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ON ENTIRE DIRICHLET SERIES OF ZERO ORDER

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1. Introduction. The entire Dirichlet series of this papers are series of the form

$$f(s) = \sum_{1}^{\infty} a_n e^{\lambda_n s} ,$$

where

$$s = \sigma + it, \quad 0 < \lambda_n < \lambda_{n+1} \quad (n \geqq 1), \quad \lim \lambda_n = \infty,$$

which are given to be not only entire functions of s in the sense that they are absolutely convergent for all finite s but also entire functions of order $\rho = 0, \rho$ being understood in the ususal sense of Ritt. A finer distinction may be introduced among such entire functions by means of their logarithmic order $\rho(\mathscr{L})$, defined as in this paper and having implications analogous to those of ρ. Among the implications of ρ is the relation of ρ to certain other orders which are, firstly the Sugimura order denoted by ρ_* in an earlier paper [4], secondly, the order defined by the coefficients a_n and denoted by ρ_C in the present paper, and, thirdly, the order defined by the rank of the maximum term of $f(s)$ and denoted by ρ_R in the present paper. The relationship between ρ, ρ_*, ρ_C and ρ_R, which is indeed quite simple, is set forth in Theorems I, II, III *infra*. One object of the present paper, suggested by the relationship last mentioned, is to investigate the relationship of the logarithmic order $\rho(\mathscr{L})$ to certain other logarithmic orders, $\rho_*(\mathscr{L})$, $\rho_C(\mathscr{L})$ and $\rho_R(\mathscr{L})$, to be presently defined in analogy with ρ_*, ρ_C and ρ_R respectively. Furthermore, corresponding to the mutually related orders ρ, ρ_*, ρ_C and ρ_R, we have the mutually related lower orders λ, λ_*, λ_C and λ_R, related (to be precise) as in Theorems I A, II A, III A, B *infra*. And it is the relationship between the various lower orders, in these theorems, which has suggested the second object of this paper, namely, to examine the relation between the lower logarithmic orders $\lambda_*(\mathscr{L})$, $\lambda_C(\mathscr{L})$ and $\lambda_R(\mathscr{L})$, corresponding to $\rho_*(\mathscr{L})$, $\rho_C(\mathscr{L})$ and $\rho_R(\mathscr{L})$ respectively.

2. Notation. In the usual notation, let

$$M(\sigma) = \underset{-\infty<t<\infty}{\text{l.u.b.}} |f(\sigma + it)|, \ \mu(\sigma) = \max | a_n e^{(\sigma+it)\lambda_n}| \equiv |a_\nu| e^{\sigma\lambda_\nu}$$

where ν, and hence λ_ν, is a function of σ, i.e.

$$\lambda_\nu = \lambda_{\nu(\sigma)} \equiv \Lambda(\sigma),$$

Furthermore, let us adopt a notation indicated in Section 1 and

$$(1)\quad \begin{cases} \text{(i)} & \lim_{\sigma\to\infty} {\sup \atop \inf} \dfrac{\log \log M(\sigma)}{\sigma} = {\rho \atop \lambda} & (0 \leqq \lambda \leqq \rho \leqq \infty), \\ \text{(ii)} & \lim_{\sigma\to\infty} {\sup \atop \inf} \dfrac{\log \log \mu(\sigma)}{\sigma} = {\rho_* \atop \lambda_*} & (0 \leqq \lambda_* \leqq \rho_* \leqq \infty), \\ \text{(iii)} & \lim_{\sigma\to\infty} {\sup \atop \inf} \dfrac{\log \Lambda(\sigma)}{\sigma} = {\rho_R \atop \lambda_R} & (0 \leqq \lambda_R \leqq \rho_R \leqq \infty), \\ \text{(iv)} & \lim_{n\to\infty} {\sup \atop \inf} \dfrac{\lambda_n \log \lambda_n}{\log |a_n|^{-1}} = {\rho_C \atop \lambda_C} & (0 \leqq \lambda_C \leqq \rho_C \leqq \infty). \end{cases}$$

Then the following theorems are known.

