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CLOSED GEODESICS ON CERTAIN RIEMANNIAN MANIFOLDS OF POSITIVE CURVATURE

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1. Introduction. To investigate the relations among geodesics, curvature and manifold structures is very interesting and important. The following theorem is well known (cf. Klingenberg [5]).

THEOREM. *Let M be a 2-dimensional complete simply connected Riemannian manifold with Gaussian curvature K, $0 < k \leqq K \leqq 1$, where k is a constant. Let G be a simple closed geodesic on M and $L(G)$ be its length. Then the following inequalities are satisfied:*

$$2\pi \leqq L(G) \leqq 2\pi/\sqrt{k}\ .$$

In particular, if there exists a closed geodesic of length 2π on M, then M is isometric to the sphere with constant curvature 1.

And if there exists a simple closed geodesic of length $2\pi/\sqrt{k}$ on M, then M is isometric to the sphere with constant curvature k.

In this paper we shall prove similar results in the higher dimensional case. By a geodesic triangle, we always mean a geodesic triangle composed of three shortest geodesic arcs.

THEOREM A. *Let M be an n-dimensional complete simply connected Riemannian manifold with sectional curvature K, $0 < k \leqq K \leqq 1$, where k is a constant. Let G be a closed geodesic which can be decomposed into a geodesic triangle and $L(G)$ be its length.*

Then the following inequalities are satisfied

$$2\pi \leqq L(G) \leqq 2\pi/\sqrt{k}\ .$$

In particular, if there exists a closed geodesic of length $2\pi/\sqrt{k}$ on M which can be decomposed into a geodesic triangle, then M is isometric to the *sphere with constant curvature k.*

REMARK 1. The latter half of Theorem A holds good without the assumption of simply connectedness.

THEOREM B. *Let M be an n-dimensional complete simply connected Riemannian manifold with sectional curvature K, $\frac{1}{4} \leqq k \leqq K \leqq 1$, where k is a constant. If there exists a simple closed geodesic of length $2\pi/\sqrt{k}$, then M is isometric to the sphere with constant curvature k.*

REMARK 2. Let M be an n-dimensional complete simply connected Riemannian manifold with sectional curvature K, $\frac{1}{4} < k \leqq K \leqq 1$, where k is a constant. If there exists a closed geodesic of length $2\pi/\sqrt{k}$, it is a simple closed geodesic, because we have two inequalities $2\pi/\sqrt{k} < 4\pi$ and $d(p,C(p)) \geqq \pi$ for all $p \in M$, (cf. §3 Theorem 1). Then M is isometric to the sphere with constant curvature k.

THEOREM C. *Let M be an n-dimensional complete simply connected Riemannian manifold with sectional curvature K, $\frac{1}{4} \leqq K \leqq 1$. If there exists a closed geodesic of length 2π on M, then M is isometric to one of the compact symmetric spaces of rank 1 with usual metric.*

THEOREM D. *Let M be an n-dimensional complete simply connected Riemannian manifold with sectional curvature K, $\frac{1}{4} < K \leqq 1$. If there exists a closed geodesic of length 2π on M, then M is isometric to the sphere with constant curvature 1.*

2. Notations and definitions. Let M be a Riemannian manifold of dimension n ($n \geqq 2$). We denote by $< \ , \ >$ (resp. $\| \ \ \|$) the scalar product (resp. norm) which defines the Riemannian structure of M. All the geodesics considered on M are parametrized by the arc-length measured from their origin.

If $\Lambda = \{\lambda(s)\}$ $(0 \leqq s \leqq l)$ is such a geodesic, then $\lambda'(s)$ denotes its tangent vector at $\lambda(s)$ and we have $\|\lambda'(s)\| = 1$ for all s. We denote by $d(p, q)$ the distance between two points p and q of M, with respect to the structure of metric space associated canonically to its Riemannian structure. If the manifold M is compact, we denote by $d(M)$ its diameter, that is the upper bound of $d(p, q)$ when p and q vary on M.

We denote by $G(p, q)$ the set of geodesics on M each of which joins p to q and whose length is equal to $d(p, q)$. We denote by $C(p)$ the cutlocus of a point p on M.

3. Review of the known results

THEOREM 1. (*Klingenberg* [4], [6], *Toponogov* [12]) *Let M be a compact simply connected Riemannian manifold with sectional curvature K, $0 < K \leqq 1$. Then the inequality $d(p, C(p)) \geqq \pi$ is satisfied for all points p of M.*

THEOREM 2. (*Myers* [8]) *Let M be a complete Riemannian manifold with sectional curvature K, $K \geqq k > 0$, where k is a constant.*
Then M is compact and we have the inequality $d(M) \leqq \pi/\sqrt{k}$.

