ON HARMONIC TENSORS IN COMPACT SASAKIAN SPACES

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A Riemannian space with a contact form $\eta = \eta_\lambda dx^\lambda$ is called Sasakian if the contact structure satisfies certain conditions (Cf. § 3). In § 4 and § 5 of this paper we shall discuss harmonic vectors in compact Sasakian spaces and obtain some analogous results to compact Kählerian spaces (Cf. Theorem 4.2, 4.3, 5.1). In § 7 we shall prove that any harmonic p-form in n dimensional compact Sasakian spaces is orthogonal to $\eta^\lambda = g^{\lambda\mu}\eta_\mu$ if $p < (1/2)(n+1)$. In § 8, we shall introduce an operator Φ and prove that Φu is harmonic for a harmonic p-form u if $p < (1/2)(n+1)$. Preliminary facts and lemmas are given in the other sections.

1. Preliminaries.[1] Consider an n dimensional Riemannian space M whose positive definite metric tensor is given by $g_{\lambda\mu}$. We denote by $R_{\lambda\mu\nu}{}^{\omega}$ the Riemannian curvature tensor

$$R_{\lambda\mu\nu}{}^{\omega} = \partial_\lambda \left\{ {\omega \atop \mu\nu} \right\} - \partial_\mu \left\{ {\omega \atop \lambda\nu} \right\} + \left\{ {\omega \atop \lambda\alpha} \right\}\left\{ {\alpha \atop \mu\nu} \right\} - \left\{ {\omega \atop \mu\alpha} \right\}\left\{ {\alpha \atop \lambda\nu} \right\}, \quad \partial_\lambda = \partial/\partial x^\lambda ,$$

and by $R_{\mu\nu} = R_{\lambda\mu\nu}{}^{\lambda}$ the Ricci tensor, where $\left\{ {\lambda \atop \mu\nu} \right\}$ are the Christoffel's symbols. The operator of covariant derivation with respect to $\left\{ {\lambda \atop \mu\nu} \right\}$ is denoted by ∇_λ.

It is easy to have

$$R_{\lambda\mu\nu}{}^{\omega} u^{\lambda\mu\nu} = 0 \tag{1.1}$$

for any skew-symmetric tensor $u^{\lambda\mu\nu}$, by virtue of the first Bianchi's identity.

If a vector field η^λ satisfies

$$\theta(\eta)\, g_{\lambda\mu} \equiv \nabla_\lambda \eta_\mu + \nabla_\mu \eta_\lambda = 0\,, \quad (\eta_\lambda \equiv g_{\lambda\mu}\eta^\mu)\,,$$

then it is called a Killing vector, where $\theta(\eta)$ is the operator of Lie derivation with respect to η^λ. It is well known that the equations

1) As to the notations we follow Yano, K., [5].

$$\nabla^\alpha \nabla_\alpha \eta_\lambda + R_\lambda{}^\alpha \eta_\alpha = 0\,, \qquad \nabla^\lambda \eta_\lambda = 0$$

are valid for any Killing vector η^λ.

We shall recall various operators on differential forms. A skew-symmetric tensor $u_{\lambda_1\cdots\lambda_p}$ may be regarded as the coefficients of a differential p-form

$$u = \frac{1}{p!} u_{\lambda_1\cdots\lambda_p} dx^{\lambda_1} \wedge \cdots \wedge dx^{\lambda_p}.$$

We represent this fact by

$$u \;:\; u_{\lambda_1\cdots\lambda_p}.$$

If $p=0$, then a p-form u is nothing but a scalar function which we shall usually denote by f.

The exterior differential du and the codifferential δu of u are given by

$$du : \begin{cases} \nabla_\mu u_{\lambda_1\cdots\lambda_p} - \sum_{i=1}^{p} \nabla_{\lambda_i} u_{\lambda_1\cdots\lambda_{i-1}\,\mu\lambda_{i+1}\cdots\lambda_p}\,, & p \geqq 1\,, \\ \nabla_\mu f\,, \quad p = 0\,, \end{cases}$$

$$\delta u : \begin{cases} \nabla^\alpha u_{\alpha\lambda_2\cdots\lambda_p}\,, \quad p \geqq 1\,, \\ 0\,, \quad p = 0\,. \end{cases}$$

