Tôhoku Math. Journ.
Vol. 18, No. 1, 1966

ON THE APPROXIMATION OF FUNCTIONS BY SOME SINGULAR INTEGRALS

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(Received September 3, 1965)

1. Introduction. Throughout this paper we assume, unless otherwise stated explicitly, that a real-valued function $f(t)$, defined on the whole real axis $-\infty < t < \infty$, is continuous and periodic with period 2π and has finite second derivative $f''(x)$ at $t = x$. Let

$$u_n(t) = \frac{1}{2} + \sum_{k=1}^{n} \rho_k^{(n)} \cos kt \geqq 0$$

be a non-negative even trigonometric polynomial of order n, and let

$$L_n(f\,;x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x+t)u_n(t)\,dt$$

be a positive linear operator with the kernel $u_n(t)$. Our main theorems may be stated as follows.

THEOREM 1. *Let p and q be positive integers such that $p \geqq q \geqq 2$ and let*

$$L_n^{(p,q)}(f\,;x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x+t)u_n(t)\,dt$$

with the kernel

$$u_n(t) = \frac{1}{A_{p,q}(n)} \frac{\sin^{2p}\dfrac{nt}{2}}{\sin^{2q}\dfrac{t}{2}} = \frac{1}{2} + \sum_{k=1}^{pn-q} \rho_k^{(pn-q)} \cos kt$$

where

$$A_{p,q}(n) = \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{\sin^{2p}\frac{nt}{2}}{\sin^{2q}\frac{t}{2}}\,dt$$

$$(1) \qquad = \frac{2^{-2p+2q+1}}{(2q-1)!}\sum_{\nu=1}^{p}(-1)^{q+\nu}\binom{2p}{p-\nu}$$

$$\times(\nu n+q-1)(\nu n+q-2)\cdots(\nu n-q+2)(\nu n-q+1)\,.$$

Then for any continuous function $f(t)$, defined on $-\infty<t<\infty$ and periodic with period 2π, we have

$$\lim_{n\to\infty} L_n^{(p,q)}(f\,;x) = f(x)$$

uniformly in x.

THEOREM 2. *Under the same condition as in Theorem 1 concerning the kernel $u_n(t)$, the following asymptotic formula holds for any $f(t)$:*

$$L_n^{(p,q)}(f\,;x) = f(x) + c_{p,q}f''(x)\frac{1}{n^2} + o\left(\frac{1}{n^2}\right) \quad (n\to\infty)$$

where $c_{p,q}$ is a numerical constant depending only on p and q and is defined by means of

$$c_{p,q} = -(2q-1)(q-1)\frac{\sum_{\nu=1}^{p}(-1)^{\nu}\binom{2p}{p-\nu}\nu^{2q-3}}{\sum_{\nu=1}^{p}(-1)^{\nu}\binom{2p}{p-\nu}\nu^{2q-1}}\,.$$

The special cases of Theorem 2 have been treated by various authors. In [6] I. P. Natanson proved, in our notation, that $c_{2,2}=3/2$. In [4] Y. Matsuoka established that $c_{3,3}=10/11$ and $c_{4,4}=105/151$ and later independently in [7] F. Schurer obtained the same result. Further in [8] F. Schurer proved that $c_{5,5}=8820/15619$ and $c_{6,6}=311465/655177$.

For the purpose of numerical calculation and investigation of the behaviour of $c_{p,q}$ it is convenient to express $c_{p,q}$, q being fixed, as a rational function of p.

THEOREM 3. *The constant $c_{p,q}$ in Theorem 2 is also given by the formula*

$$c_{p,q} = (2q-1)(q-1)(2p-2q+1)\frac{f_{q-2}(p)}{f_{q-1}(p)},$$

where the sequence $\{f_k(p)\}$ *of polynomials in* p *is defined recurssively by means of*

$$f_0(p) = 1$$

$$f_1(p) = p$$

$$f_2(p) = p(4p-1)$$

$$\cdots\cdots\cdots\cdots\cdots$$

$$f_k(p) = k(2k+1)(p-1)f_{k-1}(p-1) - \sum_{\nu=1}^{k-1}(2\nu-1)\binom{2k+1}{2\nu}B_\nu$$
$$\times (2p-2k-1)(2p-2k+1)\cdots(2p-2k+2\nu-5)(2p-2k+2\nu-3)f_{k-\nu}(p)$$
$$- 2k(2k+1)B_k(2p-2k-1)(2p-2k+1)\cdots(2p-5)(2p-3)f_0(p),$$

denoting $B_1=1/6,\ B_2=1/30,\ B_3=1/42,\cdots$ *Bernoulli numbers.*

The first nine polynomials $f_k(p)$ $(k=0, 1, \cdots, 8)$ are

$$f_0(p) = 1$$

$$f_1(p) = p$$

$$f_2(p) = p(4p-1)$$

$$f_3(p) = p(34p^2-24p+5)$$

$$f_4(p) = p(496p^3-672p^2+344p-63)$$

$$f_5(p) = p(11{,}056p^4-24{,}256p^3+22{,}046p^2-9{,}476p+1{,}575)$$

$$f_6(p) = p(349{,}504p^5-1{,}126{,}296p^4+1{,}576{,}360p^3-1{,}164{,}960p^2$$
$$+444{,}196p-68{,}409)$$

$$f_7(p) = p(14{,}873{,}104p^6-66{,}046{,}080p^5+131{,}033{,}440p^4-145{,}603{,}200p^3$$
$$+93{,}792{,}946p^2-32{,}674{,}800p+4{,}729{,}725)$$

$$f_8(p) = p(819{,}786{,}496p^7-4{,}794{,}238{,}720p^6+12{,}769{,}986{,}304p^5$$
$$-19{,}739{,}968{,}960p^4+18{,}846{,}542{,}944p^3-10{,}967{,}224{,}000p^2$$
$$+3{,}555{,}926{,}256p-488{,}783{,}295).$$

By making use of these polynomials, we obtain the following results:

$c_{2,2} = 3/2,$ $c_{3,2} = 3,$ $c_{4,2} = 15/4,$ $c_{5,2} = 21/5,$

$c_{6,2} = 9/2,$ $c_{7,2} = 33/7,$ $c_{8,2} = 39/8,$ $c_{9,2} = 5,$

$c_{3,3} = 10/11,$ $c_{4,3} = 2,$ $c_{5,3} = 50/19,$ $c_{6,3} = 70/23,$

$c_{7,3} = 10/3,$ $c_{8,3} = 110/31,$ $c_{9,3} = 26/7,$

$c_{4,4} = 105/151,$ $c_{5,4} = 57/35,$ $c_{6,4} = 69/35,$

$c_{7,4} = 441/167,$ $c_{8,4} = 651/221,$ $c_{9,4} = 8{,}085/2{,}543,$

$c_{5,5} = 8{,}820/15{,}619,$ $c_{6,5} = 1{,}116/809,$ $c_{7,5} = 6{,}012/3{,}101,$

$c_{8,5} = 7{,}956/3{,}391,$ $c_{9,5} = 91{,}548/34{,}465,$

$c_{6,6} = 311{,}465/655{,}177,$ $c_{7,6} = 2{,}215/1{,}851,$

$c_{8,6} = 16{,}955/9{,}871,$ $c_{9,6} = 172{,}325/81{,}613,$

$c_{7,7} = 11{,}117{,}106/27{,}085{,}381,$ $c_{8,7} = 59{,}226/56{,}057,$

$c_{9,7} = 489{,}678/317{,}615,$

$c_{8,8} = 841{,}695{,}855/2{,}330{,}931{,}341,$ $c_{9,8} = 33{,}349{,}575/35{,}263{,}201,$

$c_{9,9} = 685{,}798{,}733{,}048/2{,}127{,}599{,}641{,}825.$

THEOREM 4. *Let p be a positive integer such that $p \geqslant 2$, and let*

$$J_n^{(p)}(f\,;x) = \alpha_{2p}\int_{-\infty}^{\infty} f\left(x+\frac{2}{n}t\right)\left(\frac{\sin t}{t}\right)^{2p} dt$$

where

$$\alpha_{2p}^{-1} = \int_{-\infty}^{\infty}\left(\frac{\sin t}{t}\right)^{2p} dt\,.$$

Then the following asymptotic formula holds:

$$J_n^{(p)}(f\,;x) = f(x)+c_{p,p}f''(x)\frac{1}{n^2}+o\left(\frac{1}{n^2}\right) \quad (n\to\infty)$$

where the constant $c_{p,p}$ has the same meaning as in Theorem 2.

2. Lemmas. In order to prove the theorems we require the following lemmas.

LEMMA 1. [Korovkin; 2] *Let*

$$u_n(t) = \frac{1}{2} + \sum_{k=1}^{n} \rho_k^{(n)} \cos kt \geqslant 0$$

be a non-negative even trigonometric polynomial of order n, and let

$$L_n(f; x) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x+t)\, u_n(t)\, dt .$$

Further we suppose that the condition

$$\lim_{n\to\infty} \rho_1^{(n)} = 1 \tag{2}$$

is satisfied.

Then for any continuous function, defined on $-\infty < t < \infty$ *and periodic with period* 2π, *we have*

$$\lim_{n\to\infty} L_n(f; x) = f(x)$$

uniformly in x.