THEOREM I. ([1] Theorem). *Under the condition*

$$\lim_{n\to\infty} \frac{\log n}{\lambda_n \log \lambda_n} = 0,$$

we have

$$\rho = \rho_*.$$

The result actually proved in [1] is $\rho = \rho_C$[1] which combined with Theorem III below, gives us Theorem I.

THEOREM II. ([4], Lemma 3(a)). $\rho_* = \rho_R$.

THEOREM III. ([5], case $k = 1$ of Theorem 1). $\rho_* = \rho_C$.

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1) This result is included in a theorem of Tanaka's [7], p. 68, Theorem 1)

The following theorems are complementary to the above.

THEOREM I A. ([4], Theorem 1). *Under the condition*

$$\limsup_{n\to\infty}\frac{\log n}{\lambda_n} = D < \infty ,$$

we have

$$\lambda = \lambda_* .$$

THEOREM II A. ([4], Lemma 3(b)). $\lambda_* = \lambda_R$.

THOREM III A. ([6], Case $k=1$ of Theorem 1 A; [2], Theorem 1)[2]. *Under the condition*

$$\log \lambda_n \sim \log \lambda_{n-1} ,$$

we have

$$\lambda_* \geqq \lambda_C .$$

THEOREM III B. ([6], Case $k=1$ of Theorem 1 B. [2], Theorem 2)[3]. *Under the condition*

$$\frac{\log |a_n/a_{n-1}|}{\lambda_n - \lambda_{n-1}} \textit{ is monotonic decreasing and tends to } -\infty,$$

we have

$$\lambda_* \leqq \lambda_C .$$

In this paper we define for entire Dirichlet series with $\rho = 0$, in analogy with (1):

$$(2)\quad \begin{cases} \text{(i)} \quad \lim_{\sigma\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \dfrac{\log \log M(\sigma)}{\log \sigma} = \dfrac{\rho(\mathfrak{L})}{\lambda(\mathfrak{L})} \quad (1 \leqq \lambda(\mathfrak{L}),\ \rho(\mathfrak{L}) \leqq \infty), \\[2ex] \text{(ii)} \quad \lim_{\sigma\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \dfrac{\log \log \mu(\sigma)}{\log \sigma} = \dfrac{\rho_*(\mathfrak{L})}{\lambda_*(\mathfrak{L})} \quad (1 \leqq \lambda_*(\mathfrak{L}),\ \rho_*(\mathfrak{L}) \leqq \infty), \end{cases}$$

2) Rahman ([2], Theorem 1) gives the conclusion of Theorem III A as $\lambda \geqq \lambda_C$. But this is included in Theorem III A since $\lambda \geqq \lambda_*$ universally.

3) Rahman ([2], Therem 2) gives the conclusion of Theorem III B as $\lambda \leqq \lambda_C$, assuming additionally the condition on $\{\lambda_n\}$ stated in Theorem I A *supra*. Rahman's result is included in our Theorem III B on account of Theorem I A. There is a critical examination of Rahman's arguments (*loc. cit*) in paper [6], Section 3.

(iii) $$\lim_{\sigma\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log \Lambda(\sigma)}{\log \sigma} = \frac{\rho_R(\mathfrak{L})}{\lambda_R(\mathfrak{L})} \qquad (0 \leqq \lambda_R(\mathfrak{L}),\ \rho_R(\mathfrak{L}) \leqq \infty),$$

(iv) $$\lim_{n\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log \lambda_n}{\log\left(\frac{1}{\lambda_n}\log\frac{1}{|a_n|}\right)} = \frac{\rho_C(\mathfrak{L})}{\lambda_C(\mathfrak{L})} \quad (0 \leqq \lambda_C(\mathfrak{L}),\ \rho_C(\mathfrak{L}) \leqq \infty).$$

The condition $1 \leqq \lambda_*(\mathfrak{L}),\ \rho_*(\mathfrak{L}) \leqq \infty$ in (2)(ii), and hence the corresponding condition in (2)(i), is derived from Theorems 1, 2, 1 A, 2 A, *infra*, taken in conjunction with the condition $0 \leqq \lambda_R(\mathfrak{L}),\ \rho_R(\mathfrak{L}) \leqq \infty$ in (2)(iii).