The Laplacian operator Δ is given by $\Delta = d\delta + \delta d$. For a p-form u we have explicitly

$$(\Delta u)_{\lambda_1\cdots\lambda_p} = \nabla^\alpha \nabla_\alpha u_{\lambda_1\cdots\lambda_p} - \sum_{i=1}^{p} R_{\lambda_i}{}^\sigma u_{\lambda_1\cdots\sigma\cdots\lambda_p}$$
$$- \sum_{j<i} R_{\lambda_j\lambda_i}{}^{\rho\sigma} u_{\lambda_1\cdots\rho\cdots\sigma\cdots\lambda_p}\,, \qquad p \geqq 2\,,$$

where the subscripts σ appears at the i-th position and ρ at the j-th position, and

$$(\Delta u)_\lambda = \nabla^\alpha \nabla_\alpha u_\lambda - R_\lambda{}^\alpha u_\alpha\,, \qquad p = 1\,,$$

$$\Delta f = \nabla^\alpha \nabla_\alpha f\,, \qquad p = 0\,.$$

A p-form u is called to be harmonic if $du = 0$ and $\delta u = 0$ are satisfied. Thus if u is harmonic, then we have $\Delta u = 0$.

Let $\eta = \eta_\lambda dx^\lambda$ be a 1-form, then we shall naturally identify η with the contravariant vector field $\eta^\lambda = \eta_\mu g^{\lambda\mu}$. Hence, for instance, if η is closed or η^λ is a Killing vector, then we shall say that η^λ is closed or the 1-form $\eta = \eta_\lambda dx^\lambda$ is a Killing form, respectively.

For a Killing form η we have

$$(\Delta\eta)_\lambda = -2R_\lambda{}^\alpha \eta_\alpha , \quad \delta\eta = 0 . \tag{1.2}$$

For a 1-form η the operator $i(\eta)$ is defined by

$$(i(\eta)u)_{\lambda_2\cdots\lambda_p} = \eta^\alpha u_{\alpha\lambda_2\cdots\lambda_p} , \quad i(\eta)f = 0$$

for any p-form u and any scalar function f.

The Lie derivation with respect to η satisfies

$$\theta(\eta)u = (di(\eta) + i(\eta)d)u \tag{1.3}$$

for any p-form u, $(p \geqq 0)$. For a p-form u, $\theta(\eta)u$ is given explicitly by

$$(\theta(\eta)u)_{\lambda_1\cdots\lambda_p} = \eta^\alpha \nabla_\alpha u_{\lambda_1\cdots\lambda_p} + \sum_{i=1}^{p} u_{\lambda_1\cdots\alpha\cdots\lambda_p} \nabla_{\lambda_i} \eta^\alpha ,$$

$$\theta(\eta)f = \eta^\alpha \nabla_\alpha f, \quad p = 0 .$$

The exterior product of a 1-form η or a 2-form φ with a p-form u, $(p \geqq 2)$, is given respectively by

$$\eta \wedge u : \quad \eta_\alpha u_{\lambda_1\cdots\lambda_p} - \sum_{j=1}^{p} \eta_{\lambda_j} u_{\lambda_1\cdots\alpha\cdots\lambda_p} ,$$

$$\varphi \wedge u : \quad \varphi_{\alpha\beta} u_{\lambda_1\cdots\lambda_p} - \sum_{i} \varphi_{\alpha\lambda_i} u_{\lambda_1\cdots\beta\cdots\lambda_p} - \sum_{j} \varphi_{\lambda_j\beta} u_{\lambda_1\cdots\alpha\cdots\lambda_p}$$

$$+ \sum_{j<i} \varphi_{\lambda_j\lambda_i} u_{\lambda_1\cdots\alpha\cdots\beta\cdots\lambda_p} ,$$

where the subscripts α appears at the j-th position and β at the i-th position.

2. Harmonic tensors in a compact orientable Riemannian space. In this section we shall always consider a compact orientable Riemannian space M. For any p-forms u and v the global inner product (u, v) is defined by

$$(u, v) = \frac{1}{p!} \int_M u_{\lambda_1 \cdots \lambda_p} v^{\lambda_1 \cdots \lambda_p} \, d\sigma ,$$

where $d\sigma$ means the volume element of M.

For any p-form u and $(p+1)$-form v the following integral formulae are well known

$$(du, v) + (u, \delta v) = 0 , \tag{2.1}$$

$$(\Delta u, u) + (du, du) + (\delta u, \delta u) = 0 . \tag{2.2}$$

If a p-form u satisfies $(\Delta u, u) \geqq 0$, then u is harmonic, by virtue of (2.2).

The following lemma is also well known.

LEMMA 2.1. *In a compact orientable Riemannian space*

$$\theta(\eta) u = 0$$

is valid for any Killing vector η^λ *and any harmonic* p*-form* u.

