CONTACT HYPERSURFACES IN CERTAIN KAEHLERIAN MANIFOLDS

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Introduction. An odd dimensional differentiable manifold M^{2n+1} is called an almost contact manifold if the structure group of its tangent bundle can be reduced to $U(n) \times 1$, where $U(n)$ means the real representation of the unitary group of n complex variables. Recently S. Sasaki [12] proved that the condition of almost contact manifold can be represented by a set of tensor fields ϕ, ξ, η which satisfy certain conditions. From this fact it is known that any orientable hypersurface of an almost complex manifold admits the structure of an almost contact manifold.¹⁾ The study of hypersurfaces of an almost complex manifold is perhaps one of the most fruitful aspects of the theory of an almost contact manifold. In fact, M. Kurita [8], Y. Tashiro and S. Tachibana [15] and the present author [9, 10, 11] proved many theorems on certain hypersurface of an almost complex manifold. If the induced almost contact metric of a hypersurface of almost Hermitian manifold is contact metric we call the hypersurface a contact hypersurface.

In this paper, we mainly study the differential geometric properties of contact hypersurfaces of Kaehlerian manifold of constant holomorphic sectional curvature. In particular, as an even-dimensional Euclidean space E^{2n} can be regarded as a Kaehlerian manifold with vanishing sectional curvature, the results obtained apply to contact hypersurface of a Euclidean space.

In §1, we give first of all the definition of an almost contact metric manifold and, in §2, a short summary of those parts of the theory of hypersurfaces in an almost Hermitian manifold which are necessary for what follows. Moreover in §2 some general properties of contact hypersurfaces of a Kaehlerian manifold are derived and in §3 some examples of contact hypersurfaces of a Kaehlerian manifold are exhibited.

After these preliminaries we consider in §4 hypersurfaces of Kaehlerian manifold of constant holomorphic sectional curvature and prove that there are at most three distinct principal curvatures of a contact hypersurface in

1) Y. Tashiro [14].

this Kaehlerian manifold. Since we can show that the principal curvatures of contact hypersurfaces of a Kaehlerian manifold of constant holomorphic sectional curvature are all constants, the characteristic vector spaces of vectors corresponding to the principal curvatures define distributions over the hypersurfaces. The integrability of these distributions is discussed in §5 and the following fact is proved: in any Kaehlerian manifold of positive constant holomorphic sectional curvature, the only contact hypersurface is C -umbilical hypersurface.

Finally in §6 we give a complete classification of the contact hypersurfaces in a Euclidean space.

1. Almost contact metric manifold. On a $(2n-1)$ -dimensional real differentiable manifold M^{2n-1} with local coordinate system $\{x^i\}$, if there exist a tensor field ϕ_j^i , contravariant and covariant vector fields ξ^i and η_i satisfying the relations

$$(1.1) \quad \xi^i \eta_i = 1,$$

$$(1.2) \quad \text{rank}(\phi_j^i) = 2n-2,$$

$$(1.3) \quad \phi_j^i \xi^j = 0, \quad \phi_j^i \eta_i = 0,$$

$$(1.4) \quad \phi_j^i \phi_k^j = -\delta_k^i + \xi^i \eta_k,$$

then the set $(\phi_j^i, \xi^i, \eta_j)$ is called an almost contact structure. The manifold with such a structure is called an almost contact manifold. S. Sasaki [12] proved recently that this definition of the almost contact manifold is equivalent to that used in J. W. Gray's paper [6]. Furthermore he proved that an almost contact manifold always admits a positive definite Riemannian metric tensor g_{ji} satisfying

$$(1.5) \quad g_{ji} \xi^i = \eta_j,$$

$$(1.6) \quad g_{ji} \phi_k^j \phi_h^i = g_{kh} - \eta_h \eta_k.$$

The Riemannian metric g_{ji} with above properties is called an associated Riemannian metric and the almost contact manifold with this metric is called an almost contact metric manifold. In this paper, we use always the associated Riemannian metric, consequently we use a notation η^i in stead of ξ^i .

If M^{2n-1} is a differentiable manifold of dimension $2n-1$ and there is defined over M^{2n-1} a differentiable pfaffian form η having the property that

$$(1.7) \quad \eta \wedge \overbrace{d\eta \wedge \cdots \wedge d\eta}^{n-1} \neq 0$$

on M^{2n-1} , then M^{2n-1} is called a contact manifold.²⁾

In an almost contact metric manifold if there is a function ρ which takes a value zero nowhere and satisfies a relation

$$(1.8) \quad 2\rho\phi_{ji} = \partial_j\eta_i - \partial_i\eta_j,$$

then the rank of the matrix (ϕ_{ji}) being $2n-2$, we have

$$\eta \wedge \overbrace{\phi \wedge \cdots \wedge \phi}^{n-1} = \rho^{-n} \eta \wedge \overbrace{d\eta \wedge \cdots \wedge d\eta}^{n-1} \neq 0,$$

from which we can regard that the manifold is a contact manifold. So, we call such an almost contact manifold a contact metric manifold. On the other hand, if in M^{2n-1} there exists a 1-form η satisfying (1.7) and we put ϕ_{ji} by (1.8), we can find³⁾ a Riemannian metric g_{ji} such that the set $(g^{ir}\phi_{jr}, g^{ir}\eta_r, \eta_i)$ defines an almost contact structure and the Riemannian metric is the associated one. This means that any contact manifold is regarded as an almost contact metric manifold.

2. Hypersurfaces in Kaehlerian manifold and the induced almost contact metric structure. Let $\bar{M}^{2n}$ be a $2n$ -dimensional differentiable manifold with local coordinates $\{X^\kappa\}$. If in $\bar{M}^{2n}$ there is a tensor field F_λ^κ which satisfies

$$(2.1) \quad F_\lambda^\mu F_\mu^\kappa = -\delta_\lambda^\kappa,$$

the tensor F_λ^κ is called an almost complex structure and $\bar{M}^{2n}$ with this structure an almost complex manifold. It is well known that in an almost complex manifold $\bar{M}^{2n}$ there always exists a Riemannian metric tensor satisfying

$$(2.2) \quad G_{\kappa\lambda} F_\mu^\kappa F_\nu^\lambda = G_{\mu\nu},$$

which is called the Hermitian metric. The pair $(F_\lambda^\kappa, G_{\lambda\kappa})$ with the above properties is called an almost Hermitian structure and the manifold $\bar{M}^{2n}$ an almost Hermitian manifold. In an almost Hermitian manifold, if the almost complex structure F_λ^κ satisfies the condition

$$(2.3) \quad \bar{\nabla}_\mu F_{\lambda\kappa} = 0,$$

where $\bar{\nabla}$ denotes the covariant differentiation with respect to the Hermitian

2) W. M. Boothby and H. C. Wang [2].

3) S. Sasaki [13], Y. Hatakeyama [7].

metric, the manifold is said to be Kaehlerian.

Let $\bar{M}^{2n}$ be an almost Hermitian manifold with local coordinates $\{X^\kappa\}$ and $(F_\lambda^\kappa, G_{\lambda\kappa})$ be the almost Hermitian structure. We denote by M^{2n-1} the differentiable hypersurface which is defined by the parametric representation $X^\kappa = X^\kappa(x^i)$.

Assuming that the hypersurface M^{2n-1} be orientable, we put $B_i^\kappa = \partial_i X^\kappa$ ($\partial_i = \partial/\partial x^i$). Then $2n-1$ vectors B_i^κ span the tangent hyperplane of M^{2n-1} at each point of M^{2n-1} . We denote by C^κ the unit normal vector to the hypersurface. The $2n$ vectors B_i^κ, C^κ being linearly independent, we regard that they form a local basis of a vector space at each point of M^{2n-1} . If we denote by (B^i_κ, C_κ) the dual basis of (B_i^κ, C^κ) , it follows that

$$(2.4) \quad B_i^\kappa B^j_\kappa = \delta^j_i, \quad B_i^\kappa C_\kappa = 0, \quad B^i_\kappa C^\kappa = 0, \quad C^\kappa C_\kappa = 1,$$

$$(2.5) \quad B_i^\kappa B^i_\lambda + C^\kappa C_\lambda = \delta^\kappa_\lambda.$$

The Riemannian metric g_{ji} on M^{2n-1} is given by

$$(2.6) \quad g_{ji} = G_{\lambda\kappa} B_j^\lambda B_i^\kappa.$$

This is called the induced Riemannian metric. In the following discussions we always use the induced Riemannian metric.