3. Theorems of the Present Paper. In the notation (2), the following theorems will be proved analogous to the results comprised in Theorem I—Theorem III of Section 2.

THEOREM 1. *Under the condition*

$$\limsup \frac{\log n}{\lambda_n} = D < \infty,$$

we have

$$\rho(\mathfrak{L}) = \rho_*(\mathfrak{L}).$$

THEOREM 2. $$\rho_*(\mathfrak{L}) = \rho_R(\mathfrak{L}) + 1.$$

THEOREM 3. $$\rho_*(\mathfrak{L}) = \rho_C(\mathfrak{L}) + 1.$$

THEOREM 1 A. *Under the condition on* $\{\lambda_n\}$ *in Theorem* 1, *we have also*

$$\lambda(\mathfrak{L}) = \lambda_*(\mathfrak{L}).$$

THEOREM 2 A. $$\lambda_*(\mathfrak{L}) = \lambda_R(\mathfrak{L}) + 1.$$

THEOREM 3 A. *Under the condition*

$$\log \lambda_n \sim \log \lambda_{n-1},$$

we have

$$\lambda_*(\mathfrak{L}) \geqq \lambda_C(\mathfrak{L}) + 1.$$

THEORM 3 B. *Under the condition*

$$\frac{\log|a_n/a_{n-1}|}{\lambda_n - \lambda_{n-1}} \textit{ is monotonic decreasing and tends to } -\infty,$$

we have

$$\lambda_*(\mathcal{L}_0) \leqq \lambda_c(\mathcal{L}_0) + 1 .$$

Rahman ([3], relations (2), (2.3), (2.4) in p. 109) has proved Theorems 2, 2 A. Proofs of these theorems are given below, much shorter and simpler than his.

PROOF OF THEOREMS 1, 1 A. It is known ([8], p.68) that

$$\mu(\sigma) \leqq M(\sigma) < k\mu(\sigma + D + \varepsilon)$$

where $\varepsilon > 0$ may be as small as we please and k is a constant depending on D and ε. Hence

$$(3)\ \frac{\log\log\mu(\sigma)}{\log\sigma} \leqq \frac{\log\log M(\sigma)}{\log\sigma} < [1+o(1)]\frac{\log\log\mu(\sigma+D+\varepsilon)}{\log(\sigma+D+\varepsilon)} \times \frac{\log(\sigma+D+\varepsilon)}{\log\sigma},$$

as $\sigma \to \infty$.

First taking upper limits as $\sigma \to \infty$ of all members of (3) and then lower limits, we get

$$\rho_*(\mathcal{L}_0) \leqq \rho(\mathcal{L}_0) \leqq \rho_*(\mathcal{L}_0),$$

$$\lambda_*(\mathcal{L}_0) \leqq \lambda(\mathcal{L}_0) \leqq \lambda_*(\mathcal{L}_0),$$

whence Theorems 1, 1 A follow immediately.

PROOF OF THEOREMS 2, 2 A. We start with the known result ([8], p. 67) that $\Lambda(\sigma)$ is a (positive) monotonic increasing function of σ related to $\mu(\sigma)$ as follows:

$$(4) \qquad \log\mu(\sigma) = \log\mu(\sigma_0) + \int_{\sigma_0}^{\sigma} \Lambda(x)\,dx \quad (\sigma > \sigma_0).$$

From (4) we get in succession

$$\log\mu(\sigma) \leqq \log\mu(\sigma_0) + (\sigma - \sigma_0)\Lambda(\sigma)$$

$$\sim \sigma\Lambda(\sigma), \text{ as } \sigma \to \infty,$$

$$\frac{\log\log\mu(\sigma)}{\log\sigma}<\frac{\log\sigma+\log\Lambda(\sigma)+o(1)}{\log\sigma}.$$

We then obtain, first taking upper limits of both sides as $\sigma\to\infty$ and then lower limits,

$$(5)\qquad \begin{cases}\rho_*(\mathfrak{L})\leqq 1+\rho_R(\mathfrak{L}),\\ \lambda_*(\mathfrak{L})\leqq 1+\lambda_R(\mathfrak{L}).\end{cases}$$