From this lemma and (1.3) we have easily the following

LEMMA 2.2. *In a compact orientable Riemannian space,*

$$i(\eta) u \quad \textit{is closed}$$

for any Killing vector η^λ *and any harmonic* p*-form* u.

Now let η be a Killing form and u be a harmonic p-form, then we have

$$\begin{aligned}
(\delta(\eta \wedge u))_{\lambda_1 \cdots \lambda_p} &= \nabla^\alpha (\eta \wedge u)_{\alpha \lambda_1 \cdots \lambda_p} \\
&= \nabla^\alpha (\eta_\alpha u_{\lambda_1 \cdots \lambda_p} - \sum \eta_{\lambda_i} u_{\lambda_1 \cdots \alpha \cdots \lambda_p}) \\
&= \eta^\alpha \nabla_\alpha u_{\lambda_1 \cdots \lambda_p} - \sum u_{\lambda_1 \cdots \alpha \cdots \lambda_p} \nabla^\alpha \eta_{\lambda_i} \\
&= (\theta(\eta) u)_{\lambda_1 \cdots \lambda_p} .
\end{aligned}$$

Thus taking account of Lemma 2.1 we get

LEMMA 2.3. *In a compact orientable Riemannian space,*

$$\eta \wedge u \qquad \textit{is coclosed}$$

for any Killing form η *and any harmonic* p*-form* u.

By Lemma 2.2 and Lemma 2.3 we can obtain

LEMMA 2.4. *In a compact orientable Riemannian space, if*

$$i(\eta)u \qquad \textit{is coclosed}$$

for a Killing form η *and a harmonic* p*-form* u, *then the* p*-form*

$$\eta \wedge i(\eta)u \qquad \textit{is coclosed}.$$

3. Identities in a Sasakian space.[2] An n dimensional Sasakian space (or normal contact metric space) is a Riemannian space which admits a unit Killing vector field η^λ satisfying

$$\nabla_\lambda \nabla_\mu \eta_\nu = \eta_\mu g_{\lambda\nu} - \eta_\nu g_{\lambda\mu}. \tag{3.1}$$

It is well known that a Sasakian space is orientable and odd dimensional.

In this section we prepare identities in an n dimensional Sasakian space.

Now if we define $\varphi_\mu^{\ \nu}$ by

$$\varphi_\mu^{\ \nu} = \nabla_\mu \eta^\nu,$$

then we have

$$\varphi_\alpha^{\ \lambda} \varphi_\mu^{\ \alpha} = -\delta_\mu^\lambda + \eta_\mu \eta^\lambda, \quad \varphi_\alpha^{\ \lambda} \eta^\alpha = 0,$$

$$\varphi_{\mu\nu} = -\varphi_{\nu\mu}, \quad (\varphi_{\mu\nu} \equiv \varphi_\mu^{\ \alpha} g_{\alpha\nu}).$$

(3.1) is then written as

$$\nabla_\lambda \varphi_{\mu\nu} = \eta_\mu g_{\lambda\nu} - \eta_\nu g_{\lambda\mu} \tag{3.2}$$

and we have easily

$$\nabla^\lambda \varphi_{\lambda\nu} = -(n-1)\eta_\nu, \tag{3.3}$$

2) Okumura, M., [1], Sasaki, S and Y. Hatakeyama, [2]. Examples of Sasakian space have been given in [2] and [4].

(3.4) $$\nabla^{\lambda}\nabla_{\lambda}\varphi_{\mu\nu} = -2\varphi_{\mu\nu}.$$

Applying the Ricci's identity to η_λ we have

$$\nabla_\nu \nabla_\mu \eta_\lambda - \nabla_\mu \nabla_\nu \eta_\lambda = -R_{\nu\mu\lambda}{}^{\varepsilon}\eta_\varepsilon ,$$

from which we can get

(3.5) $$R_{\nu\mu\lambda}{}^{\varepsilon}\eta_\varepsilon = \eta_\nu g_{\lambda\mu} - \eta_\mu g_{\lambda\nu} ,$$

(3.6) $$R_\nu{}^{\varepsilon}\eta_\varepsilon = (n-1)\eta_\nu .$$

Next, applying the Ricci's identity to $\varphi_\lambda{}^\alpha$ we have

$$\nabla_\rho \nabla_\sigma \varphi_\lambda{}^\alpha - \nabla_\sigma \nabla_\rho \varphi_\lambda{}^\alpha = R_{\rho\sigma\varepsilon}{}^{\alpha}\varphi_\lambda{}^\varepsilon - R_{\rho\sigma\lambda}{}^{\varepsilon}\varphi_\varepsilon{}^\alpha .$$