Denoting by ∇_j the covariant differentiation with respect to the induced metric, we have the following Gauss and Weingarten equations for the hypersurface respectively :

$$(2.7) \quad \nabla_j B_i^\kappa = H_{ji} C^\kappa,$$

$$(2.8) \quad \nabla_j C_\kappa = -H_{ji} B^i_\kappa,$$

where H_{ji} is the second fundamental tensor of the hypersurface.

Now we put

$$(2.9) \quad \phi_j^i = B_j^\lambda F_\lambda^\kappa B_i^\kappa,$$

$$(2.10) \quad \eta_j = B_j^\mu F_\mu^\lambda C_\lambda = B_j^\mu F_{\mu\lambda} C^\lambda,$$

then the set $(\phi_j^i, g^{ir} \eta_r, \eta_j)$ defines an almost contact structure and the induced Riemannian metric g_{ji} is the associated one.⁴⁾

In this paper, we call a hypersurface with the induced almost contact structure an almost contact hypersurface.

4) Y. Tashiro [14].

In order to express the covariant derivatives of η_i and ϕ_{ji} in terms of the second fundamental tensor H_{ji} , we differentiate (2.9), (2.10) covariantly and we get

$$\nabla_j \eta_i = B_i^\mu B_j^\nu \bar{\nabla}_\nu F_{\mu\lambda} C^\lambda - \phi_i^r H_{jr},$$

$$\nabla_j \phi_{ih} = B_i^\mu B_j^\kappa \bar{\nabla}_\kappa F_{\mu\lambda} B_h^\lambda + \eta_i H_{jh} - \eta_h H_{ji}.$$

If the Hermitian manifold $\bar{M}^{2n}$ is Kaehlerian, the above two equations reduce to the following formulas:

$$(2.11) \quad \nabla_j \eta_i = -\phi_i^r H_{jr},$$

$$(2.12) \quad \nabla_j \phi_{ih} = \eta_i H_{jh} - \eta_h H_{ji}.$$

Now, in what follows, we always denote by $\bar{M}^{2n}$ a Kaehlerian manifold and by M^{2n-1} its almost contact hypersurface. If M^{2n-1} is a contact metric manifold, that is, M^{2n-1} satisfies the relation (1.8) we call M^{2n-1} the contact hypersurface. In an almost contact hypersurface of a Kaehlerian manifold, the condition (1.8) is equivalent to

$$(2.13) \quad H_j^r \phi_r^i + \phi_j^r H_r^i = 2\rho \phi_j^i.$$

First we prove the

LEMMA 2.1. *In a contact hypersurface M^{2n-1} of a Kaehlerian manifold, the vector η^i is a characteristic vector of the second fundamental tensor H_j^i at each point of M^{2n-1} .*

PROOF. Transvecting (2.13) with η^j , we get

$$\phi_r^i H_j^r \eta^j = 0.$$

Operating ϕ_i^h to thus obtained vector, we have

$$(2.14) \quad H_{ji} \eta^j = \alpha \eta_i, \quad (\alpha = H_{ji} \eta^j \eta^i).$$

This concludes the proof.

Next we try to express the mean curvature of a contact hypersurface of a Kaehlerian manifold by means of the functions ρ and α . Transvecting (2.13) with ϕ_{ik} , we obtain

$$(2.15) \quad H_{jk} + H_{ri} \phi_j^r \phi_k^i - 2\rho g_{jk} - (\alpha - 2\rho) \eta_j \eta_k = 0,$$

because of (2.14). Hence we have

$$(2.16) \quad H_r^r = \alpha + 2(n-1)\rho.$$

3. Examples of contact hypersurfaces. Before we discuss contact hypersurfaces of a Kaehlerian manifold, we show some examples of contact hypersurfaces.

Let E^{2n} be a Euclidean space with rectangular coordinates X^k ($k=1, 2, \dots, 2n$) and M^{2n-1} be a hypersurface defined by the equation

$$(3.1) \quad \sum_{\alpha=1}^n (X^{n+\alpha})^2 = a^2.$$

This is a product manifold of $(n-1)$ -dimensional sphere S^{n-1} with n -dimensional Euclidean space E^n . The unit normal vector C^k to the hypersurface has the components:

$$(3.2) \quad C^k = (0, \dots, 0, a^{-1}X^{n+1}, \dots, a^{-1}X^{2n}).$$

Differentiating (3.1), we have

$$(3.3) \quad \sum_{\alpha=1}^n X^{n+\alpha} B_j^{n+\alpha} = 0,$$

that is, from (3.2),

$$(3.4) \quad \sum_{\alpha=1}^n B_j^{n+\alpha} C^{n+\alpha} = 0.$$

By virtue of (3.1) and (3.2), it follows that

$$(3.5) \quad H_{kj} = -a^{-1} \sum_{\alpha=1}^n B_k^{n+\alpha} B_j^{n+\alpha}.$$

So we know that

$$(3.6) \quad \begin{aligned} \nabla_k \eta_j - \nabla_j \eta_k &= -H_{kr} \phi_j^r + H_{jr} \phi_k^r \\ &= a^{-1} \sum_{\alpha=1}^n (B_k^{n+\alpha} B_r^{n+\alpha} \phi_j^r - B_j^{n+\alpha} B_r^{n+\alpha} \phi_k^r). \end{aligned}$$

However, in general, it follows that

$$B_r^\lambda \phi_j^r = B_r^\lambda B_r^\mu F_\mu^\nu B_j^\nu = (\delta_\mu^\lambda - C^\lambda C_\mu) F_\mu^\nu B_j^\nu = F_\nu^\lambda B_j^\nu - C^\lambda \eta_j,$$

and consequently

$$\sum_{\alpha=1}^n B_k^{n+\alpha} B_j^{n+\alpha} \phi_j^r = \sum_{\alpha=1}^n B_k^{n+\alpha} (F_\mu^{n+\alpha} B_j^\mu - C^{n+\alpha} \eta_j).$$

In E^{2n} the almost complex structure F_μ^λ having the following components

$$(F_\mu^\lambda) = \begin{pmatrix} 0 & \delta_\beta^\alpha \\ -\delta_\beta^\alpha & 0 \end{pmatrix} \quad (\alpha, \beta = 1, \dots, n),$$

the above equation is reduced to

$$\sum_{\alpha=1}^n B_k^{n+\alpha} B_r^{n+\alpha} \phi_j^r = \sum_{\alpha, \beta=1}^n B_k^{n+\alpha} \delta_\beta^\alpha B_j^\beta = \sum_{\alpha=1}^n B_k^{n+\alpha} B_j^\alpha,$$

because of (3.4). Substituting this into (3.6), we have

$$(3.7) \quad \partial_k \eta_j - \partial_j \eta_k = a^{-1} \sum_{\alpha=1}^n (B_k^{n+\alpha} B_j^\alpha - B_j^{n+\alpha} B_k^\alpha).$$

On the other hand we have

$$\begin{aligned} \phi_{kj} &= B_k^\lambda F_{\lambda\mu} B_j^\mu = \sum_{\alpha, \beta=1}^n (B_k^{n+\alpha} \delta_{\alpha\beta} B_j^\beta - B_j^{n+\beta} \delta_{\alpha\beta} B_k^\alpha) \\ &= \sum_{\alpha=1}^n (B_k^{n+\alpha} B_j^\alpha - B_k^\alpha B_j^{n+\alpha}). \end{aligned}$$

Thus we get

$$\partial_k \eta_j - \partial_j \eta_k = a^{-1} \phi_{kj}.$$

THEOREM 3.1. *The hypersurface defined by (3.1) in Euclidean space E^{2n} , that is, a product manifold of n -dimensional Euclidean space with $(n-1)$ -dimensional sphere is a contact hypersurface of E^{2n} .*

In the second place, we consider a totally umbilical hypersurface M^{2n-1}

of a Kaehlerian manifold $\bar{M}^{2n}$. Then it satisfies that

$$H_{ji} = \sigma g_{ji},$$

from which

$$\nabla_k \eta_j - \nabla_j \eta_k = -\phi_j^r H_{rk} + \phi_k^r H_{rj} = 2\sigma \phi_{kj}.$$

Consequently, we have

THEOREM 3.2.⁵⁾ *A totally umbilical hypersurface of a Kaehlerian manifold with non-zero mean curvature is a contact hypersurface.*

As a special case of Theorem 3.2, we get

THEOREM 3.3.⁶⁾ *A $(2n-1)$ -dimensional sphere is a contact hypersurface of a $2n$ -dimensional Euclidean space.*

Finally, we mention an example of a contact hypersurface in a Kaehlerian manifold of constant holomorphic sectional curvature. Let K be a complex number space of complex dimension n and denote its coordinates by z^α and their conjugates by $\bar{z}^\alpha$.