LEMMA 2. [Korovkin; 3] *Let*

$$u_n(t) = \frac{1}{2} + \sum_{k=1}^{n} \rho_k^{(n)} \cos kt \geqslant 0$$

be a non-negative even trigonometric polynomial of order n, and let

$$L_n(f; x) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x+t)\, u_n(t)\, dt .$$

Then in order that the asymptotic expansion of the form

$$L_n(f; x) = f(x) + (1-\rho_1^{(n)}) f''(x) + o(1-\rho_1^{(n)})$$

be valid, it is necessary and sufficient that

$$\lim_{n\to\infty} \frac{1-\rho_2^{(n)}}{1-\rho_1^{(n)}} = 4 . \tag{3}$$

LEMMA 3. *If*

$$\left(4\sin^2\frac{nt}{2}\right)\left(\frac{1}{2}r_0+\sum_{k=1}^{\lambda} r_k\cos kt\right)=\frac{1}{2}\rho_0+\sum_{k=1}^{\lambda+n}\rho_k\cos kt \qquad (\lambda\geqslant 2n).$$

Then we have

$$\begin{aligned}
\rho_k &= -r_{n-k}+2r_k-r_{n+k} && (0\leqslant k\leqslant n)\\
\rho_k &= -r_{k-n}+2r_k-r_{k+n} && (n\leqslant k\leqslant \lambda-n)\\
\rho_k &= 2r_k-r_{k-n} && (\lambda-n+1\leqslant k\leqslant \lambda)\\
\rho_k &= -r_{k-n} && (\lambda+1\leqslant k\leqslant \lambda+n).
\end{aligned}$$

PROOF. The assumption yields

$$\begin{aligned}
\frac{1}{2}\rho_0+\sum_{k=1}^{\lambda+n}\rho_k\cos kt &= (1-\cos nt)\left(r_0+\sum_{k=1}^{\lambda}2r_k\cos kt\right)\\
&= (r_0-r_n)+\sum_{k=1}^{n}(2r_k-r_{n-k}-r_{n+k})\cos kt+\sum_{k=n+1}^{\lambda-n}(2r_k-r_{k-n}-r_{k+n})\cos kt\\
&\quad+\sum_{k=\lambda-n+1}^{\lambda}(2r_k-r_{k-n})\cos kt-\sum_{k=\lambda+1}^{\lambda+n}r_{k-n}\cos kt
\end{aligned}$$

from which the result follows at once.

LEMMA 4. *If*

$$\left(4\sin^2\frac{t}{2}\right)\left(\frac{1}{2}r_0+\sum_{1}^{\lambda} r_k\cos kt\right)=\frac{1}{2}\rho_0+\sum_{1}^{\lambda+1}\rho_k\cos kt.$$

Then

$$\begin{aligned}
\rho_0 &= 2r_0-2r_1\\
\rho_k &= 2r_k-r_{k-1}-r_{k+1} \qquad (1\leqslant k\leqslant \lambda+1)
\end{aligned}$$

with a natural convention $r_{\lambda+1}=r_{\lambda+2}=0$.

PROOF. The assumption implies

$$\frac{1}{2}\rho_0+\sum_{1}^{\lambda+1}\rho_k\cos kt=(1-\cos t)\left(r_0+\sum_{1}^{\lambda}2r_k\cos kt\right)$$

$$= (r_0 - r_1) + \sum_{1}^{\lambda+1} (2r_k - r_{k-1} - r_{k+1}) \cos kt$$

which implies in turn the required result.

LEMMA 5. *Let p and q be positive integers such that $p \geqslant q \geqslant 1$. Then the following identity holds*:

$$(4) \qquad (2q-1)!\, 2^{2p-2q-1} \frac{\sin^{2p} \frac{nt}{2}}{\sin^{2q} \frac{t}{2}}$$

$$= \frac{1}{2} r_0^{(p,q)} + \sum_{1}^{pn-q} r_k^{(p,q)} \cos kt \equiv \frac{1}{2} r_0 + \sum_{1}^{pn-q} r_k \cos kt$$

where

$$(5) \qquad r_k = \sum_{\nu=1}^{p} (-1)^{q+\nu+1} \binom{2p}{p-\nu} (k-\nu n+q-1)(k-\nu n+q-2)$$

$$\times \cdots (k-\nu n-q+2)(k-\nu n-q+1) \qquad (0 \leqslant k \leqslant n+q-1)$$

$$(6) \qquad r_k = \sum_{\nu=\mu}^{p} (-1)^{q+\nu+1} \binom{2p}{p-\nu} (k-\nu n+q-1)(k-\nu n+q-2)$$

$$\times \cdots (k-\nu n-q+2)(k-\nu n-q+1)$$

$$((\mu-1)\, n-(q-1) \leqslant k \leqslant \mu n+(q-1)\, ;\, \mu = 2, 3, 4, \cdots, p)\, .$$

Alternatively, for the sake of brevity, we shall say that the identity (4) holds for (p, q). Actually we need the validity of (4) for the case $p \geqslant q \geqslant 2$, but we shall prove that of (4) for $p \geqslant q \geqslant 1$.

PROOF. We divide the proof into two steps. First we prove the

PROPOSITION A. *If the identity* (4) *holds for* (p, q), *then it holds also for* $(p+1, q)$, *and vice versa.*

PROOF OF PROPOSITION A. By hypothesis we have

$$(2q-1)!\, 2^{2p-2q-1} \frac{\sin^{2p} \frac{nt}{2}}{\sin^{2q} \frac{t}{2}} = \frac{1}{2} r_0 + \sum_{1}^{pn-q} r_k \cos kt$$

where r_k are given by (5) and (6). Multiplying both members of the above identity by $4\sin^2\frac{nt}{2}$ we have

$$(2q-1)!\,2^{2p-2q+1}\frac{\sin^{2p+2}\frac{nt}{2}}{\sin^{2q}\frac{t}{2}} = \left(4\sin^2\frac{nt}{2}\right)\left(\frac{1}{2}r_0+\sum_1^{pn-q} r_k\cos kt\right)$$

$$= \frac{1}{2}\rho_0 + \sum_1^{(p+1)n-q}\rho_k\cos kt\,,\quad \text{say.}$$

We consider the three cases.

1°) $0\leqslant k\leqslant n$. Applying Lemma 3 we have

$$\rho_k = -r_{n-k}+2r_k-r_{n+k}$$

$$=\sum_{\nu=1}^{p}(-1)^{q+\nu+1}\binom{2p}{p-\nu}(k+(\nu-1)n+q-1)(k+(\nu-1)n+q-2)$$

$$\times\cdots(k+(\nu-1)n-q+2)(k+(\nu-1)n-q+1)$$

$$+\,2\sum_{\nu=1}^{p}(-1)^{q+\nu+1}\binom{2p}{p-\nu}(k-\nu n+q-1)(k-\nu n+q-2)$$

$$\times\cdots(k-\nu n-q+2)(k-\nu n-q+1)$$

$$-\sum_{\nu=2}^{p}(-1)^{q+\nu+1}\binom{2p}{p-\nu}(k-(\nu-1)n+q-1)(k-(\nu-1)n+q-2)$$

$$\times\cdots(k-(\nu-1)n-q+2)(k-(\nu-1)n-q+1)$$

$$=\textstyle\sum_1+\sum_2+\sum_3,\quad \text{say.}$$

Put

$$f(i)=(k-in+q-1)(k-in+q-2)\cdots(k-in-q+2)(k-in-q+1)\,.$$

Obviously $f(i)$ is a polynomial in i of order $2q-1$. Observing that $2p\geqslant 2q > 2q-1$ and denoting

$$\Delta f(x)=f(x+1)-f(x),\quad \Delta^{m+1}f(x)=\Delta(\Delta^m f(x))\qquad (m=1,2,3,\cdots)$$

we have

$$\Delta^{2p}f(-p+1)=0\,,$$

i.e.,

$$\sum_{\nu=0}^{2p}(-1)^{\nu}\binom{2p}{\nu}f(-p+1+\nu)=0. \tag{7}$$

By a simple change of indices, we have

$$\begin{aligned}\sum\nolimits_1 &= (-1)^{q+1}\sum_{\nu=1}^{p}(-1)^{\nu}\binom{2p}{p-\nu}f(-\nu+1) \qquad (\nu=p-\mu)\\ &= (-1)^{q+1}\sum_{\mu=0}^{p-1}(-1)^{p-\mu}\binom{2p}{\mu}f(-p+\mu+1)\\ &= (-1)^{p+q+1}\sum_{\nu=0}^{p-1}(-1)^{\nu}\binom{2p}{\nu}f(-p+\nu+1).\end{aligned}$$

Hence, employing (7), and changing the indices, we have

$$\begin{aligned}\sum\nolimits_1 &= (-1)^{p+q}\sum_{\nu=1}^{2p}(-1)^{\nu}\binom{2p}{\nu}f(-p+\nu+1) \qquad (\nu=p+\mu)\\ &= (-1)^{q}\sum_{\mu=0}^{p}(-1)^{\mu}\binom{2p}{p-\mu}f(\mu+1).\end{aligned}$$

Therefore, for $0\leqslant k\leqslant n$, we have

$$\begin{aligned}\rho_k &= (-1)^{q}\sum_{\nu=0}^{p}(-1)^{\nu}\binom{2p}{p-\nu}f(\nu+1)\\ &\quad + 2(-1)^{q}\sum_{\nu=1}^{p}(-1)^{\nu+1}\binom{2p}{p-\nu}f(\nu)+(-1)^{q}\sum_{\nu=2}^{p}(-1)^{\nu}\binom{2p}{p-\nu}f(\nu-1)\\ &= \sum_{\nu=1}^{p-1}(-1)^{q+\nu+1}f(\nu)\left\{\binom{2p}{p-\nu+1}+2\binom{2p}{p-\nu}+\binom{2p}{p-\nu-1}\right\}\\ &\quad + (-1)^{p+q-1}f(p)\left\{\binom{2p}{1}+2\binom{2p}{0}\right\}+(-1)^{p+q}\binom{2p}{0}f(p+1)\\ &= \sum_{\nu=1}^{p+1}(-1)^{q+\nu+1}f(\nu)\binom{2p+2}{p-\nu+1},\end{aligned}$$

since

$$\binom{2p}{p-\nu+1}+2\binom{2p}{p-\nu}+\binom{2p}{p-\nu-1}=\binom{2p+2}{p-\nu+1}.$$

Hence the first part of Proposition A is proved in this case. The other cases can be treated in a similar manner. We sketch the proof.