Substituting (3.2) into the left hand member of the last equation we get

(3.7) $$R_{\rho\sigma\varepsilon}{}^{\alpha}\varphi_\lambda{}^\varepsilon - R_{\rho\sigma\lambda}{}^{\varepsilon}\varphi_\varepsilon{}^\alpha = \varphi_{\rho\lambda}\delta_\sigma^\alpha - \varphi_\rho{}^\alpha g_{\sigma\lambda} - \varphi_{\sigma\lambda}\delta_\rho^\alpha + \varphi_\sigma{}^\alpha g_{\rho\lambda} ,$$

from which it follows that

(3.8) $$\varphi_\lambda{}^\varepsilon R_{\varepsilon\alpha\rho\sigma} = -R_{\rho\sigma\lambda\varepsilon}\varphi_\alpha{}^\varepsilon + \varphi_{\rho\lambda} g_{\sigma\alpha} - \varphi_{\rho\alpha} g_{\sigma\lambda} - \varphi_{\sigma\lambda} g_{\rho\alpha} + \varphi_{\sigma\alpha} g_{\rho\lambda} .$$

Contracting (3.7) with respect to ρ and α we can get

(3.9) $$\frac{1}{2}\varphi^{\alpha\beta}R_{\alpha\beta\mu\lambda} = R_{\mu\varepsilon}\varphi_\lambda{}^\varepsilon + (n-2)\varphi_{\mu\lambda} ,$$

(3.10) $$R_{\mu\varepsilon}\varphi_\lambda{}^\varepsilon = -R_{\lambda\varepsilon}\varphi_\mu{}^\varepsilon , \quad R_\mu{}^\varepsilon\varphi_\varepsilon{}^\lambda = R_\varepsilon{}^\lambda\varphi_\mu{}^\varepsilon .$$

From (3.9) we have

(3.11) $$\varphi^{\alpha\beta}\nabla_\alpha\nabla_\beta u_\mu = -[R_{\mu\varepsilon}\varphi_\lambda{}^\varepsilon + (n-2)\varphi_{\mu\lambda}]u^\lambda$$

for any vector u_μ.

4. Harmonic vectors in a compact Sasakian space.[3] Let u be a harmonic 1-form in a compact Sasakian space, then we have $df=0$ for the scalar f defined by $f = i(\eta)u$, by virtue of Lemma 2. 2.

3) Throughout the paper we assume that $n=\dim M>1$.

Hence f is constant. If we define β by

$$u = f\eta + \beta\,, \tag{4.1}$$

then β is a 1-form orthogonal to η, i.e., $i(\eta)\beta = 0$. Operating Δ to (4.1) we get $\Delta\beta = -f\Delta\eta$ and as η is Killing we have

$$(\Delta\beta)_\lambda = -f(\Delta\eta)_\lambda = 2fR_\lambda^{\ \varepsilon}\eta_\varepsilon = 2(n-1)f\eta_\lambda$$

by virtue of (1.2) and (3.6). Hence β is harmonic, because we have $(\Delta\beta, \beta)=0$. Thus we have $f=0$ and obtain the following

THEOREM 4.1. *Any harmonic* 1*-form* u *in a compact Sasakian space is orthogonal to* η, *i.e.*, $i(\eta)u=0$.

Next we introduce an operator J : p-form $u \rightarrow$ tensor of type $(0, p)$ $\widetilde{u} = Ju$ by

$$\widetilde{u}_{\lambda_1\cdots\lambda_p} = \varphi_{\lambda_1}^{\ \alpha}\ u_{\alpha\lambda_2\cdots\lambda_p}$$

and we shall compute $\Delta\widetilde{u}$ for a harmonic 1-form u in a compact Sasakian space.

Taking account of $i(\eta)u=0$, $\Delta u=0$, (3.4) and (3.10) we can get

$$\begin{aligned}\nabla^\mu\nabla_\mu\widetilde{u}_\lambda &= -2\nabla_\lambda(\eta^\mu u_\mu) + \varphi_\lambda^{\ \alpha}\nabla^\mu\nabla_\mu u_\alpha \\ &= \varphi_\lambda^{\ \alpha}\nabla^\mu\nabla_\mu u_\alpha = \varphi_\lambda^{\ \alpha}R_\alpha^{\ \varepsilon}u_\varepsilon = R_\lambda^{\ \alpha}\widetilde{u}_\alpha\,.\end{aligned}$$

Thus we have $\Delta\widetilde{u} = 0$ and hence we get

THEOREM 4.2. *In a compact Sasakian space,* $\widetilde{u} = Ju$ *is a harmonic* 1*-form for any harmonic* 1*-form* u.