For arbitrary k , we put

$$\Phi = \left\{ \log(1 + 2k \sum_{\alpha=1}^n z^\alpha \bar{z}^\alpha) \right\} / 2k \equiv (\log S) / 2k,$$

and

$$G_{\alpha\bar{\beta}} = \partial_\alpha \partial_{\bar{\beta}} \Phi, \quad G_{\alpha\beta} = G_{\bar{\alpha}\bar{\beta}} = 0 \quad (\alpha, \beta = 1, \dots, n),$$

that is

$$G_{\alpha\bar{\beta}} = G_{\bar{\beta}\alpha} = (S\delta_{\alpha\beta} - 2kz^\alpha \bar{z}^\beta) / S^2, \quad G_{\alpha\beta} = G_{\bar{\alpha}\bar{\beta}} = 0.$$

Then it is well known⁷⁾ that the connected component of K , where S does not vanish, is a Kaehlerian manifold with respect to the metric $G_{\lambda\kappa}$. We call the Kaehlerian manifold Fubinian manifold and denote it by $\bar{M}$. A Fubinian manifold has constant holomorphic sectional curvature. The non-trivial components of the Christoffel symbols of $\bar{M}$ are

5) Y. Tashiro [14].

6) S. Sasaki and Y. Hatakeyama [13].

7) For example, S. Bochner [1].

$$(3.8) \quad \left\{ \begin{matrix} \alpha \\ \beta\gamma \end{matrix} \right\} = -2k(\delta_\gamma^\alpha z^\beta + \delta_\beta^\alpha z^\gamma)/S$$

and their conjugates. Consider in $\bar{M}$ a hypersurface M defined by

$$(3.9) \quad \sum_{\alpha=1}^n z^\alpha z^{\bar{\alpha}} = a^2.$$

The unit normal vector C^κ and the induced Riemannian metric on M are respectively given by

$$(3.10) \quad C_\kappa = (z^{\bar{\alpha}}/\sqrt{2} aS, \quad z^\alpha/\sqrt{2} aS),$$

$$(3.11) \quad g_{ji} = S^{-1} \sum_{\alpha=1}^n (B_i^{\bar{\alpha}} B_j^\alpha + B_i^\alpha B_j^{\bar{\alpha}}) - 4a^2 k \sum_{\alpha, \beta=1}^n (B_j^\alpha C_\alpha B_i^{\bar{\beta}} C_\beta + B_j^{\bar{\alpha}} C_{\bar{\alpha}} B_i^\beta C_\beta).$$

Since F_μ^λ has the components

$$(F_\mu^\lambda) = \begin{pmatrix} \sqrt{-1} \delta_\beta^\alpha & 0 \\ 0 & -\sqrt{-1} \delta_\beta^\alpha \end{pmatrix}$$

in our coordinates, we have from (2.10),

$$(3.12) \quad \eta_j = \sqrt{-1} \sum_{\alpha=1}^n (B_j^\alpha C_\alpha - B_j^{\bar{\alpha}} C_{\bar{\alpha}}).$$

On the other hand, differentiating (3.9), we get

$$\sum_{\alpha=1}^n (B_i^\alpha z^{\bar{\alpha}} + B_i^{\bar{\alpha}} z^\alpha) = 0,$$

from which

$$\sum_{\alpha=1}^n \left[H_{ji} (C^\alpha z^{\bar{\alpha}} + C^{\bar{\alpha}} z^\alpha) + B_i^\alpha B_j^{\bar{\alpha}} + B_i^{\bar{\alpha}} B_j^\alpha - B_i^\kappa B_j^\lambda \left(\left\{ \begin{matrix} \alpha \\ \kappa\lambda \end{matrix} \right\} z^{\bar{\alpha}} + \left\{ \begin{matrix} \bar{\alpha} \\ \kappa\lambda \end{matrix} \right\} z^\alpha \right) \right] = 0.$$

Substituting (3.8) and (3.10) into the above and making use of the fact that C^κ is the unit normal vector to M , we get

$$(3.13) \quad H_{ji} = \frac{-1}{\sqrt{2} aS} \left\{ \sum_{\alpha=1}^n (B_i^\alpha B_j^{\bar{\alpha}} + B_i^{\bar{\alpha}} B_j^\alpha) + \sum_{\beta, \gamma=1}^n 8ka^2 S (B_i^\beta C_\beta B_j^\gamma C_\gamma + B_i^{\bar{\beta}} C_{\bar{\beta}} B_j^{\bar{\gamma}} C_{\bar{\gamma}}) \right\}.$$

By means of (3.11), (3.12) and (3.13), we obtain

$$(3.14) \quad H_{ji} = -a^{-1}(g_{ji} + 2a^2 k \eta_j \eta_i) / \sqrt{2}.$$

Consequently it follows that

$$\partial_j \eta_i - \partial_i \eta_j = \nabla_j \eta_i - \nabla_i \eta_j = -\sqrt{2} a^{-1} \phi_{ji},$$

because of (2.11). This means that the hypersurface M is a contact hypersurface.

THEOREM 3.4.⁸⁾ *The hypersurface of a Fubinian manifold which is defined by (3.9) is a contact hypersurface.*

4. Kaehlerian manifolds of constant holomorphic sectional curvature.

Let Y^* be a vector in $\bar{M}^{2n}$, then the holomorphic sectional curvature with respect to the vector Y^* is given by

$$(4.1) \quad K = - \frac{\bar{R}_{v\mu\lambda\kappa} F_\pi^\mu F_\sigma^\kappa Y^v Y^\pi Y^\lambda Y^\sigma}{G_{\lambda\mu} Y^\lambda Y^\mu G_{\kappa\nu} Y^\kappa Y^\nu}$$

where $\bar{R}_{v\mu\lambda\kappa}$ is the covariant components of the curvature tensor of $\bar{M}^{2n}$. When the holomorphic sectional curvature is always constant with respect to any vector at each point of $\bar{M}^{2n}$, we call the manifold $\bar{M}^{2n}$ that of constant holomorphic sectional curvature. It has been proved that the Kaehlerian manifold of constant holomorphic sectional curvature has the curvature tensor of the form:⁹⁾

$$(4.2) \quad \bar{R}_{v\mu\lambda\kappa} = k(G_{\mu\lambda} G_{v\kappa} - G_{v\lambda} G_{\mu\kappa} + F_{\mu\lambda} F_{v\kappa} - F_{v\lambda} F_{\mu\kappa} - 2F_{v\mu} F_{\lambda\kappa}),$$

where $k=K/4$ is constant.

Substituting (4.2) into the Gauss and Codazzi equations

$$(4.3) \quad R_{kji h} = B_k^v B_j^\mu B_i^\lambda B_h^\kappa \bar{R}_{v\mu\lambda\kappa} + H_{kh} H_{ji} - H_{jh} H_{ki},$$

$$(4.4) \quad \nabla_k H_{ji} - \nabla_j H_{ki} = B_k^v B_j^\mu B_i^\lambda C^\kappa \bar{R}_{v\mu\lambda\kappa},$$

8) M. Kurita [8]. Y. Tashiro and S. Tachibana [15].