2°) $n\leqslant k\leqslant(p-1)n$. We can find a positive integer μ such that $(\mu-1)n \leqslant k\leqslant \mu n$ $(2\leqslant\mu\leqslant p-1)$. Making use of Lemma 3 we have

$$\begin{aligned}
\rho_k &= -r_{k-n}+2r_k-r_{k+n}\\
&= -\sum_{\nu=\mu-1}^{p}(-1)^{q+\nu+1}\binom{2p}{p-\nu}f(\nu+1)+2\sum_{\nu=\mu}^{p}(-1)^{q+\nu+1}\binom{2p}{p-\nu}f(\nu)\\
&\quad -\sum_{\nu=\mu+1}^{p}(-1)^{q+\nu+1}\binom{2p}{p-\nu}f(\nu-1)\\
&= \sum_{\nu=\mu}^{p+1}(-1)^{q+\nu+1}\binom{2p}{p-\nu+1}f(\nu)+2\sum_{\nu=\mu}^{p}(-1)^{q+\nu+1}\binom{2p}{p-\nu}f(\nu)\\
&\quad +\sum_{\nu=\mu}^{p-1}(-1)^{q+\nu+1}\binom{2p}{p-\nu-1}f(\nu)\\
&= \sum_{\nu=\mu}^{p+1}(-1)^{q+\nu+1}\binom{2p+2}{p-\nu+1}f(\nu).
\end{aligned}$$

3°) $(p-1)n+1\leqslant k\leqslant pn$. By Lemma 3

$$\begin{aligned}
\rho_k &= 2r_k-r_{k-n}=2(-1)^{p+q+1}\binom{2p}{0}f(p)-\sum_{\nu=p-1}^{p}(-1)^{q+\nu+1}\binom{2p}{p-\nu}f(\nu+1)\\
&= \sum_{\nu=p}^{p+1}(-1)^{q+\nu+1}\binom{2p+2}{p-\nu+1}f(\nu).
\end{aligned}$$

4°) $pn+1\leqslant k\leqslant(p+1)n$. On the basis of Lemma 3

$$\rho_k=-r_{k-n}=-(-1)^{p+q+1}\binom{2p}{0}f(p+1)=\sum_{\nu=p+1}^{p+1}(-1)^{q+\nu+1}\binom{2p+2}{p+1-\nu}f(\nu).$$

Therefore the first part of Proposition A is established. The converse statement is obvious.

PROPOSITION B. *If the identity* (4) *holds for* $(p, p-1)$ *then it holds also for* (p, p), *and vice versa.*

PROOF OF PROPOSITION B. It will suffice to prove the converse proposition, hence by hypothesis

$$(2p-1)!\,\frac{1}{2}\,\frac{\sin^{2p}\frac{nt}{2}}{\sin^{2p}\frac{t}{2}}=\frac{1}{2}r_0+\sum_{k=1}^{pn-p}r_k\cos kt,$$

where

$$r_k=\sum_{\nu=1}^{p}(-1)^{p+\nu+1}\binom{2p}{p-\nu}(k-\nu n+p-1)(k-\nu n+p-2)\cdots(k-\nu n-p+1)$$
$$(0\leqslant k\leqslant n+p-1)$$

$$r_k=\sum_{\nu=\mu}^{p}(-1)^{p+\nu+1}\binom{2p}{p-\nu}(k-\nu n+p-1)(k-\nu n+p-2)\cdots(k-\nu n-p+1)$$
$$((\mu-1)n-(p-1)\leqslant k\leqslant \mu n+(p-1)\,;\,\mu=2,3,4,\cdots,p).$$

Incidentally, we note that

$$r_{pn-p+1}=r_{pn-p+2}=\cdots=r_{pn+p-1}=0. \tag{8}$$

Multiplying the both members of above identity by $4\sin^2\frac{t}{2}$, we have

$$2(2p-1)!\,\frac{\sin^{2p}\frac{nt}{2}}{\sin^{2p-2}\frac{t}{2}}=\left(4\sin^2\frac{t}{2}\right)\left(\frac{1}{2}r_0+\sum_{1}^{pn-p}r_k\cos kt\right) \tag{9}$$
$$\equiv\frac{1}{2}\rho_0+\sum_{1}^{pn-p+1}\rho_k\cos kt,$$

where, by Lemma 4

$$\rho_k=-r_{k-1}+2r_k-r_{k+1} \tag{10}$$

for $0\leqslant k\leqslant pn-p+1$. Suppose that k satisfies the condition $(\mu-1)n\leqslant k\leqslant \mu n$ for a suitable pcsitive integer μ $(1\leqslant\mu\leqslant p)$. Then (10) becomes

$$\rho_k=\sum_{\nu=\mu}^{p}(-1)^{p+\nu}\binom{2p}{p-\nu}(k-\nu n+p-2)(k-\nu n+p-3)\cdots(k-\nu n-p+2) \tag{11}$$

$$\times \{(k-\nu n-p+1)(k-\nu n-p) - 2(k-\nu n+p-1)(k-\nu n-p+1) + (k-\nu n+p)(k-\nu n+p-1)\}$$

$$= (2p-1)(2p-2)\sum_{\nu=\mu}^{p}(-1)^{p+\nu}\binom{2p}{p-\nu}(k-\nu n+p-2)(k-\nu n+p-3) \times \cdots (k-\nu n-p+3)(k-\nu n-p+2).$$

From (9) and (11) we conclude that

$$2(2p-3)!\frac{\sin^{2p}\frac{nt}{2}}{\sin^{2p-2}\frac{t}{2}} = \frac{1}{2}\tilde{\rho}_0 + \sum_{1}^{pn-p+1}\tilde{\rho}_k \cos kt,$$

where

$$\tilde{\rho}_k = \sum_{\nu=\mu}^{p}(-1)^{p+\nu}\binom{2p}{p-\nu}(k-\nu n+p-2)(k-\nu n+p-3) \times \cdots (k-\nu n-p+3)(k-\nu n-p+2) \quad (0 \leqslant k \leqslant pn-p+1). \tag{12}$$

Obviously from (8) and (10)

$$\tilde{\rho}_k = 0 \qquad (k = pn-p+2, pn-p+3, \cdots, pn+p-2).$$

Hence (12) is also true for $pn-p+2 \leqslant k \leqslant pn+p-2$. Thus (5) and (6) are established for $(p, p-1)$.

From the validity of Propositions A and B together with the fact that the expansion (4) is true for the case $p = q = 1$, we conclude that the expansion (4) is valid for any pair of integers p and q such that $p \geqslant q \geqslant 1$.

LEMMA 6. *The following relations hold*:

$$r_k = r_{-k} \qquad (k = 1, 2, 3, \cdots, q-1)$$

where r_k *and* r_{-k} *are defined by means of* (5) *in Lemma* 5.

PROOF. According to (5) we have

$$r_{-k} = \sum_{\nu=1}^{p}(-1)^{q+\nu+1}\binom{2p}{p-\nu}(-k-\nu n+q-1)(-k-\nu n+q-2) \times \cdots (-k-\nu n-q+2)(-k-\nu n-q+1)$$

$$= \sum_{\nu=1}^{p} (-1)^{q+\nu} \binom{2p}{p-\nu} (k+\nu n+q-1)(k+\nu n+q-2)$$

$$\times \cdots (k+\nu n-q+2)(k+\nu n-q+1)$$

$$= \sum_{\nu=-p}^{-1} (-1)^{q-\nu} \binom{2p}{p-\nu} (k-\nu n+q-1)(k-\nu n+q-2)$$

$$\times \cdots (k-\nu n-q+2)(k-\nu n-q+1).$$

Thus we obtain

$$r_k - r_{-k} = \sum_{\nu=-p}^{p}{}' (-1)^{q+\nu+1} \binom{2p}{p-\nu} f(\nu),$$

where the prime indicates that the term corresponding to $\nu=0$ is to be omitted. Therefore

$$r_k - r_{-k} = \sum_{\nu=-p}^{p} (-1)^{q+\nu+1} \binom{2p}{p-\nu} f(\nu) - (-1)^{q+1} \binom{2p}{p} f(0)$$

$$= \sum_{\nu=0}^{2p} (-1)^{q+\nu-p+1} \binom{2p}{\nu} f(\nu-p) + (-1)^{q} \binom{2p}{p} f(0)$$

$$= (-1)^{q-p+1} \Delta^{2p} f(-p) + (-1)^{q} \binom{2p}{p} (k+q-1)(k+q-2)$$

$$\times \cdots (k-q+2)(k-q+1).$$

The first term on the right is zero because $\Delta^{2p} f(-p) = 0$ as in the proof of Proposition A in Lemma 5 and the second term vanishes since k is equal to one of the values $1, 2, 3, \cdots, q-1$. Hence the result follows.

