*(f)=\hat{f}\circ\tau$. τ^* is an algebraic homomorphism into $H^\infty(D)$ and norm-decreasing. Moreover, it is actually onto $H^\infty(D)$, which we prove by the method of Theorem 8 of [4]. Let $h\in H^\infty(D)$. We can select a sequence $\{p_n\}$ of polynomials in z such that

$$|p_n(\lambda)|\leqq K,\ |\lambda|\leqq 1,\ n=1,2,3,\cdots;\ p_n(\lambda)\to h(\lambda),\ \lambda\in D.$$

Since $p_n\circ Z\in H^\infty(dm)$ and $\|p_n\circ Z\|\leqq K$, a subnet $\{p_{n_\alpha}\circ Z\}$ and $f\in H^\infty(dm)$ exist such that $p_{n_\alpha}\circ Z\to f$ in the w^*-topology. Let $\theta\in P$. Then $d\theta=kdm$ with k bounded, so we have

$$(p_{n_\alpha}\circ Z)^\wedge(j(\theta))=\int_X(p_{n_\alpha}\circ Z)kdm\to\int_X fkdm=\hat{f}(j(\theta)).$$

On the other hand,

$$(p_{n_\alpha}\circ Z)^\wedge(j(\theta))=p_{n_\alpha}(\hat{Z}(j(\theta)))\to h[\hat{Z}(j(\theta))]$$

Hence, $\hat{f}=h\circ\hat{Z}$ on $\mathfrak{P}$, so $h=\hat{f}\circ\tau$ on D.

Let ${}^t\tau^*$ be the adjoint of τ^* which is a one-to-one w^*-continuous mapping of $H^\infty(D)^*$ into $H^\infty(dm)^*$. We restrict ${}^t\tau^*$ on $\mathfrak{K}$, and again denote by ${}^t\tau^*$. Then ${}^t\tau^*$ is a homeomorphism of $\mathfrak{K}$ into $\mathfrak{M}$ where $\mathfrak{K}$ and $\mathfrak{M}$ are endowed with the Gelfand topologies. Since $({}^t\tau^*(\varphi))(f)=\varphi(\hat{f}\circ\tau)$ for $\varphi\in\mathfrak{K}$, $f\in H^\infty(dm)$, and each $\varphi\in\Delta$ is an evaluation at λ, we have ${}^t\tau^*(\Delta)=\mathfrak{P}$. Further, it follows that ${}^t\tau^*(\mathfrak{K})=\bar{\mathfrak{P}}$ from the fact that Δ is dense in $\mathfrak{K}$ and ${}^t\tau^*(\mathfrak{K})$ is compact.

Let $\mu'=j(\mu){}^t\tau^{*^{-1}}$. μ' is a positive measure on $\mathfrak{P}$. Let $f\in H^\infty(dm)$. Then we have

$$\int_{\mathfrak{m}}\hat{f}d\mu'=\int_{\mathfrak{K}}\hat{f}({}^t\tau^*(\varphi))dj(\mu)(\varphi)=\int_C(\hat{f}\circ\tau)^\sim d\mu$$

$$=(\hat{f}\circ\tau)(0)=\int_Y\hat{f}dj(m).$$

It follows that the support of μ' lies in Y and, since $H^\infty(dm)$ is logmodular on Y, $\mu'=j(m)$. Thus, $j(m)=j(\mu){}^t\tau^{*^{-1}}$ and $j(m)$ is the measure on $\bar{\mathfrak{P}}$. To prove that

τ^* is isometric, it is sufficient to show that τ^* is one-to-one. For this, let $\tau^* f=0$. Then $\hat{f}$ vanishes on $\overline{\mathfrak{P}}$, so $\hat{f}=0$ a.e.$dj(m)$. Let $h \in A+\bar{A}$ where $\bar{A}$ denotes the complex conjugate of A. Then we have by Lemma that

$$\int_X fhdm=\int_Y \hat{f}\hat{h}dj(m)=0.$$

By Theorem 6.7 in [1], we conclude that $f=0$ a.e.dm. Thus, $H^\infty(dm)\cong H^\infty(D)$.