9) K. Yano and I. Mogi [18].

we have

$$(4.5) \quad R_{kji h} = k(g_{ji} g_{kh} - g_{ki} g_{jh} + \phi_{kh} \phi_{ji} - \phi_{jh} \phi_{ki} - 2\phi_{kj} \phi_{ih}) + H_{ji} H_{kh} - H_{ki} H_{jh},$$

$$(4.6) \quad \nabla_k H_{ji} - \nabla_j H_{ki} = k(\eta_k \phi_{ji} - \eta_j \phi_{ki} - 2\phi_{kj} \eta_i),$$

from which we get the following identities:

$$(4.7) \quad \nabla^r H_{jr} - \nabla_j H_r{}^r = 0,$$

$$(4.8) \quad (\nabla_k H_{ji} - \nabla_j H_{ki}) \eta^i = -2k\phi_{kj},$$

$$(4.9) \quad (\nabla_k H_{ji} - \nabla_j H_{ki}) \eta^k = k\phi_{ji},$$

$$(4.10) \quad \nabla_k H_{ji} \phi^{ki} = -2(n-1)k\eta_j.$$

Now we consider a contact hypersurface in Kaehlerian manifold of constant holomorphic sectional curvature. Differentiating (2.14) covariantly and making use of (2.11), we get

$$\nabla_k H_{ji} \eta^i - H_j{}^i \phi_i{}^r H_{rk} = \nabla_k \alpha \eta_j + \alpha \nabla_k \eta_j,$$

which implies that the exterior derivative of a 1-form

$$\alpha \eta = (\alpha \eta_i) dx^i$$

is given by, in the components,

$$\begin{aligned} & (\nabla_k H_{ji} - \nabla_j H_{ki}) \eta^i - H_j{}^i \phi_i{}^r H_{rk} + H_k{}^i \phi_i{}^r H_{rj} \\ & = \nabla_k \alpha \eta_j - \nabla_j \alpha \eta_k + \alpha (\nabla_k \eta_j - \nabla_j \eta_k), \end{aligned}$$

or

$$(4.11) \quad -2k\phi_{kj} - H_j{}^i \phi_i{}^r H_{rk} + H_k{}^i \phi_i{}^r H_{rj} = \nabla_k \alpha \eta_j - \nabla_j \alpha \eta_k + 2\rho \alpha \phi_{kj}$$

because of (1.8) and (4.8).

Transvecting (4.11) with η^j and taking account of Lemma 2.1, we have

$$\nabla_j \alpha = \beta \eta_j, \quad (\beta = \eta^r \nabla_r \alpha),$$

that is

$$(4.12) \quad d\alpha = \beta \eta,$$

from which

$$d\beta \wedge \eta + \beta d\eta = 0,$$

or in the components form,

$$\nabla_j \beta \eta_i - \nabla_i \beta \eta_j + 2\rho \beta \phi_{ji} = 0,$$

because of (1.8). Transvecting this with ϕ^{ji} , we get

$$4(n-1)\rho\beta = 0,$$

from which, ρ being non-zero valued function, we have $\alpha = \text{const.}$

LEMMA 4.1. *In a contact hypersurface of a Kaehlerian manifold of constant holomorphic sectional curvature, the characteristic root α in (2.14) is constant.*

In the following discussions, we denote the constant by a .

The tensor $H_{ji}H_{rk} + H_{ki}H_{rj}$ being symmetric with respect to i and r , we have

$$\phi^{ir}(H_{ji}H_{rk} + H_{ki}H_{rj}) = H_j^i \phi_i^r H_{rk} + H_k^i \phi_i^r H_{rj} = 0,$$

from which (4.11) is reduced to

$$(4.13) \quad (a\rho + k)\phi_{kj} + H_j^i \phi_i^r H_{rk} = 0,$$

because of Lemma 4.1. Substituting (2.13) into (4.13), we have

$$(a\rho + k)\phi_{kj} + (2\rho\phi_j^r - \phi_j^s H_s^r)H_{rk} = 0.$$

Transvecting this with ϕ_i^j , we get

$$(4.14) \quad H_{is}H_j^s - 2\rho H_{ji} + (a\rho + k)g_{ji} + (a\rho - a^2 - k)\eta_j\eta_i = 0.$$

Now we state the

LEMMA 4.2. *In a contact hypersurface M^{2n-1} of a Kaehlerian manifold of constant holomorphic sectional curvature, the function ρ is constant.*

PROOF. Differentiating (2.13) covariantly and making use of (2.12), we have

$$\begin{aligned} \nabla_k H_j^r \phi_r^i + H_j^r (\eta_r H_k^i - \eta^i H_{rk}) + \nabla_k H_r^i \phi_j^r + H_r^i (\eta_j H_k^r - \eta^r H_{kj}) \\ = 2\nabla_k \rho \phi_j^i + 2\rho (\eta_j H_k^i - \eta^i H_{jk}). \end{aligned}$$

By means of (2.14) this can be written as

$$\begin{aligned} \nabla_k H_j^r \phi_r^i + \nabla_k H_r^i \phi_j^r + \eta_j (a H_k^i + H_r^i H_k^r - 2\rho H_k^i) - \eta^i (a H_{kj} + H_j^r H_{rk} - 2\rho H_{jk}) \\ = 2\nabla_k \rho \phi_j^i. \end{aligned}$$

Substituting (4.14) into the above equation, we get

$$2\nabla_k \rho \phi_j^i = \nabla_k H_j^r \phi_r^i + \nabla_k H_r^i \phi_j^r + \{a H_k^i - (a\rho + k)\delta_k^i\} \eta_j - \{a H_{kj} - (a\rho + k)g_{kj}\} \eta^i,$$

from which, together with (2.14), it follows that

$$2\nabla_r \rho \phi_j^r = \nabla_i H_{jr} \phi^{ri} + \nabla_i H_r^i \phi_j^r + \{a H_r^r - (2n-1)(a\rho + k) - a^2 + (a\rho + k)\} \eta_j,$$

which implies that

$$(4.15) \quad (2\nabla_r \rho - \nabla_r H_i^i) \phi_j^r = 0,$$

by virtue of (2.16), (4.7) and (4.10). Transvecting this with ϕ_i^j , we have, for a suitable scalar function σ ,

$$(4.16) \quad \nabla_i (H_r^r - 2\rho) = \sigma \eta_i,$$

that is,

$$d(H_r^r - 2\rho) = (\nabla_i (H_r^r - 2\rho)) dx^i = (\sigma \eta_i) dx^i = \sigma \eta,$$

from which

$$d\sigma \wedge \eta + \sigma d\eta = 0.$$

By means of (1.8) we have in the components,

$$\eta_j \nabla_i \sigma - \eta_i \nabla_j \sigma + 2\rho \sigma \phi_{ji} = 0,$$

from which by contraction with ϕ^{ji} ,

$$4(n-1)\rho\sigma = 0.$$

Since at any point of M^{2n-1} , ρ can not vanish, this means that $\sigma=0$ and consequently

$$\nabla_i(H_r^r - 2\rho) = 0.$$

On the other hand, (2.16) and Lemma 4.1 tell us that

$$\nabla_i\{H_r^r - 2(n-1)\rho\} = 0.$$

From the last two equations it follows that ρ is constant and this proves the lemma.

Combining Lemma 4.1, 4.2 and equation (2.16), we get the

THEOREM 4.3. *The mean curvature of a contact hypersurface M^{2n-1} of a Kaehlerian manifold of constant holomorphic sectional curvature is constant.*

In the following we denote the constant by c instead of ρ .

Let λ be a characteristic root of the second fundamental tensor H_j^i which is distinct to that corresponding to η^i and let v^i be a corresponding characteristic vector to the root λ . Then, by virtue of (4.14), it follows that

$$(4.17) \quad \lambda^2 - 2c\lambda + ca + k = 0.$$

Thus the characteristic roots of the second fundamental tensor must satisfy the following algebraic equation of the third order:

$$(4.18) \quad (\lambda - a)(\lambda^2 - 2c\lambda + ca + k) = 0,$$

which shows that the hypersurface admits at most three distinct principal curvatures. Furthermore, the coefficients c , a and k being constants, we know that the principal curvatures are all constants.