Again, from (4) with σ changed to 2σ, we get successively:

$$\log\mu(2\sigma)\geqq\log\mu(\sigma_0)+\int_\sigma^{2\sigma}\Lambda(x)\,dx$$

$$\geqq\log\mu(\sigma_0)+\sigma\,\Lambda(\sigma),$$

$$\frac{\log\log\mu(2\sigma)}{\log 2\sigma}\,\frac{\log 2\sigma}{\log\sigma}\geqq\frac{\log\sigma+\log\Lambda(\sigma)+o(1)}{\log\sigma}\,(\sigma\to\infty).$$

As before, taking first upper limits of both sides as $\sigma\to\infty$ and then lower limits, we see that

$$(6)\qquad \begin{cases}\rho_*(\mathfrak{L})\geqslant 1+\rho_R(\mathfrak{L}),\\ \lambda_*(\mathfrak{L})\geqslant 1+\lambda_R(\mathfrak{L}).\end{cases}$$

(5) and (6) together establish Theorems 2, 2 A.

PROOF OF THEOREM 3. We suppose, to begin with, that $0<\rho_C(\mathfrak{L})<\infty$. Then, given any small $\varepsilon>0$, we write for brevity $\Delta=\rho_C(\mathfrak{L})-\varepsilon$ and use definition (2) (iv) to infer that

$$(7)\qquad |a_n|>\exp(-\lambda_n^{1+\Delta^{-1}})\qquad\begin{cases}n=n_p,\ p=1,2,\cdots\\ n_1<n_2<\cdots n_p\to\infty.\end{cases}$$

By the definition of $\mu(\sigma)$ followed by the use of (7),

$$\mu(\sigma)\geqq|a_n|\,e^{\lambda_n\sigma}\qquad(n=1,2,\cdots)$$

$$>\exp(-\lambda_n^{1+\Delta^{-1}}+\lambda_n\sigma)\qquad(n=n_p,\ p=1,2,\cdots)$$

or

$$\log \mu(\sigma) > -\lambda_n^{1+\Delta^{-1}} + \lambda_n \sigma \equiv g(\sigma, \lambda_n) \qquad (n = n_p,\ p = 1, 2, \cdots). \tag{8}$$

Now $g(\sigma, x)$, considered as a function of $x > 0$, has absolute maximum when

$$x = \left(\frac{\sigma}{1+\Delta^{-1}}\right)^{\Delta}.$$

And so, if $\{\sigma_{n_\gamma}\}$ is defined as the sequence

$$\lambda_{n_\gamma} = \left(\frac{\sigma_{n_\gamma}}{1+\Delta^{-1}}\right)^{\Delta} \qquad (\gamma = 1, 2, \cdots), \tag{9}$$

we see that

$$g(\sigma_{n_\gamma}, \lambda_{n_\gamma}) \geqq g(\sigma_{n_\gamma}, \lambda_{n_p}) \qquad (\text{any } \gamma \geqq 1,\ p = 1, 2, \cdots). \tag{10}$$

On the other hand, (8) with $\sigma = \sigma_{n_\gamma}$ gives us the inequality

$$\log \mu(\sigma_{n_\gamma}) > g(_{n_\gamma}, \lambda_{n_p}) \qquad (\text{any } \gamma \geqq 1,\ p = 1, 2, \cdots)$$

whose best form, with the largest possible right-hand member, occurs when $p = \gamma$ as shown by a comparison with (10). The best form in question is thus (8) with $\sigma = \sigma_{n_\gamma}, n = n_\gamma$ and given by

$$\left\{\begin{aligned} \log \mu(\sigma_{n_\gamma}) &> g(\sigma_{n_\gamma}, \lambda_{n_\gamma}) \\ &= -\lambda_{n_\gamma}^{1+\Delta^{-1}} + \lambda_{n_\gamma} \sigma_{n_\gamma} \\ &= \left(\frac{\sigma_{n_\gamma}}{1+\Delta^{-1}}\right) \left(-\frac{\sigma_{n_\gamma}}{1+\Delta^{-1}} + \sigma_{n_\gamma}\right) \end{aligned}\right. \tag{11}$$