In view of Lemma 6 we can restate Lemma 5 as follows:

LEMMA 7. *Let p and q be positive integers such that $p \geqslant q \geqslant 1$. Then the following identity holds*:

$$(2q-1)!\, 2^{2p-2q-1} \frac{\sin^{2p} \dfrac{nt}{2}}{\sin^{2q} \dfrac{t}{2}} = \frac{1}{2} r_0 + \sum_{1}^{pn-q} r_k \cos kt$$

where

$$r_k=\sum_{\nu=\mu}^{p}(-1)^{q+\nu+1}\binom{2p}{p-\nu}(k-\nu n+q-1)(k-\nu n+q-2) \tag{13}$$
$$\times\cdots(k-\nu n-q+2)(k-\nu n-q+1)$$
$$((\mu-1)n-(q-1)\leqslant k\leqslant \mu n+(q-1)\,;\,\mu=1,2,3,\cdots,p),$$

or in a more concise form

$$r_k=\sum_{\nu=\mu}^{p}(-1)^{q+\nu+1}\binom{2p}{p-\nu}(k-\nu n+q-1)(k-\nu n+q-2) \tag{14}$$
$$\times\cdots(k-\nu n-q+2)(k-\nu n-q+1)$$
$$((\mu-1)n\leqslant k\leqslant \mu n\,;\,\mu=1,2,3,\cdots,p).$$

LEMMA 8. *Suppose that*

$$f(t)=(1-\cos t)^{\lambda}P(t),$$

where $P(t)$ is a trigonometric polynomial of order μ; λ,μ being integers such that $\lambda\geqslant 1$, $\mu\geqslant 0$. Then we have

$$f''(t)=(1-\cos t)^{\lambda-1}Q(t),$$

where $Q(t)$ is a trigonometric polynomial of order $\mu+1$.

PROOF. It is easily seen that the assumption yields

$$f''(t)=(1-\cos t)^{\lambda-1}\{\lambda(\lambda-1)P(t)+\lambda^2P(t)\cos t$$
$$+2\lambda P'(t)\sin t+P''(t)-P''(t)\cos t\},$$

which guarantees the assertion of the lemma.

LEMMA 9. *The following identity holds:*

$$\binom{2p-1}{p-1}+\sum_{\nu=1}^{p}(-1)^{\nu}\binom{2p}{p-\nu}\cos\nu t=2^{p-1}(1-\cos t)^{p}\qquad(p=1,2,3,\cdots). \tag{15}$$

PROOF. The proof proceeds by induction. In the case $p=1$ the lemma is obviously true. Suppose that it holds for p. Multiplying the both members of (15) by $2(1-\cos t)$, we have

$$2\binom{2p-1}{p-1}(1-\cos t) + \sum_{\nu=1}^{p} 2(-1)^\nu \binom{2p}{p-\nu}\cos \nu t$$

$$(16) \qquad - \sum_{\nu=1}^{p} (-1)^\nu \binom{2p}{p-\nu}\{\cos(\nu+1)t + \cos(\nu-1)t\}$$

$$= 2^p(1-\cos t)^{p+1}.$$

The left hand side of (16) is equal to

$$2\binom{2p-1}{p-1} - 2\binom{2p-1}{p-1}\cos t + \sum_{\nu=1}^{p} 2(-1)^\nu\binom{2p}{p-\nu}\cos\nu t$$

$$+ \sum_{\nu=2}^{p+1}(-1)^\nu\binom{2p}{p-\nu+1}\cos\nu t + \sum_{\nu=0}^{p-1}(-1)^\nu\binom{2p}{p-\nu-1}\cos\nu t$$

$$= 2\binom{2p-1}{p-1} + \binom{2p}{p-1} - \left\{2\binom{2p-1}{p-1} + 2\binom{2p}{p-1} + \binom{2p}{p-2}\right\}\cos t$$

$$+ \sum_{\nu=2}^{p-1}(-1)^\nu\left\{2\binom{2p}{p-\nu} + \binom{2p}{p-\nu+1} + \binom{2p}{p-\nu-1}\right\}\cos\nu t$$

$$+ (-1)^p\left\{2+\binom{2p}{1}\right\}\cos pt + (-1)^{p+1}\cos(p+1)t$$

$$= \binom{2p+1}{p} + \sum_{\nu=1}^{p+1}(-1)^\nu\binom{2p+2}{p+1-\nu}\cos\nu t.$$

Thus (16) reduces to

$$\binom{2p+1}{p} + \sum_{\nu=1}^{p+1}(-1)^\nu\binom{2p+2}{p+1-\nu}\cos\nu t = 2^p(1-\cos t)^{p+1},$$

which shows that the assertion is true for $p+1$. This completes the proof.

LEMMA 10. *Denoting*

$$S(p,\lambda) = \sum_{\nu=1}^{p}(-1)^\nu\binom{2p}{p-\nu}\nu^\lambda,$$

we have

$$S(p, 2k) = 0 \qquad (p \geqslant k+1).$$

PROOF. Differentiating $2k$ times the both members of (15) with respect to t, and applying Lemma 8, we have

$$\sum_{\nu=1}^{p}(-1)^{\nu+k}\binom{2p}{p-\nu}\nu^{2k}\cos\nu t=(1-\cos t)^{p-k}Q(t), \tag{17}$$

where $Q(t)$ is a trigonometric polynomial of order k. Putting $t=0$ in (17) and noting $p-k\geqq 1$, we have

$$\sum_{\nu=1}^{p}(-1)^{\nu}\binom{2p}{p-\nu}\nu^{2k}=0,$$

which was to be proved.

LEMMA 11. *Denoting*

$$s_n=s_n(k)=\sum_{\nu=1}^{n}\nu^{2k-1}, \qquad s_0=0,$$

$$\sigma_n=\sigma_n(k)=\sum_{\nu=1}^{n}s_\nu, \qquad \sigma_0=0,$$

we have

$$2k(2k+1)\sigma_{n-1}=n^{2k+1}+\sum_{\nu=1}^{k-1}(-1)^{\nu}(2\nu-1)\binom{2k+1}{2\nu}B_\nu n^{2k+1-2\nu} + (-1)^k\,2k(2k+1)B_k n, \tag{18}$$

where B_ν denote Bernoulli numbers: $B_1=1/6$, $B_2=1/30$, $B_3=1/42,\cdots$.

PROOF. It is known [1, p. 5] that

$$\sum_{\nu=1}^{n}\nu^{2k-1}=\frac{1}{2k}n^{2k}+\frac{1}{2}n^{2k-1}+\frac{1}{2}\binom{2k-1}{1}B_1n^{2k-2}-\frac{1}{4}\binom{2k-1}{3}B_2n^{2k-4} +\frac{1}{6}\binom{2k-1}{5}B_3n^{2k-6}-\cdots+(-1)^k\frac{1}{2k-2}\binom{2k-1}{2k-3}B_{k-1}n^2$$

and

$$\sum_{\nu=1}^{n}\nu^{2k}=\frac{1}{2k+1}n^{2k+1}+\frac{1}{2}n^{2k}+\frac{1}{2}\binom{2k}{1}B_1n^{2k-1}-\frac{1}{4}\binom{2k}{3}B_2n^{2k-3} +\frac{1}{6}\binom{2k}{5}B_3n^{2k-5}-\cdots+(-1)^{k-1}\frac{1}{2k}\binom{2k}{2k-1}B_kn.$$

Using these relations we have

$$\sigma_{n-1} = \sum_{\nu=1}^{n} (n-\nu)\,\nu^{2k-1} = n\sum_{\nu=1}^{n} \nu^{2k-1} - \sum_{\nu=1}^{n} \nu^{2k}$$

$$= \frac{1}{2k(2k+1)} n^{2k+1} - \frac{1}{2} B_1 \left\{ \binom{2k}{1} - \binom{2k-1}{1} \right\} n^{2k-1}$$

$$+ \frac{1}{4} B_2 \left\{ \binom{2k}{3} - \binom{2k-1}{3} \right\} n^{2k-3}$$

$$- \cdots + (-1)^{k-1} \frac{1}{2k-2} B_{k-1} \left\{ \binom{2k}{2k-3} - \binom{2k-1}{2k-3} \right\} n^3$$

$$+ (-1)^k \frac{1}{2k} B_k \binom{2k}{2k-1} n$$

$$= \frac{1}{2k(2k+1)} n^{2k+1} - \frac{1}{2} B_1 \binom{2k-1}{0} n^{2k-1} + \frac{1}{4} B_2 \binom{2k-1}{2} n^{2k-3}$$

$$- \cdots + (-1)^{k-1} \frac{1}{2k-2} B_{k-1} \binom{2k-1}{2k-4} n^3 + (-1)^k \frac{1}{2k} B_k \binom{2k}{2k-1} n$$

$$= \frac{1}{2k(2k+1)} n^{2k+1} + \sum_{\nu=1}^{k-1} (-1)^\nu \frac{1}{2\nu} \binom{2k-1}{2\nu-2} B_\nu n^{2k+1-2\nu} + (-1)^k B_k n\,.$$