THEOREM 4.4. *In a contact hypersurface of a Kaehlerian manifold of constant holomorphic sectional curvature, the second fundamental tensor has at most three distinct characteristic roots and they are all constants.*

5. Contact hypersurfaces in Kaehlerian manifolds of constant holomorphic sectional curvature. In order to get further results on contact hypersurfaces of Kaehlerian manifold of constant holomorphic sectional curvature, we consider the vector spaces spanned by the characteristic vectors of the second fundamental tensor. We denote the principal curvatures by a , λ_1 , λ_2 , their respective multiplicities by ν_0 , ν_1 , ν_2 and the characteristic vector spaces of vectors corresponding to the principal curvatures a , λ_1 , λ_2 by V_a , V_{λ_1} , V_{λ_2} . Then it follows that

$$(5.1) \quad \begin{cases} \dim V_a = \nu_0, \quad \dim V_{\lambda_i} = \nu_i, \quad (i=1, 2) \\ \nu_0 + \nu_1 + \nu_2 = 2n-1, \\ T_p(M^{2n-1}) = V_a \oplus V_{\lambda_1} \oplus V_{\lambda_2} \text{ (direct sum)}, \end{cases}$$

and that V_a , V_{λ_1} and V_{λ_2} are orthogonal to each other. Since the principal curvatures are all constants, we know that the multiplicities ν_0 , ν_1 and ν_2 are also constants and so the correspondence $P \in M^{2n-1}$ to V_a at P and the correspondence $P \in M^{2n-1}$ to V_{λ_i} ($i=1, 2$) at P define ν_0 and ν_i ($i=1, 2$) dimensional distributions over M^{2n-1} respectively. We denote these distributions by D_a and D_{λ_i} ($i=1, 2$). These distributions are complementary in the sense that they are disjoint and their Whitney sum is $T(M^{2n-1})$.

LEMMA 5.1. *For any v^i belonging to D_{λ_1} (or D_{λ_2}), the vector field $\phi_j^i v^j$ belongs to D_{λ_2} (or D_{λ_1}).*

PROOF. From (2.13) we have for a vector field belonging to D_{λ_1} ,

$$\phi_r^i H_j^r v^j + H_r^i \phi_j^r v^j = 2c \phi_j^i v^j.$$

On the other hand, (4.17) gives that $\lambda_1 + \lambda_2 = 2c$. Consequently $\phi_j^i v^j$ belongs to D_{λ_2} .

LEMMA 5.2. *Let M^{2n-1} be a contact hypersurface in a Kaehlerian manifold of constant holomorphic sectional curvature and D_a be the distribution spanned by the vectors corresponding to the principal curvature a . If $\dim D_a \geq 2$, M^{2n-1} admits at most two distinct principal curvatures.*

PROOF. Let v^i be a characteristic vector corresponding to the principal curvature a . Transvecting (4.14) with v^j and making use of $H_j^i v^j = a v^i$, we get

$$(a^2 - ca + k)(v_i - \eta_r v^r \eta_i) = 0.$$

If v^i is orthogonal to η^i , we have

$$(5.2) \quad k = ca - a^2.$$

On the other hand since the characteristic roots of the second fundamental tensor must be roots of the algebraic equation (4.18), the distinct roots to a are given by

$$(5.3) \quad \lambda_1 = c + \sqrt{c^2 - ca - k}, \quad \lambda_2 = c - \sqrt{c^2 - ca - k}.$$

Substituting (5.2) into (5.3), we obtain

$$\lambda_1 = 2c - a, \quad \lambda_2 = a.$$

Therefore it follows that H_j^i admits at most two distinct principal curvatures a and $2c - a$. This completes the proof.

According to Lemma 5.2, if a contact hypersurface of a Kaehlerian manifold of constant holomorphic sectional curvature admits three distinct principal curvatures, $\nu_0 = \dim D_a = 1$. In the first place, we consider a contact hypersurface which admits three distinct principal curvatures.

Now we prove the

LEMMA 5.3. *The distributions D_{λ_1} and D_{λ_2} are both integrable.*

PROOF. Let v^i and u^i be vector fields belonging to D_{λ_1} . Then we get

$$H_i^j v^i = \lambda_1 v^j, \quad H_i^j u^i = \lambda_1 u^j.$$

Therefore, for the bracket $[u, v]^i$ of two vector fields u^i and v^i , we have

$$\begin{aligned} H_j^i [u, v]^j &= H_j^i (u^r \nabla_r v^j - v^r \nabla_r u^j) \\ &= u^r \{ \nabla_r (H_j^i v^j) - \nabla_r H_j^i v^j \} - v^r \{ \nabla_r (H_j^i u^j) - \nabla_r H_j^i u^j \} \\ &= u^r (\lambda_1 \nabla_r v^i - \nabla_r H_j^i v^j) - v^r (\lambda_1 \nabla_r u^i - \nabla_r H_j^i u^j) \\ &= \lambda_1 [u, v]^i + (\nabla_r H_j^i - \nabla_j H_r^i) v^r u^j. \end{aligned}$$

Substituting (4.6) into the above, we get

$$H_j^i [u, v]^j = \lambda_1 [u, v]^i + k(\eta_r \phi_j^i - \eta_j \phi_r^i - 2\phi_{rj} \eta^i) v^r u^j.$$

The vector field η^i belonging to D_a and $\phi_r^i v^r$ belonging to D_{λ_2} , we have

$$H_j^i [u, v]^j = \lambda_1 [u, v]^i,$$

which shows that the distribution D_{λ_1} is integrable. In exactly the same way, we can show that D_{λ_2} is also integrable.

Lemma 5.1, 5.2 and (5.1) show us that if a contact hypersurface of a Kaehlerian manifold of constant holomorphic sectional curvature admits three

distinct principal curvatures, $\dim D_{\lambda_1} = \dim D_{\lambda_i} = n-1$. Now let $v_{(a)}^i$ be $n-1$ orthonormal vector fields belonging to D_{λ_1} , then $\phi_r^i v_{(a)}^r$ are $n-1$ orthonormal vector fields belonging to D_{λ_i} . So, the second fundamental tensor H_{ji} can be represented in the following form :

$$(5.4) \quad H_{ji} = a\eta_j\eta_i + \sum_{a=1}^{n-1} \lambda_1 v_{(a)}^i v_{(a)}^j + \sum_{a=1}^{n-1} \lambda_2 \phi_{(a)}^i v_{(a)}^r \phi_{(a)}^s v_{(a)}^s.$$

The distribution D_{λ_1} being integrable, at each point P of M^{2n-1} , we can find a neighbourhood U of P and $n-1$ functions f^a ($a=1, \dots, n-1$) such that

$$H_j^i \frac{\partial f^a}{\partial x^i} = \lambda_1 \frac{\partial f^a}{\partial x^j},$$

that is, $\frac{\partial f^a}{\partial x^i}$ belongs to D_{λ_1} . Consequently, without loss of generality, we regard that $v_{(a)}^i$ be gradient vectors in the neighbourhood U of P .

Differentiating (5.4) covariantly and making use of (2.12), we get

$$\begin{aligned} \nabla_k H_{ji} &= a(\nabla_k \eta_j \eta_i + \nabla_k \eta_i \eta_j) + \sum_{a=1}^{n-1} \lambda_1 (\nabla_k v_{(a)}^j v_{(a)}^i + \nabla_k v_{(a)}^i v_{(a)}^j) \\ &+ \sum_{a=1}^{n-1} \lambda_2 \{ (\eta_j H_k^r - \eta^r H_{jk}) v_{(a)}^r \phi_{(a)}^s v_{(a)}^s + \phi_j^r \nabla_k v_{(a)}^r \phi_{(a)}^s v_{(a)}^s \\ &+ \phi_j^r v_{(a)}^r (\eta_i H_k^s - \eta^s H_{ik}) v_{(a)}^s + \phi_j^r v_{(a)}^r \phi_{(a)}^s \nabla_k v_{(a)}^s \}. \end{aligned}$$