by (9). From the last step we get

$$\log \log \mu(\sigma_{n_\gamma}) > (\Delta + 1) \log \sigma_{n_\gamma} - \Delta \log (1 + \Delta^{-1}) - \log(1 + \Delta), \tag{12}$$

whence it is obvious that

$$\begin{aligned} \limsup_{\sigma\to\infty} \frac{\log \log \mu(\sigma)}{\log \sigma} &\geqq \limsup_{\gamma\to\infty} \frac{\log \log \mu(\sigma_{n_\gamma})}{\log \sigma_{n_\gamma}} \\ &\geqq \Delta + 1 = \rho_c(\mathcal{L}) - \varepsilon + 1 . \end{aligned}$$

Since ε being arbitrarily small,

(13) $$\rho_*(\mathfrak{L}) \geqq \rho_c(\mathfrak{L}) + 1\,.$$

(13) is thus proved in the case $0 < \rho_c(\mathfrak{L}) < \infty$; while it is trivial in the case $\rho_c(\mathfrak{L}) = 0$, since $\rho_*(\mathfrak{L}) \geqq 1$ universally by Theorem 2. In the remaining case $\rho_c(\mathfrak{L}) = \infty$, (13) is ture in the form $\rho_*(\mathfrak{L}) = \infty$, since Δ in (12) is arbitrarily large, i. e. tending to infinity and so making the right-hand member of (12) asymptotically equal to $(\Delta + 1)\log\sigma_{n_\gamma}$ for every σ_{n_γ} after a stage (say, greater than e).

We proceed to establish (12) as an equality whether $\rho_c(\mathfrak{L})$ is infinite or not. The case of infinite $\rho_c(\mathfrak{L})$ having been disposed of, we now write $\Delta = \rho_c(\mathfrak{L}) + \varepsilon$ and note that we have, besides (7):

$$|a_n| < \exp(-\lambda_n^{1+\Delta^{-1}}) \quad (n > n_0(\varepsilon)).$$

Hence, remembering that $\mu(\sigma) = |a_\nu|\, e^{\sigma\lambda_\nu}$, where $\nu = \nu(\sigma)$ tends to ∞ with σ, we see that

$$\mu(\sigma) = |a_\nu|\, e^{\sigma\lambda_\nu} < \exp(-\lambda_\nu^{1+\Delta^{-1}} + \lambda_\nu\sigma) \equiv g(\sigma, \lambda_\nu) \quad (\sigma > \sigma_0)\,.$$

In the above inequality we may replace $g(\sigma, \lambda_\nu)$ by $\max g(\sigma, x)$ for $x > 0$ and obtain as a result of calculations similar to those leading up to (12):

$$\log\log\mu(\sigma) < (\Delta + 1)\log\sigma - \Delta\log(1 + \Delta^{-1}) - \log(1 + \Delta)\,.$$

Hence follows, as (13) from (12),

(14) $$\rho_*(\mathfrak{L}) \leqq \rho_c(\mathfrak{L}) + 1\,,$$

where $0 \leqq \rho_c(\mathfrak{L}) < \infty$. The case $\rho_c(\mathfrak{L}) = 0$ is included since then $\Delta = \varepsilon$ arbitrarily small. Combining (13) and (14) we have the result sought.

PROOF OF THEOREM 3 A. As in the proof of Theorem 3, we first suppose that $0 < \lambda_c(\mathfrak{L}) < \infty$ and write $H = \lambda_c(\mathfrak{L}) - \varepsilon$ where $\varepsilon > 0$ is arbitrarily small. We then obtain

$$|a_n| > \exp(-\lambda_n^{1+H^{-1}}) \qquad (n \geqq n_0(\varepsilon)),$$

(15) $$\mu(\sigma) \geqq |a_n|\, e^{\sigma\lambda_n} > \exp(-\lambda_n^{1+H^{-1}} + \lambda_n\sigma) \equiv \exp h(\sigma, \lambda_n) \quad (n \geqq n_0)\,.$$