Hence

$$2k(2k+1)\sigma_{n-1} = n^{2k+1} + \sum_{\nu=1}^{k-1} (-1)^\nu (2\nu-1) \binom{2k+1}{2\nu} B_\nu n^{2k+1-2\nu} + (-1)^k 2k(2k+1) B_k n\,,$$

since

$$2k(2k+1)\frac{1}{2\nu}\binom{2k-1}{2\nu-2} = (2\nu-1)\binom{2k+1}{2\nu}.$$

LEMMA 12. *There hold the relations*

$$S(p, 2k-1) = \sum_{\nu=1}^{p+1} (-1)^{\nu-1} \binom{2p+2}{p-\nu+1} \sigma_{\nu-1}\,, \tag{19}$$

$$\begin{aligned} S(p+1, 2k+1) = & -2k(2k+1) S(p, 2k-1) \\ & + \sum_{\nu=1}^{k-1} (-1)^{\nu-1} (2\nu-1) \binom{2k+1}{2\nu} B_\nu\, S(p+1, 2k-2\nu+1) \\ & + (-1)^{k-1}\, 2k(2k+1)\, B_k\, S(p+1, 1)\,. \end{aligned} \tag{20}$$

PROOF. The first relation is easily proved by applying twice Abel's transformation:

$$S(p,2k-1)=\sum_{\nu=1}^{p}(-1)^{\nu}\binom{2p}{p-\nu}\nu^{2k-1}=\sum_{\nu=1}^{p}(-1)^{\nu}\binom{2p}{p-\nu}(s_{\nu}-s_{\nu-1})$$

$$=\sum_{\nu=1}^{p}(-1)^{\nu}\binom{2p+1}{p-\nu}s_{\nu}=\sum_{\nu=1}^{p}(-1)^{\nu}\binom{2p+1}{p-\nu}(\sigma_{\nu}-\sigma_{\nu-1})$$

$$=\sum_{\nu=1}^{p}(-1)^{\nu}\binom{2p+2}{p-\nu}\sigma_{\nu}=\sum_{\nu=1}^{p+1}(-1)^{\nu-1}\binom{2p+2}{p-\nu+1}\sigma_{\nu-1}.$$

We turn to the proof of the second relation. Combining (18) and (19), we have

$$2k(2k+1)\,S(p,2k-1)=\sum_{\nu=1}^{p+1}(-1)^{\nu-1}\binom{2p+2}{p-\nu+1}\,2k(2k+1)\,\sigma_{\nu-1}$$

$$=\sum_{\nu=1}^{p+1}(-1)^{\nu-1}\binom{2p+2}{p-\nu+1}\left\{\nu^{2k+1}+\sum_{\mu=1}^{k-1}(-1)^{\mu}(2\mu-1)\binom{2k+1}{2\mu}B_{\mu}\nu^{2k+1-2\mu}\right.$$
$$\left.+(-1)^{k}\,2k(2k+1)\,B_{k}\,\nu\right\}$$

$$=-S(p+1,2k+1)+\sum_{\mu=1}^{k-1}(-1)^{\mu-1}(2\mu-1)\binom{2k+1}{2\mu}B_{\mu}S(p+1,2k-2\mu+1)$$
$$+(-1)^{k-1}\,2k(2k+1)\,B_{k}\,S(p+1,1),$$

which is the desired result.

LEMMA 13. *Let*

$$\frac{d^{2p}}{d\theta^{2p}}\sin^{-2}\theta=(2p+1)!\sum_{\nu=1}^{p+1}a_{\nu}^{(2p)}\sin^{-2\nu}\theta. \tag{21}$$

Then

$$\frac{n}{2}\int_{-\pi}^{\pi}f(x+t)\left(\frac{\sin\frac{nt}{2}}{n\sin\frac{t}{2}}\right)^{2p+2}\left\{\sum_{\nu=1}^{p+1}a_{\nu}^{(2p)}\sin^{2p+2-2\nu}\frac{t}{2}\right\}dt$$

$$=\int_{-\infty}^{\infty}f\left(x+\frac{2}{n}t\right)\left(\frac{\sin t}{t}\right)^{2p+2}dt,$$

where $f(t)$ denotes any continuous, periodic function with period 2π.

PROOF. It is well known that

$$\sin^{-2}\theta=\sum_{k=-\infty}^{\infty}\frac{1}{(\theta+k\pi)^2}.$$

Differentiating this equation $2p$ times, we have

$$\frac{d^{2p}}{d\theta^{2p}}\sin^{-2}\theta=\sum_{k=-\infty}^{\infty}\frac{(2p+1)!}{(\theta+k\pi)^{2p+2}}. \tag{22}$$

By making use of (21) and (22), we have

$$\begin{aligned}\left(\frac{\sin n\theta}{n\sin\theta}\right)^{2p+2}\sum_{v=1}^{p+1}a_v^{(2p)}\sin^{2p+2-2v}\theta&=\left(\frac{\sin n\theta}{n}\right)^{2p+2}\sum_{v=1}^{p+1}a_v^{(2p)}\sin^{-2v}\theta\\ &=\left(\frac{\sin n\theta}{n}\right)^{2p+2}\sum_{k=-\infty}^{\infty}\frac{1}{(\theta+k\pi)^{2p+2}}=\frac{1}{n^{2p+2}}\sum_{k=-\infty}^{\infty}\frac{\sin^{2p+2}n\theta}{(\theta+k\pi)^{2p+2}}.\end{aligned} \tag{23}$$

Hence, in view of (23), we obtain

$$\begin{aligned}&\frac{n}{2}\int_{-\pi}^{\pi}f(x+t)\left(\frac{\sin\frac{nt}{2}}{n\sin\frac{t}{2}}\right)^{2p+2}\left\{\sum_{v=1}^{p+1}a_v^{(2p)}\sin^{2p+2-2v}\frac{t}{2}\right\}dt\\ &=\frac{n}{2}\int_{0}^{2\pi}f(x+t)\left(\frac{\sin\frac{nt}{2}}{n\sin\frac{t}{2}}\right)^{2p+2}\left\{\sum_{v=1}^{p+1}a_v^{(2p)}\sin^{2p+2-2v}\frac{t}{2}\right\}dt\\ &=\frac{n}{2}\int^{2\pi}f(x+t)\frac{1}{n^{2p+2}}\sum_{k=-\infty}^{\infty}\frac{\sin^{2p+2}\frac{nt}{2}}{\left(\frac{t}{2}+k\pi\right)^{2p+2}}dt.\end{aligned}$$

Putting $t/2=u$, we infer that the last integral equals to

$$\frac{1}{n^{2p+1}}\int_{0}^{\pi}f(x+2u)\sum_{k=-\infty}^{\infty}\frac{\sin^{2p+2}nu}{(u+k\pi)^{2p+2}}du$$

$$= \frac{1}{n^{2p+1}} \sum_{k=-\infty}^{\infty} \int_{k\pi}^{(k+1)\pi} f(x+2u-2k\pi) \frac{\sin^{2p+2} nu}{u^{2p+2}} du$$

$$= \frac{1}{n^{2p+1}} \int_{-\infty}^{\infty} f(x+2u) \left(\frac{\sin nu}{u}\right)^{2p+2} du$$

$$= \int_{-\infty}^{\infty} f\left(x + \frac{2}{n} t\right)\left(\frac{\sin t}{t}\right)^{2p+2} dt .$$

LEMMA 14. *The following recurrence relation holds*:

$$a_k^{(2n+2)} = \frac{1}{(2n+3)(n+1)} \{(2k-1)(k-1)\, a_{k-1}^{(2n)} - 2k^2 a_k^{(2n)}\}$$

$$(k = 1, 2, 3, \cdots, n+2),$$

with a convention $a_{n+2}^{(2n)} = 0$, *where* $a_k^{(2n)}$ *denotes the coefficient introduced in Lemma* 13.

PROOF. By hypothesis

$$\frac{d^{2n}}{dt^{2n}} \sin^{-2} t = (2n+1)! \sum_{k=1}^{n+1} a_k^{(2n)} \sin^{-2k} t .$$

Differentiating this equality with respect to t, we obtain successively

$$\frac{d^{2n+1}}{dt^{2n+1}} \sin^{-2} t = -2(2n+1)! \cos t \sum_{k=1}^{n+1} k a_k^{(2n)} \sin^{-2k-1} t ,$$

$$\frac{d^{2n+2}}{dt^{2n+2}} \sin^{-2} t = 2(2n+1)! \left\{\sum_{k=1}^{n+1} (2k+1)\, k a_k^{(2n)} \sin^{-2k-2} t - \sum_{k=1}^{n+1} 2k^2 a_k^{(2n)} \sin^{-2k} t\right\}$$

$$= (2n+3)! \left\{\sum_{k=1}^{n+1} \frac{(2k-1)(k-1) a_{k-1}^{(2n)} - 2k^2 a_k^{(2n)}}{(2n+3)(n+1)} \sin^{-2k} t + a_{n+1}^{(2n)} \sin^{-2n-4} t\right\}.$$

The last relation yields the result.