This implies that

$$\begin{aligned} \nabla_k H_{ji} - \nabla_j H_{ki} &= a(2c\eta_i \phi_{kj} + \eta_j \nabla_k \eta_i - \eta_k \nabla_j \eta_i) \\ &+ \sum_{a=1}^{n-1} \lambda_1 (\nabla_k v_{(a)}^j v_{(a)}^i - \nabla_j v_{(a)}^i v_{(a)}^k) + \sum_{a=1}^{n-1} \lambda_2 (\lambda_1 \eta_j v_{(a)}^k \phi_{(a)}^s v_{(a)}^s \\ &+ \phi_j^r \nabla_k v_{(a)}^r \phi_{(a)}^s v_{(a)}^s + \lambda_1 \eta_i \phi_j^r v_{(a)}^r v_{(a)}^k + \phi_j^r v_{(a)}^r \phi_{(a)}^s \nabla_k v_{(a)}^s \\ &- \lambda_1 \eta_k v_j \phi_{(a)}^s v_{(a)}^s - \phi_k^r \nabla_j v_{(a)}^r \phi_{(a)}^s v_{(a)}^s - \lambda_1 \eta_i \phi_k^r v_{(a)}^r v_{(a)}^j - \phi_k^r v_{(a)}^r \phi_{(a)}^s \nabla_j v_{(a)}^s) \end{aligned}$$

from which

$$\eta^k (\nabla_k H_{ji} - \nabla_j H_{ki}) = -a \nabla_j \eta_i + \sum_{a=1}^{n-1} \lambda_1 v_j \eta_k \nabla_i v_{(a)}^k$$

$$\begin{aligned}
& + \sum_{a=1}^{n-1} \lambda_2 (\phi_j^r \eta_k \nabla_r v^k \phi_i^s v_s + \phi_j^r v_r \phi_i^s \eta_k \nabla_s v^k - \lambda_1 v_j \phi_i^s v_s) \\
& = a \phi_i^r H_{rj} - \lambda_1 \sum_{a=1}^{n-1} v_j v^k \nabla_i \eta_k \\
& + \sum_{a=1}^{n-1} \lambda_2 (-\phi_j^r v^k \nabla_r \eta_k \phi_i^s v_s - \phi_j^r v_r \phi_i^s v^k \nabla_s \eta_k - \lambda_1 v_j \phi_i^s v_s) \\
& = a \phi_i^r H_{rj} + \lambda_1 \sum_{a=1}^{n-1} v_j v^l \phi_l^r H_{ri} \\
& + \lambda_2 \sum_{a=1}^{n-1} (\phi_j^r v^k \phi_k^l H_{lr} \phi_i^s v_s + \phi_j^r v_r \phi_i^s v^k \phi_k^l H_{ls} - \lambda_1 v_j \phi_i^s v_s).
\end{aligned}$$

By means of (4.9) and Lemma 5.1, this is reduced to

$$\begin{aligned}
k \phi_{ji} & = a \phi_i^r H_{rj} + \lambda_1 \lambda_2 \sum_{a=1}^{n-1} v_j \phi_{ki} v^k \\
& + \lambda_2 \sum_{a=1}^{n-1} (\lambda_2 \phi_i^s v_j v_s + \lambda_2 v_i \phi_j^r v_r - \lambda_1 v_j \phi_i^s v_s).
\end{aligned}$$

Transvecting this with v^j which belongs to D_{λ_1} , we have

$$\begin{aligned}
k \phi_{ji} v^j & = a \lambda_1 \phi_i^r v_r + \lambda_1 \lambda_2 \sum_{a=1}^{n-1} v_j v^j \phi_{ki} v^k \\
& + \lambda_2 \sum_{a=1}^{n-1} (\lambda_2 \phi_i^s v_s v_j v^j - \lambda_1 v_j v^j \phi_i^s v_s) \\
& = a \lambda_1 \phi_i^r v_r + \lambda_1 \lambda_2 \phi_{ki} v^k + \lambda_2^2 \phi_i^s v_s - \lambda_1 \lambda_2 \phi_i^s v_s.
\end{aligned}$$

Hence it follows that

$$(\lambda_2^2 - 2\lambda_1 \lambda_2 + a \lambda_1 + k) \phi_i^s v_s = 0.$$

Since $\phi_i^s v_s$ is non-zero vector, we get

$$(5.5) \quad \lambda_2^2 - 2\lambda_1 \lambda_2 + a \lambda_1 + k = 0.$$

This equation is valid at a point $P \in U$ where there are functions f^a satisfying

$\frac{\partial f^a}{\partial x^i} H_j^i = \lambda_1 \frac{\partial f^a}{\partial x^j}$. However, λ_i , a and k being all constants, the left hand members of (5.5) are also constant. Therefore (5.5) is also valid in the hypersurface M^{2n-1} .

Substituting (5.3) into (5.5), we have

$$2(c^2 - ac - k) = (2c - a)\sqrt{c^2 - ac - k}.$$

If $k = c^2 - ac$, then $\lambda_1 = \lambda_2 = c$. So, the hypersurface admits only two principal curvatures. Thus in the considering case we may assume that $k \neq c^2 - ac$ and get, after some elementary computations,

$$(5.6) \quad k = -a^2/4.$$

Thus we obtain the

THEOREM 5.4. *Let M^{2n-1} be a contact hypersurface of a Kaehlerian manifold of constant holomorphic sectional curvature. If M^{2n-1} admits three distinct principal curvatures, the enveloping Kaehlerian manifold has negative constant holomorphic sectional curvature which is given by (5.6).*

COROLLARY 5.5. *A contact hypersurface of a flat Kaehlerian manifold admits at most two principal curvatures. Furthermore, if it admits two distinct principal curvatures, one of them must be zero curvature.*

PROOF. From (5.6), it follows that $a=0$. Consequently $\lambda_1=2c$ and $\lambda_2=0$. This completes the proof.

Next we consider the integral submanifold of D_{λ_i} . Let v^i and w^i be any two mutually orthonormal vectors belonging to D_{λ_i} . Then the sectional curvature $K'_{(i)}$ with respect to the vectors v^i and w^i is given by

$$K'_{(i)} = -R_{kjih} v^k w^j v^i w^h,$$

because that v^i and w^i are unit vectors. One easily derive from (4.5) that the sectional curvature becomes to

$$K'_{(i)} = \{-k(g_{ji}g_{kh} - g_{ki}g_{jh} + \phi_{ji}\phi_{kh} - \phi_{ki}\phi_{jh} - 2\phi_{kj}\phi_{ih}) - H_{ji}H_{kh} + H_{ki}H_{jh}\} v^k w^j v^i w^h.$$

Since $\phi_j^i v^j$ belongs to D_{λ_i} ($j \neq i$), we have

$$K'_{(i)} = k + \lambda_i^2, \quad (i=1, 2),$$

from which

$$(5.7) \quad K'_{(1)} = 4c^2 - 2ac, \quad K'_{(2)} = 0,$$

because of (5.6).

Summing up, we state the

THEOREM 5.6. *If a contact hypersurface of a Kaehlerian manifold of constant holomorphic sectional curvature admits three distinct principal curvatures a, λ_1 and λ_2 , the sectional curvature with respect to any two vectors of the integral submanifold of D_{λ_i} is constants and is given by (5.7).*

In the next place we assume that the contact hypersurface admits only two distinct principal curvatures a and λ . Since (2.13) implies that $2c - \lambda$ is also a principal curvature of the hypersurface, we get

$$(5.8) \quad \lambda = 2c - a,$$

or

$$(5.9) \quad \lambda = c.$$

Suppose first that $\lambda = 2c - a$. We denote by V_a, V_λ and V respectively the vector spaces spanned by the vectors corresponding to the principal curvatures a, λ and by the orthogonal vectors to η^i . Then the correspondence $P \in M^{2n-1}$ to $V_a \cap V$ at P and the correspondence $P \in M^{2n-1}$ to V_λ define $(n-1)$ -dimensional distributions over M^{2n-1} . We call them D and D_λ respectively.

Let u^i and v^i be any vector fields belonging to D . Calculating $[u, v]^i \eta_i$, we have

$$\begin{aligned} [u, v]^i \eta_i &= (u^r \nabla_r v^i - v^r \nabla_r u^i) \eta_i = -u^r v^i \nabla_r \eta_i + v^r u^i \nabla_r \eta_i \\ &= \phi_i^s H_{rs} u^r v^i - \phi_i^s H_{sr} v^r u^i = a \phi_j^s u_s v^j - a \phi_j^s v_s u^j. \end{aligned}$$

By means of Lemma 5.1, the above equation becomes to

$$(5.10) \quad [u, v]^i \eta_i = 0.$$

This shows that for any $u^i, v^i \in D$, $[u, v]^i$ is an orthogonal vector field to η^i . Furthermore, in exactly the same way to the proof of Lemma 5.3, we have

$$[D, D] \subset D, \quad [D_\lambda, D_\lambda] \subset D_\lambda,$$

because of (5.10). Consequently we have

LEMMA 5.7. *The distributions D and D_λ are both integrable.*

This lemma means that our case can be treated by the analogous manner to the preceding case. In fact, we can write the second fundamental tensor in the following form :

$$H_{ji} = a \eta_i \eta_j + a \sum_{\alpha=1}^{n-1} v_{(a)}^i v_{(a)}^j + (2c-a) \sum_{\alpha=1}^{n-1} \phi_{(a)}^i v_{(a)}^r \phi_{(a)}^j v_{(a)}^s,$$

where $v_{(a)}^i$ are $n-1$ orthonormal gradient vectors belonging to $D(p)$. So, we can easily see that Theorem 5.4 and 5.6 are valid, replacing the distributions D_{λ_1} and D_{λ_2} by D and D_λ respectively.