As a corollary to this theorem we have

THEOREM 4.3. *The first Betti number of a compact Sasakian space is zero or even.*

5. *C*-analytic 1-form. It is known that in a compact Kählerian space a harmonic 1-form is holomorphic (i.e., covariant analytic) and vice versa. In this section we shall consider an analogous fact.

Now, in a Sasakian space, let u be a 1-form satisfying

$$Jdu - dJu = 0, \tag{5.1}$$

which is written explicitly as follows

$$\varphi_\lambda{}^\alpha(\nabla_\alpha u_\mu - \nabla_\mu u_\alpha) - [\nabla_\lambda(\varphi_\mu{}^\alpha u_\alpha) - \nabla_\mu(\varphi_\lambda{}^\alpha u_\alpha)] = 0. \tag{5.2}$$

It is easy to see that (5.2) is equivalent to the following equation

$$\varphi_\lambda{}^\alpha \nabla_\alpha u_\mu - \varphi_\mu{}^\alpha \nabla_\lambda u_\alpha + \eta_\lambda u_\mu - \eta_\mu u_\lambda = 0. \tag{5.3}$$

If we transvect η^λ to (5.2), we have

$$u_\lambda = (u_\alpha \eta^\alpha)\eta_\lambda + \varphi_\lambda{}^\alpha \eta^\varepsilon \nabla_\varepsilon u_\alpha \tag{5.4}$$

and transvecting (5.2) with $g^{\lambda\mu}$ we obtain

$$\varphi^{\alpha\beta} \nabla_\alpha u_\beta = 0. \tag{5.5}$$

Now we define a *C-analytic* 1-form[4] as a 1-form u satisfying (5.1) and $i(\eta)u=0$. Then we have

THEOREM 5.1. *A necessary and sufficient condition for a 1-form in a compact Sasakian space to be harmonic is that it is C-analytic.*

PROOF. For a harmonic 1-form u, we have $du=0$, $dJu=0$ and $i(\eta)u=0$ by virtue of Theorem 4.1 and Theorem 4.2. Hence it is C-analytic.

Conversely let u be C-analytic, then we have $i(\eta)u = \eta^\alpha u_\alpha = 0$. Taking account of (3.3) and (5.5) we can get

$$\eta^\alpha \nabla^\lambda \nabla_\lambda u_\alpha = 0. \tag{5.6}$$

Next, transvecting (5.3) with $\varphi_\nu{}^\mu$ we have

$$\nabla_\lambda u_\nu + \varphi_\nu{}^\beta \varphi_\lambda{}^\alpha \nabla_\alpha u_\beta - \eta_\nu \eta^\alpha \nabla_\lambda u_\alpha + \eta_\lambda \varphi_\nu{}^\alpha u_\alpha = 0.$$

Operating $\nabla^\lambda = g^{\lambda\mu}\nabla_\mu$ to the last equation we get $\Delta u = 0$, by virtue of (5.6) and (3.11). Thus u is harmonic. Q.E.D.

4) For an almost complex space, see Tachibana, S., [3].

6. A lemma. Consider a harmonic p-form u, $(p \geqq 2)$, in an n dimensional Sasakian space and define a $(p-2)$-form v by

$$v: \quad v_{\lambda_3\cdots\lambda_p} = \varphi^{\lambda_1\lambda_2} u_{\lambda_1\lambda_2\cdots\lambda_p}.$$

In the following we shall compute Δv and obtain a lemma.

First as we have

$$\nabla^\varepsilon \nabla_\varepsilon v_{\lambda_3\cdots\lambda_p} = \nabla^\varepsilon \nabla_\varepsilon \varphi^{\lambda_1\lambda_2} u_{\lambda_1\cdots\lambda_p} + 2\nabla^\varepsilon \varphi^{\lambda_1\lambda_2} \nabla_\varepsilon u_{\lambda_1\cdots\lambda_p} + \varphi^{\lambda_1\lambda_2} \nabla^\varepsilon \nabla_\varepsilon u_{\lambda_1\cdots\lambda_p},$$

if we take account of (3.2) and (3.4) we can get

$$\nabla^\varepsilon \nabla_\varepsilon v_{\lambda_3\cdots\lambda_p} = -2v_{\lambda_3\cdots\lambda_p} + \varphi^{\lambda_1\lambda_2} \nabla^\varepsilon \nabla_\varepsilon u_{\lambda_1\cdots\lambda_p}. \tag{6.1}$$