Again as in the proof of Theorem 3, max $h(\sigma, x)$ for $x > 0$ occurs when

$$x = \left(\frac{\sigma}{1+H^{-1}}\right)^{H},$$

so that, defining $\{\sigma_n\}$ by

$$\lambda_n = \left(\frac{\sigma_n}{1+H^{-1}}\right)^{H}, \tag{16}$$

we have

$$h(\sigma_n, \lambda_n) \geqq h(\sigma_n, \lambda_m) \quad (\text{any } n \geqq 1, \text{ all } m \geqq 1). \tag{17}$$

But (15) with $\sigma = \sigma_n$ gives us the inequality

$$\log \mu(\sigma_n) > h(\sigma_n, \lambda_m) \quad (\text{any } n \geqq 1, \text{ all } m \geqq n_0). \tag{18}$$

In (18) we may confine ourselves to any $n \geqq n_0$ and all $m \geqq n_0$, and then we find, comparing (18) with (17), that the best form of (18), or the form with the largest right-hand member, corresponds to $m = n$, and is given by

$$\log \mu(\sigma_n) > h(\sigma_n, \lambda_n) \quad (n \geqq n_0).$$

Treating the above inequality exactly like (11) in the proof of Theorem 3, we find that

$$\liminf_{n\to\infty} \frac{\log \log \mu(\sigma_n)}{\log \sigma_n} \geqq \lambda_C(\mathcal{L}_0) + 1\,.$$

Hence, if $\sigma_n \leqq \sigma \leqq \sigma_{n+1}$, we have

$$\liminf_{\sigma\to\infty} \frac{\log\log \mu(\sigma)}{\log \sigma} \geqq \liminf_{n\to\infty} \frac{\log\log \mu(\sigma_n)}{\log \sigma_n} \cdot \frac{\log \sigma_n}{\log \sigma_{n+1}}$$

$$= \liminf_{n\to\infty} \frac{\log\log \mu(\sigma_n)}{\log \sigma_n} \cdot \frac{\log \lambda_n + \text{const.}}{\log \lambda_{n+1} + \text{const.}}$$

by (16). Now using our hypothesis $\log \lambda_n \sim \log \lambda_{n+1}$, we get from the last two steps, as required,

$$\lambda_*(\mathcal{L}_0) \geqq \lambda_C(\mathcal{L}_0) + 1\,.$$

The proof given of this result is for the case $0 < \lambda_C(\mathcal{L}_0) < \infty$. The remaining cases, $\lambda_C(\mathfrak{L}) = 0$ and $\lambda_C(\mathfrak{L}) = \infty$, may be treated as at the end of the first part of the proof of Theorem 3.

PROOF OF THEOREM 3B. In the proof we replace the given Dirichlet series $f(s)$ by the following Dirichlet series $F(s)$ having the same $\lambda_*(\mathfrak{L})$ and $\lambda_C(\mathfrak{L})$ as $f(s)$:

$$F(s) = \sum_1^\infty |a_n|\, e^{\lambda_n s} \equiv |a_1|\, e^{\lambda_1 s} + |a_1| \sum_2^\infty \frac{e^{\lambda_n s}}{r_2^{\lambda_2-\lambda_1} \cdots r_n^{\lambda_n-\lambda_{n-1}}}, \tag{19}$$

where, by hypothesis

$$r_n \equiv \left| \frac{a_{n-1}}{a_n} \right|^{1/(\lambda_n-\lambda_{n-1})} \uparrow \infty. \tag{20}$$

From (20) it is plain that not all r_n can be equal after a stage. In other words, $r_{n+1} > r_n$ for an infinity of n and we may suppose that there is an infinity of steadily increasing pairs of n, say $(N+1, N-M)$, where $N > M \equiv M(N) \geqq 0$ such that

$$r_{N+1} > r_N = r_{N-1} = \cdots \quad r_{N-M} > r_{N-M-1}.$$

If σ is such that

$$\text{either } r_N < e^\sigma < r_{N+1} \text{ or } r_N = r_{N-1} = \cdots = r_{N-M} = e^\sigma,$$

then $\mu(\sigma)$ for $F(s)$ is given by

$$\mu(\sigma) = |a_1| \frac{e^{\lambda_N \sigma}}{r_2^{\lambda_2-\lambda_1} \cdots r_N^{\lambda_N-\lambda_{N-1}}} = |a_N|\, e^{\lambda_N \sigma} \quad (r_N < e^\sigma < r_{N+1})$$

$$= |a_{N-1}|\, e^{\lambda_{N-1}\sigma} = \cdots |a_{N-M}|\, e^{\lambda_{N-M}\sigma} \quad (e^\sigma = r_N = \cdots = r_{N-M}).$$