Suppose next that $\lambda=c$ and denote by r and s the multiplicities of the principal curvatures a and c respectively. Since the multiplicities r and s are constants, we know that, for a suitable orthonormal frame, the second fundamental tensor H_j^i has the components of the form at each point of M^{2n-1} :

$$(H_j^i) = \begin{pmatrix} \overbrace{a}^r & & & & & \\ & \ddots & & & & \\ & & \ddots & & & \\ & & & \ddots & & \\ & & & & a & \\ & & & & & c \\ 0 & & & & & & \ddots \\ & & & & & & & c \end{pmatrix} \quad r+s=2n-1,$$

from which

$$(5.11) \quad H_r^r = ra + (2n-r-1)c.$$

The trace of a matrix being invariant under the change of the frame, (5.11) does hold for any frame, especially for natural frame.

On the other hand, from (2.16), we have

$$H_r^r = a + 2(n-1)c.$$

Combining these two equations, we get

$$(r-1)(c-a) = 0.$$

Since c and a are the distinct principal curvatures we know that the distribution corresponding to the principal curvature a is one-dimensional. Consequently we have, for a suitable orthonormal frame,

$$(H_j^i) = \begin{pmatrix} a & & & \\ & c & & \\ & & \ddots & 0 \\ 0 & & & c \end{pmatrix} = c \begin{pmatrix} 1 & & & \\ & \cdot & & 0 \\ & & \ddots & \\ 0 & & & \cdot \\ & & & 1 \end{pmatrix} + (a-c) \begin{pmatrix} 1 & & & \\ & 0 & & 0 \\ & & \ddots & \\ 0 & & & \cdot \\ & & & 0 \end{pmatrix},$$

that is,

$$(5.12) \quad H_j^i = c\delta_j^i + (a-c)\eta^i\eta_j.$$

Since (5.12) is a tensor equation, it does hold for natural frame, and so the hypersurface is so called C -umbilical hypersurface. Substituting (5.12) into the Gauss equation (4.5), we obtain

$$\begin{aligned} R_{kji h} = & (k+c^2)(g_{ji}g_{kh} - g_{ki}g_{jh}) + k(\phi_{kh}\phi_{ji} - \phi_{jh}\phi_{ki} - 2\phi_{kj}\phi_{ih}) \\ & + c(a-c)(g_{kh}\eta_j\eta_i - g_{jh}\eta_k\eta_i + g_{ji}\eta_k\eta_h - g_{ki}\eta_j\eta_h). \end{aligned}$$

A contact metric manifold which has the curvature tensor of the above form is called a locally C -Fubinian manifold.¹⁰⁾ Summing up, we state the

THEOREM 5.8. *If a contact hypersurface of a Kaehlerian manifold of constant holomorphic sectional curvature admits only two distinct principal curvatures, one of the following must be satisfied.*

- 1) *The hypersurface is C -umbilical hypersurface.*
- 2) *On M^{2n-1} , there are two integral distributions D and D_λ of dimension $n-1$ and the sectional curvatures with respect to any two vectors of the integral submanifolds are both constants.*

Now we assume that the hypersurface of a Kaehlerian manifold of constant holomorphic sectional curvature admits only one principal curvature. Then the hypersurface is an umbilical hypersurface. However, we have known¹¹⁾ that there is no umbilical hypersurface in non-flat Kaehlerian manifold of constant holomorphic sectional curvature. Thus, together with Theorem 5.4 and Theorem 5.8, we get the

10) Y. Tashiro and S. Tachibana [15].

11) Y. Tashiro and S. Tachibana [15].

THEOREM 5.9. *The contact hypersurface of a Kaehlerian manifold of positive constant holomorphic sectional curvature is a C-umbilical hypersurface and consequently a locally C-Fubinian manifold.*

6. Contact hypersurfaces in a Euclidean space. In this section, we regard a Euclidean space E^{2n} as a Kaehlerian manifold with vanishing holomorphic sectional curvature and consider contact hypersurfaces of a Euclidean space.

First of all, we recall that we have shown in §2 that a hypersphere S^{2n-1} of E^{2n} and the product manifold of $(n-1)$ -dimensional sphere S^{n-1} with n -dimensional Euclidean space E^n are contact hypersurfaces in E^{2n} . In the next place it is natural to ask in which other cases the induced almost contact metric structure is contact metric one. In the following, we try to give a complete solution of this problem and show the impossibility of a hypersurface in E^{2n} , other than the above two examples, whose induced almost contact metric structure is contact metric. For this purpose some of the results of the preceding discussions are now applied and we obtain a complete classification of the contact hypersurfaces M^{2n-1} in $2n$ -dimensional Euclidean space E^{2n} .

According to Corollary 5.5, we have known that a contact hypersurface of a Euclidean space admits at most two distinct principal curvatures. If the hypersurface admits only one principal curvature, the hypersurface is a totally umbilical hypersurface. Consequently it is a portion of a $(2n-1)$ -dimensional sphere. So, in this section, we mainly discuss the case that the hypersurface admits two distinct principal curvatures.

We begin with the

LEMMA 6.1. *Let M^{2n-1} be a hypersurface with constant mean curvature of a Euclidean space. If M^{2n-1} satisfy*

$$(6.1) \quad H_j{}^r H_{ri} = b H_{ji} ,$$

for a constant b , we have

$$(6.2) \quad \nabla_j H_{ih} = 0 .$$

PROOF. Applying Ricci's identity to H_{ji} , we get

$$(6.3) \quad \nabla_k \nabla_j H_{ih} - \nabla_j \nabla_k H_{ih} = -R_{kji}{}^r H_{rh} - R_{kjh}{}^r H_{ir} .$$

Since, for a hypersurface in E^{2n} , the Gauss equation being reduced to

$$R_{kji h} = H_{ji} H_{kh} - H_{ki} H_{jh} ,$$

it follows that

$$\nabla_k \nabla_j H_{ih} - \nabla_j \nabla_k H_{ih} = -(H_{ji} H_k^r - H_{ki} H_j^r) H_{rh} - (H_{jh} H_k^r - H_{kh} H_j^r) H_{ir}.$$

Substituting (6.1) into the above, we have

$$\nabla_k \nabla_j H_{ih} - \nabla_j \nabla_k H_{ih} = -b H_{ji} H_{kh} + b H_{ki} H_{jh} - b H_{jh} H_{ki} + b H_{kh} H_{ji} = 0,$$

from which, by contraction with g^{kh} ,

$$(6.4) \quad \nabla^r \nabla_j H_{ir} = \nabla_j \nabla^r H_{ir} = \nabla_j \nabla_i H_r^r = 0,$$

because that M^{2n-1} has constant mean curvature. Since the condition (6.1) implies that

$$H_{ji} H^{ji} = b H_r^r = \text{const.},$$

we have

$$\nabla_k H_{ji} H^{ji} = 0,$$

and this gives

$$(6.5) \quad \nabla_k H_{ji} \nabla^k H^{ji} + \nabla^k \nabla_k H_{ji} H^{ji} = 0.$$

On the other hand, for a hypersurface of E^{2n} , the Codazzi equation being reduced to

$$\nabla_k H_{ji} - \nabla_j H_{ki} = 0,$$

it follows that

$$(6.6) \quad \nabla^k \nabla_k H_{ji} = \nabla^k \nabla_j H_{ki}.$$

Thus, according to (6.4), (6.5) and (6.6), we have

$$\nabla_k H_{ji} \nabla^k H^{ji} = 0.$$

The Riemannian metric being positive definite, this shows that our assertion is true. This completes the proof.