From the arguement above, we have ${}^t\tau^*(\Gamma)=Y$, and hence $C(Y)\cong C(\Gamma)$. From this, an isomorphism U of $L^\infty(dm)$ onto $L^\infty(d\mu)$ is induced. Let $g \in L^\infty(dm)$. The transformation: $g \to \hat{g}\circ {}^t\tau^*$ gives the correspondence between $L^\infty(dm)$ and $C(\Gamma)$, and we have $\widehat{(Ug)}=\hat{g}.{}^t\tau^*$ where $\widehat{(Ug)}$ is the representation of $Ug \in L^\infty(d\mu)$ on Γ. Thus, we have

$$\int_X gdm=\int_Y \hat{g}dj(m)=\int_\Gamma (\hat{g}\circ{}^t\tau^*)\ dj(\mu)$$

$$=\int_\Gamma \widehat{(Ug)}\ dj(\mu)=\int_C Ug\ d\mu.$$

For $f \in H^\infty(dm)$, replacing g by $|f|^p$, $p \geqq 1$, we have

$$\int_X |f|^p dm=\int_C |Uf|^p d\mu.$$

Since $H^\infty(dm)$ and $H^\infty(D)$ are L^p-dense in $H^p(dm)$ and $H^p(D)$, respectively, we have $H^p(dm)\cong H^p(D)$. This completes the proof.

REMARK. (1) We see that $\mathfrak{P}$ is open and dense in $\mathfrak{M}$, for ${}^t\tau^*(\mathfrak{K})=\mathfrak{M}$. Hence, P is open and dense when embedded in $\mathfrak{M}$. This fact may be regarded as the generalized version of the corona theorem.

(2) If m belongs to a non-trivial part, then $H^p(dm)$ theory reduces to the $H^p(D)$ theory. For example, for every $f \in H^1(dm)$, $f \neq 0$, we have $\int_X \log|f|dm > -\infty$. In fact, let $\int_X fdm=0$, and let $h=\tau^* f$. Then $h(0)=0$, hence $h=z^k h_1$ where $h_1 \in H^1(D)$ and $h_1(0)\neq 0$. We have $f=Z^k f_1$, $f_1 \in H^1(dm)$ and $\int_X f_1 dm \neq 0$. Theorem 6.4 in [1] assures that $\int_X \log|f|dm > -\infty$. In general, however, it happens that

$\int_X \log|f|dm = -\infty$ for $f \in H^1(dm)$ even if f is not identically zero. From the above we see that this phenomenon occurs only if m constitutes a one-point part. A typical example is provided by the algebra of continuous functions of analytic type on a compact abelian group G where the character group $\widehat{G}$ is a non-archimedean ordered group. Normalized Haar measure is such a representing measure (p.208 in [2]).

THEOREM 2. *Let $\theta \in P$, then the support of θ lies in $\overline{P}-P$.*

PROOF. Let $\Theta \in \mathfrak{M}$. We define the restriction mapping σ of $\mathfrak{M}$ into M by $(\sigma\Theta)(f)=\Theta(f)$ for $f \in A$. σ is continuous, and one-to-one on $\mathfrak{P}$. It is easily seen that $\sigma(\mathfrak{P})=P$ and $\sigma(\mathfrak{M})=\overline{P}$. Let $m'=j(m)\sigma^{-1}$. m' is a positive measure on $\overline{P}$. Let $f \in A$. Since

$$\int_M \hat{f}(\theta)dm'(\theta) = \int_M \hat{f}dj(m) = \int_X fdm,$$

we have $m'=m$. This implies that the support of m lies in P. But, the support of m lies in the Silov boundary X and every point of X constitutes a trivial part, hence the support of m lies in $\overline{P}-P$. On the other hand, the support of θ is identical with that of m. This completes the proof.

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