Writing $N+1 = n_{p+1}$ and $N - M = n_p$, we are thus led to a sequence of positive integers n_p $(p = 1, 2, \cdots)$ such that

$$1 \leqslant n_1 < n_2 < \cdots, \ n_p \to \infty \quad \text{as } p \to \infty,$$

$$\mu(\sigma) = |a_n|^{\lambda_n \sigma} \quad (n_p \leqq n < n_{p+1},\ r_{n_p} \leqslant e^\sigma < r_{n_{p+1}}), \tag{21}$$

$$\nu(\sigma) = n \quad \text{corresponding to } \max |a_n|\, e^{\lambda_n \sigma} \quad (r_{n_p} \leqq e^\tau < r_{n_{p+1}}) \tag{22}$$

$$= \text{any } n \text{ such that } \ n_p \leqq n < n_{p+1}.$$

Now using the definition of $\lambda_R(\mathfrak{L})$ in (2) (iii) and supposing that $0<\lambda_R(\mathfrak{L})<\infty$ we find $\sigma_0(\varepsilon)$ corresponding to any small $\varepsilon>0$ so that

$$\text{(23)} \qquad \begin{cases} \log \lambda_{\nu(\sigma)} > [\lambda_R(\mathfrak{L}) - \varepsilon] \log \sigma \\ \qquad\qquad = [\lambda_*(\mathfrak{L}) - 1 - \varepsilon] \log \sigma \end{cases} \qquad (\sigma > \sigma_0)$$

by Theorem 2 A. Recalling (21) and (22), we see that in (23) we may put $\nu(\sigma) = n$ $(n_p \leqq n < n_{p+1})$ and $e^\sigma = r_n$. We then get from (23):

$$\text{(24)} \qquad \log \lambda_n > [\lambda_*(\mathfrak{L}) - 1 - \varepsilon] \log \log r_n$$

for all sufficiently large n. On the other hand, by a summation from (20),

$$\text{(25)} \qquad \begin{cases} \log \left| \dfrac{a_1}{a_n} \right| = \sum_2^n (\lambda_m - \lambda_{m-1}) \log r_m \\ \qquad \leqq \log r_n \sum_2^n (\lambda_m - \lambda_{m-1}) \quad < \lambda_n \log r_n. \end{cases}$$

The elimination of $\log r_n$ between (24) and (25) gives us, for all sufficiently large n,

$$\log \lambda_n > [\lambda_*(\mathfrak{L}) - 1 - \varepsilon] \log \left(\frac{1}{\lambda_n} \log \left| \frac{a_1}{a_n} \right| \right).$$

Hence, ε being arbitrarily small,

$$\lambda_C(\mathfrak{L}) \equiv \liminf_{n\to\infty} \frac{\log \lambda_n}{\log \left(\dfrac{1}{\lambda_n} \log \dfrac{1}{|a_n|} \right)} \geqq \lambda_*(\mathfrak{L}) - 1\,.$$

This is the conlusion sought. In proving it we have supposed that $0 < \lambda_R(\mathfrak{L}) < \infty$, or (in virtue of Theorem 2 A) $1 < \lambda_*(\mathfrak{L}) < \infty$. When $\lambda_R(\mathfrak{L}) = 0$ or $\lambda_*(\mathfrak{L}) = 1$, the conclusion is trivial since $\lambda_C(\mathfrak{L}) \geqq 0$ universally. And when $\lambda_R(\mathfrak{L}) = \lambda_*(\mathfrak{L}) = \infty$, the conclusion follows by an adaptation of the preceding proof. The proof for all cases is thus complete.

In conclusion it must be said that the methods of proof employed in this paper are similar to those in two earlier papers of the author already mentioned ([4], [5]) and that the author has received from Prof. C. T. Rajagopal during

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