THEOREM 3. (*Rauch* [9], *Klingenberg* [4], [6], [7], *Berger* [3], *Toponogov* [11], *Tsukamoto* [13]) *Let M be a complete simply connected Riemannian manifold with sectional curvature K, $\frac{1}{4} < K \leqq 1$. Then M is homeomorphic to a sphere.*

THEOREM 4. (*Berger* [2]) *Let M be a complete simply connected Riemannian manifold with sectional curvature K, $\frac{1}{4} \leqq K \leqq 1$.*
And let $d(M) = \pi$ be satisfied. Then M is isometric to one of the compact symmetric spaces of rank one with usual metric.

THEOREM 5. (*Berger* [1]) *Let M be a complete Riemannian manifold with sectional curvature K, $\frac{1}{4} \leqq K \leqq 1$. Let G be a closed geodesic on M, of length $\leqq 2\pi$. Then for any points p on M and q on G, we have $d(p, q) \leqq \pi$.*

THEOREM 6. (*Toponogov's comparison theorem, cf.* [7], [10]) *Let M be a complete Riemannian manifold with sectional curvature K, $K \geqq k > 0$, where k is a constant. Let p, q, r be three points on M and $\Gamma = \{\gamma(s)\}$, $\Lambda = \{\lambda(s)\}$ be two geodesics on M such that $\Gamma \in G(p, q)$, $\Lambda \in G(p, r)$, $\gamma(0) = \lambda(0) = p$.*
We denote by $S_2(k)$ the 2-dimensional sphere with constant curvature k and denote by $\Delta(\hat{p}\hat{q}\hat{r})$ the triangle on $S_2(k)$ such that $\hat{d}(\hat{p}, \hat{q}) = d(p, q)$, $\hat{d}(\hat{p}, \hat{r}) = d(p, r)$ and that the angle $\hat{\alpha}$ at $\hat{p}$ verifies $\cos\hat{\alpha} = \langle \gamma'(0), \lambda'(0) \rangle$.
If $\hat{d}(\hat{q}, \hat{r})$ denote the length of third side of the triangle $\Delta(\hat{p}\hat{q}\hat{r})$ of $S_2(k)$, then we have the inequality $d(q, r) \leqq \hat{d}(\hat{q}, \hat{r})$.

THEOREM 7. (*Toponogov, cf.* [7], [10]) *Let M be a complete Riemannian* *manifold with sectional curvature* K, $K \geqq k > 0$.

And let $d(M) = \pi/\sqrt{k}$ *be satisfied. Then* M *is isometric to the sphere with constant curvature* k.

4. **Proof of Theorem A**. By using Theorem 1, we have easily $L(G) \geqq 2\pi$. Since G can be decomposed into a geodesic triangle, we can obtain the inequality $L(G) \leqq 2\pi/\sqrt{k}$ by using Theorem 6. (cf. [7]) So we have the inequalities

$$2\pi \leqq L(G) \leqq 2\pi/\sqrt{k}\,.$$

We assume that there exists a closed geodesic G of length $2\pi/\sqrt{k}$ on M which can be decomposed into a geodesic triangle $\Delta(pqr)$. Then we prove $d(M) = \pi/\sqrt{k}$. By the assumption of Theorem A we have $d(M) \leqq \pi/\sqrt{k}$. Now we assume $d(M) < \pi/\sqrt{k}$ and lead a contradiction. By this assumption we have at least two geodesic arcs of length $> \pi/2\sqrt{k}$ among three geodesic arcs pq, qr, rp which compose the closed geodesic G. Let them be pq and pr. And we divide G into two parts of same length by the two points p and p' on G. Then we can find that the point p' lies on G between q and r. Since we have $d(M) < \pi/\sqrt{k}$, we have a shortest geodesic arc $\Theta = \{\theta(v)\}$ $(0 \leqq v \leqq m, m = d(p,p'), \theta(0) = p', \theta(m) = p)$. And Θ can not be the subarc of G. Let the geodesic subarc $p'qp$ of G be $\Gamma_1 = \{\gamma_1(v)\}$ $(0 \leqq v \leqq l, l = \pi/\sqrt{k}$, $\gamma_1(0) = p'$, $\gamma_1(l) = p)$. And let the geodesic subarc $p'rp$ of G be $\Gamma_2 = \{\gamma_2(v)\}$ $(0 \leqq v \leqq l, l = \pi/\sqrt{k}$, $\gamma_2(0) = p'$, $\gamma_2(l) = p)$. Then we have either

$$< \gamma_1'(0),\ \theta'(0) > \geqq 0 \quad \text{or} \quad < \gamma_2'(0),\ \theta'(0) > \geqq 0\,.$$