LEMMA 15. *For any positive integer* p,

$$\int_{-\infty}^{\infty} f\left(x + \frac{2}{n} t\right)\left(\frac{\sin t}{t}\right)^{2p} dt \tag{24}$$

$$= \frac{1}{(2p-1)!\, n^{2p-1}} \int_{-\pi}^{\pi} f(x+t) \left\{\frac{1}{2} \rho_0 + \sum_{1}^{pn-1} \rho_k \cos kt\right\} dt ,$$

where

(25)
$$\rho_k = \sum_{\nu=\mu}^{p} (-1)^{p+\nu+1} \binom{2p}{p-\nu} (k-\nu n)^{2p-1}$$
$$((\mu-1)\, n \leqslant k \leqslant \mu n\, ;\ \mu=1, 2, 3, \cdots, p)\, .$$

PROOF. First we shall prove that

(26)
$$\frac{1}{2}(2p-1)!\left(\frac{\sin\frac{nt}{2}}{\sin\frac{t}{2}}\right)^{2p}\left(\sum_{\nu=1}^{p} a_\nu^{(2p-2)} \sin^{2p-2\nu}\frac{t}{2}\right) = \frac{1}{2}\rho_0 + \sum_{1}^{pn-1} \rho_k \cos kt\, ,$$

where the coefficients ρ_k on the right are given by (25).

By Lemma 7 the left hand side of (26) is equal to

$$\sum_{\nu=1}^{p} a_\nu^{(2p-2)} \frac{1}{2}(2p-1)!\, \frac{\sin^{2p}\frac{nt}{2}}{\sin^{2\nu}\frac{t}{2}}$$
$$= \sum_{\nu=1}^{p} a_\nu^{(2p-2)} \frac{(2p-1)!}{(2\nu-1)!}\, \frac{1}{2^{2p-2\nu}} \left\{\frac{1}{2} r_0^{(p,\nu)} + \sum_{k=1}^{pn-\nu} r_k^{(p,\nu)} \cos kt\right\}.$$

Comparing this equation with the right hand said of (26), we obtain

$$\rho_k = \sum_{\nu=1}^{p} a_\nu^{(2p-2)} \frac{(2p-1)!}{(2\nu-1)!}\, \frac{1}{2^{2p-2\nu}}\, r_k^{(p,\nu)}$$
$$= \sum_{\nu=0}^{p-1} a_{p-\nu}^{(2p-2)} \frac{(2p-1)!}{(2p-2\nu-1)!}\, \frac{1}{2^{2\nu}}\, r_k^{(p,p-\nu)}\, .$$

When $(\mu-1)\, n \leqslant k \leqslant \mu n$ $(1 \leqslant \mu \leqslant p)$, we have, by Lemma 7,

$$\rho_k = \sum_{\nu=0}^{p-1} a_{p-\nu}^{(2p-2)} \frac{(2p-1)!}{(2p-2\nu-1)!}\, \frac{1}{2^{2\nu}} \sum_{\lambda=\mu}^{p} (-1)^{p-\nu+\lambda+1} \binom{2p}{p-\lambda}$$
$$\times (k-\lambda n+p-\nu-1)(k-\lambda n+p-\nu-2)\cdots(k-\lambda n-p+\nu+1)\, .$$

Interchanging the indices λ and ν, we get

$$\rho_k=\sum_{\nu=\mu}^{p}\sum_{\lambda=0}^{p-1}(-1)^{p-\lambda+\nu+1}\binom{2p}{p-\nu}(k-\nu n+p-\lambda-1)(k-\nu n+p-\lambda-2)$$
$$\times\cdots(k-\nu n-p+\lambda+1)\,a_{p-\lambda}^{(2p-2)}\frac{(2p-1)!}{(2p-2\lambda-1)!2^{2\lambda}}\,.$$

Thus in order to prove (26), it will suffice to show that

$$(k-\nu n)^{2p-1}=\sum_{\lambda=0}^{p-1}(-1)^{\lambda}a_{p-\lambda}^{(2p-2)}\frac{(2p-1)!}{(2p-2\lambda-1)!\,2^{2\lambda}}(k-\nu n+p-\lambda-1)$$
$$\times(k-\nu n+p-\lambda-2)\cdots(k-\nu n-p+\lambda+1)\,.$$

Putting $k-\nu n=x$, the last equality becomes

$$x^{2p-1}=\sum_{\nu=0}^{p-1}(-1)^{\nu}a_{p-\nu}^{(2p-2)}\frac{(2p-1)!}{(2p-2\nu-1)!\,2^{2\nu}}(x+p-\nu-1)\times(x+p-\nu-2)\cdots(x-p+\nu+1)\,. \tag{27}$$

We shall prove (27) by induction on p. The identity (27) holds in the case $p=2$, since $a_2^{(2)}=1$, $a_1^{(2)}=-2/3$. We have to prove that (27) implies (28):

$$x^{2p+1}=\sum_{\nu=0}^{p}(-1)^{\nu}a_{p+1-\nu}^{(2p)}\frac{(2p+1)!}{(2p-2\nu+1)!\,2^{2\nu}}(x+p-\nu)\times(x+p-\nu-1)\cdots(x-p+\nu)\,. \tag{28}$$

Multiplying the both members of (27) by x^2, we have

$$x^{2p+1}=\sum_{\nu=0}^{p-1}(-1)^{\nu}a_{p-\nu}^{(2p-2)}\frac{(2p-1)!}{(2p-2\nu-1)!\,2^{2\nu}}(x+p-\nu)$$
$$\times(x+p-\nu-1)\cdots(x-p+\nu+1)(x-p+\nu)$$
$$+\sum_{\nu=0}^{p-1}(-1)^{\nu}a_{p-\nu}^{(2p-2)}\frac{(2p-1)!}{(2p-2\nu-1)!\,2^{2\nu}}(p-\nu)^2(x+p-\nu-1)$$
$$\times(x+p-\nu-2)\cdots(x-p+\nu+2)(x-p+\nu+1)$$
$$=\sum_{\nu=0}^{p}(-1)^{\nu}a_{p-\nu}^{(2p-2)}\frac{(2p-1)!}{(2p-2\nu-1)!\,2^{2\nu}}(x+p-\nu)$$
$$\times(x+p-\nu-1)\cdots(x-p+\nu+1)(x-p+\nu)$$

$$+\sum_{\nu=1}^{p}(-1)^{\nu-1}a_{p-\nu+1}^{(2p-2)}\frac{(2p-1)!}{(2p-2\nu+1)!\,2^{2\nu-2}}(p-\nu+1)^2$$

$$\times(x+p-\nu)(x+p-\nu-1)\cdots(x-p+\nu)$$

$$=\sum_{\nu=0}^{p}(-1)^{\nu}\frac{(2p+1)!}{(2p-2\nu+1)!\,2^{2\nu}}(x+p-\nu)(x+p-\nu-1)$$

$$\times\cdots(x-p+\nu+1)(x-p+\nu)\left\{\frac{(2p-2\nu+1)(p-\nu)}{p(2p+1)}a_{p-\nu}^{(2p-2)}\right.$$

$$\left.-\frac{2}{p(2p+1)}(p-\nu+1)^2a_{p-\nu+1}^{(2p-2)}\right\},$$

since $a_{p+1}^{(2p-2)}=0$. By Lemma 14 the expression in the curly brackets of the last equation is equal to $a_{p-\nu+1}^{(2p)}$. Hence we conclude (28). Thus we establish the identity (26).

On the basis of Lemma 13 and (26), we have

$$\int_{-\infty}^{\infty}f\left(x+\frac{2}{n}t\right)\left(\frac{\sin t}{t}\right)^{2p}dt$$

$$=\frac{n}{2}\int_{-\pi}^{\pi}f(x+t)\left(\frac{\sin\dfrac{nt}{2}}{n\sin\dfrac{t}{2}}\right)^{2p}\left\{\sum_{\nu=1}^{p}a_{\nu}^{(2p-2)}\sin^{2p-2\nu}\frac{t}{2}\right\}dt$$

$$=\frac{1}{(2p-1)!\,n^{2p-1}}\int_{-\pi}^{\pi}f(x+t)\left\{\frac{1}{2}\rho_0+\sum_{1}^{pn-1}\rho_k\cos kt\right\}dt,$$

where ρ_k are given by (25). This completes the proof.