According to Corollary 5.5, (4.14) is simplified as

$$(6.7) \quad H_{jr} H_i^r = 2c H_{ji},$$

so, we can apply Lemma 6.1 to a contact hypersurface M^{2n-1} of Euclidean space E^{2n} and obtain

$$(6.8) \quad \nabla_k H_{ji} = 0.$$

Under the above preparations, we put

$$(6.9) \quad J_k^i = \delta_k^i - c^{-1} H_k^i,$$

then we have

$$J_k^i J_j^k = \delta_j^i.$$

This means that J_j^i defines an almost product structure in M^{2n-1} . The characteristic roots of the tensor J_i^h are $+1$ and -1 . The characteristic vectors corresponding to the root $+1$ span the distribution D_+ and those corresponding to -1 span the distribution D_- and these distributions are mutually complementary. The projection tensors on two complementary distributions D_+ and D_- are respectively given by

$$(6.10) \quad P_i^h = \delta_i^h - \frac{1}{2c} H_i^h,$$

$$(6.11) \quad Q_i^h = \frac{1}{2c} H_i^h,$$

and satisfy

$$(6.12) \quad \begin{cases} P_i^h + Q_i^h = \delta_i^h, \\ P_i^h P_j^i = P_j^h, Q_i^h Q_j^i = Q_j^h, P_i^h Q_j^i = Q_i^h P_j^i = 0. \end{cases}$$

To show that the distribution D_+ be integrable, we note that, for any two vectors u^i and v^i on M^{2n-1} , the vectors $P_i^h u^i$ and $P_i^h v^i$ belong to D_+ and we calculate $[Pu, Pv]$ as follows:

$$\begin{aligned} [P_i^h u^i, P_i^h v^i] &= P_i^r u^i \nabla_r (P_j^h v^j) - P_i^r v^i \nabla_r (P_j^h u^j) \\ &= P_i^r P_j^h (u^i \nabla_r v^j - v^i \nabla_r u^j) \end{aligned}$$

where we have used (6.8) and (6.9). This implies, together with (6.12), that

$$(6.13) \quad Q_i^h [P_j^i u^j, P_j^i v^j] = Q_i^h P_k^i P_j^r (u^i \nabla_r v^j - v^i \nabla_r u^j) = 0,$$

from which we know that the distribution D_+ is integrable. Thus the rank of P_i^h being n , for any point of $P \in M^{2n-1}$, there is a coordinate neighbourhood U_1 containing P with coordinates y^a ($a=1, \dots, n$) such that

$$P_j^i \frac{\partial y^a}{\partial x^i} = \frac{\partial y^x}{\partial x^j},$$

or

$$Q_j^i \frac{\partial y^a}{\partial x^i} = 0.$$

Similarly, we can prove that the distribution D_- is also integrable and consequently that in the neighbourhood U_1 , there are $n-1$ functions y^x such that

$$P_j^i \frac{\partial y^x}{\partial x^i} = 0.$$

These discussions show us that the tensor J_i^h , P_i^h and Q_i^h have numerical components

$$(J_i^h) = \begin{pmatrix} \delta_h^a & 0 \\ 0 & -\delta_v^z \end{pmatrix}, (P_i^h) = \begin{pmatrix} \delta_h^a & 0 \\ 0 & 0 \end{pmatrix}, (Q_i^h) = \begin{pmatrix} 0 & 0 \\ 0 & \delta_v^z \end{pmatrix},$$

in the coordinate system (y^a, y^x) . The integral submanifolds of D_+ and D_- are respectively given by

$$(6.14) \quad y^x = \text{const.}, \quad y^a = \text{const.}$$

On the other hand

$$g_{ih} J_k^i J_j^h = g_{kj},$$

implies that

$$(g_{jk}) = \begin{pmatrix} g_{ab} & 0 \\ 0 & g_{xv} \end{pmatrix}.$$

Hence, the first fundamental form of M^{2n-1} becomes to

$$(6.15) \quad ds^2 = g_{ab}(y^a, y^x) dy^a dy^b + g_{xv}(y^a, y^x) dy^x dy^v$$

in this coordinate system. However, in a contact hypersurface of E^{2n} , (6.8)

being valid, we have $\nabla_j J_i^h = 0$. Consequently we get¹²⁾

$$ds^2 = g_{ab}(y^a) dy^a dy^b + g_{xy}(y^x) dy^x dy^y.$$

This means that the first fundamental form of M^{2n-1} is identical with that of Riemannian product of the integral submanifold defined by (6.14). So, the integral submanifolds given by (6.14) are both totally geodesic in M^{2n-1} . Thus we proved the

LEMMA 6.2.¹³⁾ *If a contact hypersurface M^{2n-1} of a Euclidean space admits two distinct principal curvatures, M^{2n-1} is locally isometric to the Riemannian product of the integral submanifolds of D_+ and D_- . Furthermore these integral submanifolds are both totally geodesic in M^{2n-1} .*

In order to get further results on the integral submanifolds, we take any vector field w^i belonging to D_+ and two arbitrary vector fields u^i and v^i belonging to D_- . Then we have

$$(6.16) \quad u^r \nabla_r (v^i B_i^\kappa) = (u^r \nabla_r v^i) B_i^\kappa + 2c u_i v^i C^\kappa,$$

$$(6.17) \quad u^r \nabla_r (w^i B_i^\kappa) = (u^r \nabla_r w^i) B_i^\kappa,$$

$$(6.18) \quad u^r \nabla_r C^\kappa = -2c u^i B_i^\kappa.$$

Since, using (6.8), (6.10) and (6.11), we can easily verify that the vector fields $u^r \nabla_r v^i$ and $u^r \nabla_r w^i$ belong to D_- and D_+ respectively, the above equations show us that the integral submanifold of D_- lies in the n -dimensional Euclidean space E^n which is orthogonal to the integral submanifold of D_+ in E^{2n} .

We take, in E^n , $n-1$ mutually orthonormal vectors $X_a^i B_i^\kappa$ ($a=1, \dots, n-1$) belonging to D_- . Then, $X_a^i B_i^\kappa$ are linearly independent tangent vectors to the integral submanifold of D_- and C^κ is a unit normal to the submanifold in E^n . The second fundamental tensor $H'_{a'}$ of the integral submanifold of D_- in E^n is, by definition, computed as follows:

$$\begin{aligned} H'_{ab} &= \left[\partial_a (X_b^j B_j^\kappa) + X_a^i B_i^\lambda X_b^j B_j^\mu \left\{ \begin{matrix} \kappa \\ \lambda \mu \end{matrix} \right\} \right] C_\kappa \\ &= X_b^j X_a^i \left(\partial_i B_j^\kappa + B_i^\lambda B_j^\mu \left\{ \begin{matrix} \kappa \\ \lambda \mu \end{matrix} \right\} \right) C_\kappa \\ &= X_b^j X_a^i H_{ji} = 2c g_{ji} X_b^j X_a^i = 2c g_{ba}. \end{aligned}$$

12) For example, K. Yano [17]. pp. 220-221.

13) The hypersurfaces in Euclidean space whose principal curvatures are all constants were studied by E. Cartan [3, 4]. Making use of his results, we can also prove Lemma 6.2.

This means that the integral submanifold of D_- is a totally umbilical hypersurface in E^n . Consequently the integral submanifold is an open submanifold of an $(n-1)$ -dimensional sphere. Similarly we can show that the integral submanifold of D_+ is an open submanifold of n -dimensional Euclidean space.

Summing up all the discussions of this section, we have that the contact hypersurface of a Euclidean space is locally isometric with an open submanifold of

$$S^{2n-1} \quad \text{or} \quad S^{n-1} \times E^n,$$

where S^r denotes an r -dimensional sphere and E^n an n -dimensional Euclidean space.

On the other hand, from (6.8), we can easily see that the contact hypersurface of a Euclidean space is a Riemannian locally symmetric space and consequently an analytic Riemannian manifold. Since any isometry between connected open subsets of connected, simply connected, complete analytic Riemannian manifolds M and M' can be uniquely extended to an isometry between these Riemannian manifolds, we get at last the

THEOREM 6.3. *A complete simply connected contact hypersurface of a Euclidean space is isometric with one of the following:*

$$S^{2n-1}, \quad S^{n-1} \times E^n,$$

where S^r denotes an r -dimensional sphere and E^n is an n -dimensional Euclidean space.

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