First we assume $< \gamma_1'(0), \theta'(0) > \geqq 0$. We use the cosine rule of spherical trigonometry and Theorem 6. We construct a geodesic triangle $\Delta(\hat{p}'\hat{p}\hat{q})$ on $S_2(k)$ such that $\hat{d}(\hat{p}', \hat{p}) = d(p', p)$, $\hat{d}(\hat{p}', \hat{q}) = d(p', q)$ and the angles $\measuredangle(\hat{p}\hat{p}'\hat{q}) = \measuredangle(pp'q)$. Then we have by using Theorem 6

$$(1) \qquad \hat{d}(\hat{p}, \hat{q}) \geqq d(p, q) = \pi/\sqrt{k} - d(p', q)\,.$$

On the other hand we have by using the cosine rule of spherical trigonometry

$$(2) \qquad \hat{d}(\hat{p}, \hat{q}) < \pi/\sqrt{k} - \hat{d}(\hat{p}', \hat{q}) = \pi/\sqrt{k} - d(p', q).$$

From (1) and (2) we lead a contradiction. In the case $< \gamma_2'(0), \theta'(0) > \geqq 0$, we can lead a contradiction by using the same argument as above.

Hence we have $d(M) = \pi/\sqrt{k}$. By using Theorem 7, we find that M is isometric to the sphere with constant curvature k. Q. E. D.

5. Proof of Theorem B. Let G be a closed geodesic of length $2\pi/\sqrt{k}$ on M. Now we divide G into four subarcs pq, qr, rs and sp of same length $\pi/2\sqrt{k}$.

Then they are all shortest arcs because of $\pi/2\sqrt{k} \leqq \pi$. Now we prove $d(M) = \pi/\sqrt{k}$. We assume $d(M) < \pi/\sqrt{k}$ and lead a contradiction. Then we have a shortest geodesic arc $\Theta = \{\theta(v)\}$ $(0 \leqq v \leqq m,\ m = d(p, r),\ \theta(0) = p,\ \theta(m) = r)$ and Θ can not be the subarc of G. Let the geodesic subarc pqr of G be $\Gamma_1 = \{\gamma_1(v)\}$ $(0 \leqq v \leqq l,\ l = \pi/\sqrt{k},\ \gamma_1(0) = p,\ \gamma_1(l) = r)$ and let the geodesic subarc psq of G be $\Gamma_2 = \{\gamma_2(v)\}$ $(0 \leqq v \leqq l,\ l = \pi/\sqrt{k},\ \gamma_2(0) = p,\ \gamma_2(l) = r)$. Then we have either

$$< \gamma_1'(0),\ \theta'(0) > \geqq 0 \quad \text{or} \quad < \gamma_2'(0),\ \theta'(0) > \geqq 0 .$$

First we assume $< \gamma_1'(0),\ \theta'(0) > \geqq 0$. We use the cosine rule of spherical trigonometry and Theorem 6. We construct a geodesic triangle $\Delta(\hat{p}\,\hat{q}\,\hat{r})$ on $S_2(k)$ such that $\hat{d}(\hat{p}, \hat{q}) = d(p, q)$, $\hat{d}(\hat{p}, \hat{r}) = d(p, r)$ and the angles $\sphericalangle(\hat{q}\,\hat{p}\,\hat{r}) = \sphericalangle(q\,p\,r)$. Then we have by using Theorem 6

$$\hat{d}(\hat{q}, \hat{r}) \geqq d(q, r) = \pi/2\sqrt{k} . \tag{1}$$

On the other hand we have by using the cosine rule of spherical trigonometry

$$d(q, r) < \pi/2\sqrt{k} . \tag{2}$$

From (1) and (2) we lead a contradiction. In the case $< \gamma_2'(0),\ \theta'(0) > \geqq 0$, we can lead a contradiction by the same argument. Hence we have $d(M) = \pi/\sqrt{k}$. By using Theorem 7, we find that M is isometric to the sphere with constant curvature k. Q. E. D.

6. Proof of Theorem C. Let G be a closed geodesic of length 2π on M. And let p be a point on G. By using Theorem 5, we have $d(p, q) \leqq \pi$ for all points q on M. Hence we have $d(p, r) \leqq \pi$ for all points r of $C(p)$.

On the other hand we have $d(p, C(p)) \geqq \pi$, by Theorem 1. So we have $d(p, r) = \pi$ for all points r of $C(p)$. By using the same argument as in Berger [2], we find that all the geodesics which start at p are closed and of length 2π. Then for any point q on M, there exists a closed geodesic of length 2π which passes through two points p and q. Then we have $d(q, r) = \pi$ for all points r of $C(q)$. Hence we have $d(M) = \pi$. By using Theorem 4, M is isometric to one of the compact symmetric spaces of rank one with usual metric. Q. E. D.

7. Proof of Theorem D. Under the assumption of Theorem D, M is homeomorphic to a sphere by using Theorem 3. On the other hand M is isometric to one of the compact symmetric spaces of rank one with usual metric, by using Theorem C. Hence M is isometric to the sphere with constant curvature 1. Q. E. D.

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