3. Proofs of the theorems.

PROOF OF THEOREM 1. First we prove the validity of (1). By Lemma 7

$$(2q-1)!\,2^{2p-2q-1}\frac{\sin^{2p}\dfrac{nt}{2}}{\sin^{2q}\dfrac{t}{2}}=\frac{1}{2}r_0+\sum_{1}^{pn-q}r_k\cos kt, \tag{29}$$

where the coefficients r_k are given by (14). On the other hand, by the assumption of the theorem

$$u_n(t) = \frac{\sin^{2p}\dfrac{nt}{2}}{A_{p,q}(n)\sin^{2q}\dfrac{t}{2}} = \frac{1}{2} + \sum_{1}^{pn-q} \rho_k^{(pn-q)} \cos kt. \tag{30}$$

Hence

$$A_{p,q}(n) = \frac{2^{-2p+2q+1}}{(2q-1)!} r_0 = \frac{2^{-2p+2q+1}}{(2q-1)!} \sum_{\nu=1}^{p} (-1)^{q+\nu} \binom{2p}{p-\nu} (\nu n+q-1)$$
$$\times (\nu n+q-2) \cdots (\nu n-q+2)(\nu n-q+1)$$

as mentioned in the theorem.

From (29) and (30) we conclude that

$$r_0 \rho_k = r_k$$

which obviously implies

$$1-\rho_1 = \frac{r_0-r_1}{r_0}, \qquad 1-\rho_2 = \frac{r_0-r_2}{r_0}. \tag{31}$$

Now by Lemma 7

$$r_0-r_1 = \sum_{\nu=1}^{p} (-1)^{q+\nu} \binom{2p}{p-\nu} (\nu n+q-1)(\nu n+q-2) \cdots (\nu n-q+2)(\nu n-q+1)$$
$$- \sum_{\nu=1}^{p} (-1)^{q+\nu} \binom{2p}{p-\nu} (\nu n+q-2)(\nu n+q-3) \cdots (\nu n-q+1)(\nu n-q)$$
$$= (2q-1) \sum_{\nu=1}^{p} (-1)^{q+\nu} \binom{2p}{p-\nu} (\nu n+q-2)(\nu n+q-3) \cdots (\nu n-q+1)$$
$$= (2q-1) \sum_{\nu=1}^{p} (-1)^{q+\nu} \binom{2p}{p-\nu} \{\nu n-(q-1)\}\, \nu n \{\nu^2 n^2-(q-2)^2\}$$
$$\times \{\nu^2 n^2-(q-3)^2\} \cdots \{\nu^2 n^2-1^2\}$$
$$= (2q-1) \sum_{\nu=1}^{p} (-1)^{q+\nu} \binom{2p}{p-\nu} \{\nu^{2q-2} n^{2q-2} - \alpha \nu^{2q-4} n^{2q-4} + \cdots\}$$
$$- (2q-1)(q-1) \sum_{\nu=1}^{p} (-1)^{q+\nu} \binom{2p}{p-\nu} \{\nu^{2q-3} n^{2q-3} - \alpha \nu^{2q-5} n^{2q-5} + \cdots\},$$

where $\alpha = 1^2+2^2+\cdots+(q-2)^2$. The first term on the right of the above equality is equal to

$$(2q-1)(-1)^q\{S(p,2q-2)n^{2q-2}-\alpha S(p,2q-4)n^{2q-4}+\cdots\}.$$

Each term in the curly brackets of this expression vanishes by Lemma 10, since $p \geqslant q$. Thus

$$(32)\quad r_0-r_1=(2q-1)(q-1)(-1)^{q+1}\{S(p,2q-3)n^{2q-3}-\alpha S(p,2q-5)n^{2q-5}+\cdots\}.$$

In a similar manner, by Lemma 7

$$(33)\quad r_0=\sum_{\nu=1}^{p}(-1)^{q+\nu}\binom{2p}{p-\nu}(\nu n+q-1)(\nu n+q-2)\cdots(\nu n-q+2)(\nu n-q+1)$$

$$=\sum_{\nu=1}^{p}(-1)^{q+\nu}\binom{2p}{p-\nu}\{\nu^2n^2-(q-1)^2\}\cdots\{\nu^2n^2-1^2\}(\nu n)$$

$$=\sum_{\nu=1}^{p}(-1)^{q+\nu}\binom{2p}{p-\nu}\{\nu^{2q-1}n^{2q-1}-\beta\nu^{2q-3}n^{2q-3}+\cdots\}$$

$$=(-1)^q\{S(p,2q-1)n^{2q-1}-\beta S(p,2q-3)n^{2q-3}+\cdots\},$$

where $\beta=1^2+2^2+\cdots+(q-1)^2$. In virtue of (31), (32) and (33), we obtain

$$(34)\qquad 1-\rho_1=-(2q-1)(q-1)\frac{S(p,2q-3)}{S(p,2q-1)n^2}+O\left(\frac{1}{n^4}\right),$$

which assures the condition (2) in Lemma 1. Thus on the basis of Lemma 1, Theorem 1 is established.

PROOF OF THEOREM 2. We prove that the coefficients $\rho_k^{(pn-q)}$ given by $u_n(t)$ in Theorem 2 satisfy the condition (3) of Lemma 2. By Lemmas 7 and 10

$$r_0-r_2=2(2q-1)\sum_{\nu=1}^{p}(-1)^{q+\nu}\binom{2p}{p-\nu}\nu n(\nu n-1)(\nu n-q+1)(\nu n-q+2)$$
$$\times\{\nu^2n^2-(q-3)^2\}\{\nu^2n^2-(q-4)^2\}\cdots\{\nu^2n^2-1^2\}$$

$$(35)\qquad =2(2q-1)\sum_{\nu=1}^{p}(-1)^{q+\nu+1}\binom{2p}{p-\nu}\{2(q-1)\nu^3n^3+(q-1)(q-2)\nu n\}$$
$$\times\{\nu^2n^2-(q-3)^2\}\{\nu^2n^2-(q-4)^2\}\cdots\{\nu^2n^2-1^2\}$$

$$=4(2q-1)(q-1)(-1)^{q+1}S(p,2q-3)n^{2q-3}$$
$$+2(2q-1)(-1)^{q+1}\gamma S(p,2q-5)n^{2q-5}+\cdots,$$

where $\gamma=(q-1)(q-2)-2(q-1)\{1^2+2^2+\cdots+(q-3)^2\}$. Hence by (31), (32) and (35)

$$\lim_{n\to\infty}\frac{1-\rho_2}{1-\rho_1}=\lim_{n\to\infty}\frac{r_0-r_2}{r_0-r_1}=4.$$

Therefore the condition (3) is fulfilled.

Now by Lemma 2 and (34) of Theorem 1, we have

$$c_{p,q}=-(2q-1)(q-1)\frac{S(p,2q-3)}{S(p,2q-1)}$$

$$=-(2q-1)(q-1)\frac{\sum_{\nu=1}^{p}(-1)^\nu\binom{2p}{p-\nu}\nu^{2q-3}}{\sum_{\nu=1}^{p}(-1)^\nu\binom{2p}{p-\nu}\nu^{2q-1}}.$$

This completes the proof of Theorem 2.

PROOF OF THEOREM 3. Define

$$f_k(p)=(-1)^{k-1}\frac{(p-1)!\,(p-k-1)!}{2^k(2p-2k-2)!}S(p,2k+1). \tag{36}$$

Then the equation (20) of Lemma 12 may be expressed in terms of $f_k(p)$:

$$f_k(p+1)=k(2k+1)\,pf_{k-1}(p)$$
$$-\sum_{\nu=1}^{k-1}(2\nu-1)\binom{2k+1}{2\nu}B_\nu(2p-2k+1)(2p-2k+3)\cdots$$
$$\times(2p-2k+2\nu-3)(2p-2k+2\nu-1)f_{k-\nu}(p+1)$$
$$-2k(2k+1)\,B_k(2p-2k+1)(2p-2k+3)\cdots(2p-3)(2p-1)f_0(p+1).$$

Writing $p-1$ in place of p, we have

$$f_k(p)=k(2k+1)(p-1)f_{k-1}(p-1)-\sum_{\nu=1}^{k-1}(2\nu-1)\binom{2k+1}{2\nu}B_\nu$$
$$\times(2p-2k-1)(2p-2k+1)\cdots(2p-2k+2\nu-3)f_{k-\nu}(p)$$
$$-2k(2k+1)B_k(2p-2k-1)(2p-2k+1)\cdots(2p-5)(2p-3)f_0(p).$$

This is the recurrence relation of $f_k(p)$ as described in the theorem.

Furthermore we can show that

$$f_0(p)=1\,,\qquad f_1(p)=p\,.$$

In order to prove these results we show that

$$(37)\qquad S(p,1)=\sum_{\nu=1}^{p}(-1)^{\nu}\binom{2p}{p-\nu}\nu=-\binom{2p-2}{p-1}\qquad(p=2,3,4,\cdots)$$

and

$$(38)\qquad S(p,3)=\sum_{\nu=1}^{p}(-1)^{\nu}\binom{2p}{p-\nu}\nu^{3}=\frac{2p(2p-4)!}{(p-1)!\,(p-2)!}\qquad(p=2,3,4,\cdots).$$

Obviously we have

$$(39)\qquad (1+x)^{2p}=\sum_{\nu=1}^{p}\binom{2p}{p-\nu}x^{p-\nu}+\sum_{\nu=0}^{p}\binom{2p}{p+\nu}x^{p+\nu}$$

and

$$(40)\qquad (1+x)^{-2}=\sum_{\nu=1}^{\infty}(-1)^{\nu-1}\nu x^{\nu-1}\qquad(|x|<1).$$

A comparison of the coefficient of x^{p-1} on the both members of the product of (39) and (40) will yield

$$\binom{2p-2}{p-1}=\sum_{\nu=1}^{p}(-1)^{\nu-1}\binom{2p}{p-\nu}\nu,$$

which implies (37). Meanwhile, by a simple calculation

$$(41)\qquad (1-4x+x^{2})(1+x)^{-4}=\sum_{\nu=1}^{\infty}(-1)^{\nu-1}\nu^{3}x^{\nu-1}.$$

In a similar manner, from (39) and (41), we have

$$\sum_{\nu=1}^{p}(-1)^{\nu-1}\binom{2p}{p-\nu}\nu^{3}=\binom{2p-4}{p-1}-4\binom{2p-4}{p-2}+\binom{2p-4}{p-3}$$

$$=-\frac{2p(2p-4)!}{(p-1)!\,(p-2)!},$$

which implies (38). The above reasoning was made under the condition $p \geqslant 3$, but an actual verification shows that (38) is valid also for $p=2$. Therefore, by (36), (37) and (38)

$$f_0(p) = -\frac{\{(p-1)!\}^2}{(2p-2)!} S(p,1) = 1\,,$$

$$f_1(p) = \frac{(p-1)!\,(p-2)!}{2(2p-4)!} S(p,3) = p\,.$$

Thus the sequence $\{f_k(p)\}$ defined by (36) coincides with that mentioned in Theorem 3.

Now by Theorem 2 we have

$$c_{p,q} = -(2q-1)(q-1)\frac{S(p,2q-3)}{S(p,2q-1)}\,.$$

We can rewrite this by employing (36) in terms of $f_k(p)$:

$$c_{p,q} = (2q-1)(q-1)(2p-2q+1)\frac{f_{q-2}(p)}{f_{q-1}(p)}\,,$$

which is the desired result.

PROOF OF THEOREM 4. By Lemma 15

$$\text{(42)} \qquad \alpha_{2p}\int_{-\infty}^{\infty} f\left(x+\frac{2}{n}t\right)\left(\frac{\sin t}{t}\right)^{2p} dt$$

$$= \frac{\alpha_{2p}\rho_0}{(2p-1)!\,n^{2p-1}}\int_{-\pi}^{\pi} f(x+t)\left(\frac{1}{2}+\sum_{1}^{pn-1}\left(\frac{\rho_k}{\rho_0}\right)\cos kt\right) dt\,.$$

On the other hand, it is known [5, p. 254] that

$$\int_0^{\infty}\left(\frac{\sin t}{t}\right)^n dt = \frac{\pi}{2^n(n-1)!}\sum_{\nu=0}^{[(n-1)/2]}(-1)^{\nu}\binom{n}{\nu}(n-2\nu)^{n-1}\,.$$

From this it follows at once that

$$\alpha_{2p}^{-1} = \int_{-\infty}^{\infty}\left(\frac{\sin t}{t}\right)^{2p} dt = \frac{\pi}{2^{2p-1}(2p-1)!}\sum_{\nu=0}^{p-1}(-1)^{\nu}\binom{2p}{\nu}(2p-2\nu)^{2p-1}$$

$$= \frac{\pi}{(2p-1)!}\sum_{\nu=1}^{p}(-1)^{p-\nu}\binom{2p}{p-\nu}\nu^{2p-1} = \frac{\pi}{(2p-1)!}(-1)^p S(p,2p-1)\,.$$

In virtue of Lemma 15

$$\rho_0 = \sum_{\nu=1}^{p} (-1)^{p+\nu+1} \binom{2p}{p-\nu} (-\nu n)^{2p-1} \tag{43}$$

$$= n^{2p-1} \sum_{\nu=1}^{p} (-1)^{p+\nu} \binom{2p}{p-\nu} \nu^{2p-1} = n^{2p-1} (-1)^p S(p, 2p-1).$$

Combination of the last two relations yields

$$\alpha_{2p} \rho_0 = \frac{1}{\pi} (2p-1)!\, n^{2p-1}.$$

Thus from (42) it follows that

$$J_n^{(2p)}(f; x) = \alpha_{2p} \int_{-\infty}^{\infty} f\left(x + \frac{2}{n} t\right) \left(\frac{\sin t}{t}\right)^{2p} dt \tag{44}$$

$$= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x+t) \left(\frac{1}{2} + \sum_{1}^{pn-1} \left(\frac{\rho_k}{\rho_0}\right) \cos kt\right) dt$$

$$= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x+t) \left(\frac{1}{2} + \sum_{1}^{pn-1} \tilde{\rho}_k \cos kt\right) dt, \qquad \text{say,}$$

where ρ_k are given by (25).

Next we show that the linear operator (44) satisfies the conditions of Lemma 2. The positivity of the operator follows from Lemma 13 and (23), namely,

$$J_n^{(2p)}(f; x) = \frac{\alpha_{2p}}{2} n \int_{-\pi}^{\pi} f(x+t) \left(\frac{\sin \frac{nt}{2}}{n \sin \frac{t}{2}}\right)^{2p} \left(\sum_{\nu=1}^{p} a_\nu^{(2p-2)} \sin^{2p-2\nu} \frac{t}{2}\right) dt$$

$$= \frac{\alpha_{2p}}{2n^{2p-1}} \int_{-\pi}^{\pi} f(x+t) \sin^{2p} \frac{nt}{2} \left(\sum_{k=-\infty}^{\infty} \frac{1}{\left(\frac{t}{2} + k\pi\right)^{2p}}\right) dt.$$

Further clearly

$$\frac{1-\tilde{\rho}_2}{1-\tilde{\rho}_1} = \frac{\rho_0 - \rho_2}{\rho_0 - \rho_1}. \tag{45}$$

From Lemma 15 it follows that

$$(46)\qquad \rho_0-\rho_2=\sum_{\nu=1}^{p}(-1)^{p+\nu+1}\binom{2p}{p-\nu}\{(\nu n-2)^{2p-1}-(\nu n)^{2p-1}\}$$

$$=\sum_{\nu=1}^{p}(-1)^{p+\nu+1}\binom{2p}{p-\nu}\left\{-2\binom{2p-1}{1}\nu^{2p-2}n^{2p-2}\right.$$

$$\left.+4\binom{2p-1}{2}\nu^{2p-3}n^{2p-3}-\cdots\right\}$$

$$=(-1)^{p+1}\left\{-2\binom{2p-1}{1}S(p,2p-2)n^{2p-2}\right.$$

$$\left.+4\binom{2p-1}{2}S(p,2p-3)\,n^{2p-3}-\cdots\right\}$$

$$=(-1)^{p+1}\left\{4\binom{2p-1}{2}S(p,2p-3)\,n^{2p-3}\right.$$

$$\left.+16\binom{2p-1}{4}S(p,2p-5)\,n^{2p-5}+\cdots\right\},$$

since by Lemma 10

$$S(p,2p-2)=S(p,2p-4)=\cdots=0\,.$$

In exactly the same manner,

$$(47)\qquad \rho_0-\rho_1=(-1)^{p+1}\left\{\binom{2p-1}{2}S(p,2p-3)\,n^{2p-3}\right.$$

$$\left.+\binom{2p-1}{4}S(p,2p-5)\,n^{2p-5}+\cdots\right\}.$$

Therefore from (45), (46) and (47) we conclude that

$$\lim_{n\to\infty}\frac{1-\tilde{\rho}_2}{1-\tilde{\rho}_1}=4\,.$$

Hence we can apply Lemma 2 to complete the proof. Evidently

$$(48)\qquad 1-\tilde{\rho}_1=\frac{\rho_0-\rho_1}{\rho_0}\,.$$

From (43), (47) and (48) we conclude that

$$1-\tilde{\rho}_1 = -(2p-1)(p-1)\frac{S(p,2p-3)}{S(p,2p-1)n^2} + o\left(\frac{1}{n^2}\right)$$

$$= c_{p,p}\frac{1}{n^2} + o\left(\frac{1}{n^2}\right).$$

On the basis of Lemma 2 the proof of our theorem is completed.

REFERENCES

[1] L. B. W. JOLLEY, Summation of Series (2nd revised edition), Dover, 1961.
[2] P. P. KOROVKIN, On convergence of linear positive operators in the space of continuous functions, Dokl. Akad. Nauk SSSR (N. S.), 90(1953), 961-964 (Russian).
[3] ————, An asymptotic property of positive methods of summation of Fourier series and best approximation to functions of class Z_2 by linear positive polynomial operators, Uspehi Mat. Nauk, 13(1958), no. 6(84), 99-103 (Russian).
[4] Y. MATSUOKA, Note on Komleva's theorem, Sci. Rep. Kagoshima Univ., 9(1960), 17-23.
[5] S. MORIGUCHI, K. UDAGAWA AND S. HITOTUMATU, A table of mathematical formulas I. Iwanami Shoten, 1956, (Japanese).
[6] I. P. NATANSON, On the exactness of representations of continuous, periodic functions by singular integrals, Dokl. Akad. Nauk SSSR, 73(1950), 273-276 (Russian).
[7] F. SCHURER, Positive linear operators in approximation theory, Mathematical Institute of the Technological University Delft, Report, 1962.
[8] ————, Some remarks on the approximation of functions by some positive linear operators, Monatsh. Math., 67 (1963), 353-358.

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