As u satisfies $\Delta u = 0$, the last term becomes as follows:

$$\begin{aligned} \varphi^{\lambda_1\lambda_2} \nabla^\varepsilon \nabla_\varepsilon u_{\lambda_1\cdots\lambda_p} &= \varphi^{\lambda_1\lambda_2}\left(\sum_{i=1}^{p} R_{\lambda_i}{}^{\varepsilon} u_{\lambda_1\cdots\varepsilon\cdots\lambda_p} + \sum_{j<i} R_{\lambda_j\lambda_i}{}^{\rho\sigma} u_{\lambda_1\cdots\rho\cdots\sigma\cdots\lambda_p}\right) \\ &= \sum_{i=3}^{p} R_{\lambda_i}{}^{\varepsilon} v_{\lambda_3\cdots\varepsilon\cdots\lambda_p} + \sum_{2<j<i} R_{\lambda_j\lambda_i}{}^{\rho\sigma} v_{\lambda_3\cdots\rho\cdots\sigma\cdots\lambda_p} \\ &\quad + 2\varphi^{\lambda_1\lambda_2} R_{\lambda_2}{}^{\varepsilon} u_{\lambda_1\varepsilon\lambda_3\cdots\lambda_p} + \varphi^{\lambda_1\lambda_2} R_{\lambda_1\lambda_2}{}^{\rho\sigma} u_{\rho\sigma\lambda_3\cdots\lambda_p} \\ &\quad + 2\varphi^{\lambda_1\lambda_2} \sum_{2<i} R_{\lambda_2\lambda_i}{}^{\rho\sigma} u_{\lambda_1\rho\cdots\sigma\cdots\lambda_p}. \end{aligned} \tag{6.2}$$

On the other hand we have

$$\begin{aligned} \varphi^{\lambda_1\lambda_2} R_{\lambda_1\lambda_2}{}^{\rho\sigma} u_{\rho\sigma\lambda_3\cdots\lambda_p} &= \varphi^{\lambda_1\lambda_2} R_{\lambda_1\lambda_2\rho\sigma} u^{\rho\sigma}{}_{\lambda_3\cdots\lambda_p} \\ &= 2[R_{\rho\alpha}\varphi_\sigma{}^\alpha + (n-2)\varphi_{\rho\sigma}] u^{\rho\sigma}{}_{\lambda_3\cdots\lambda_p} \\ &= -2\varphi^{\lambda_1\lambda_2} R_{\lambda_2}{}^{\varepsilon} u_{\lambda_1\varepsilon\lambda_3\cdots\lambda_p} + 2(n-2) v_{\lambda_3\cdots\lambda_p} \end{aligned} \tag{6.3}$$

by virtue of (3.9) and (3.10) and taking account of (3.8) and (1.1) we have

$$\begin{aligned} \varphi^{\lambda_1\lambda_2} R_{\lambda_2\lambda_i}{}^{\rho\sigma} u_{\lambda_1\rho\cdots\sigma\cdots\lambda_p} &= \varphi_{\lambda_1}{}^{\lambda_2} R_{\lambda_2\lambda_i\rho\sigma} u^{\lambda_1\rho}{}_{\cdots}{}^{\sigma}{}_{\cdots\lambda_p} \\ &= -R_{\rho\sigma\lambda_1\alpha}\varphi_{\lambda_i}{}^{\alpha} u^{\lambda_1\rho}{}_{\cdots}{}^{\sigma}{}_{\cdots\lambda_p} \\ &\quad + (\varphi_{\rho\lambda_1} g_{\sigma\lambda_i} - \varphi_{\rho\lambda_i} g_{\sigma\lambda_1} - \varphi_{\sigma\lambda_1} g_{\rho\lambda_i} + \varphi_{\sigma\lambda_i} g_{\rho\lambda_1}) u^{\lambda_1\rho}{}_{\cdots}{}^{\sigma}{}_{\cdots\lambda_p} \\ &= -2v_{\lambda_3\cdots\lambda_p}. \end{aligned} \tag{6.4}$$

Substituting (6.3) and (6.4) into (6.2) we have

$$\varphi^{\lambda_1\lambda_2}\nabla^{\varepsilon}\nabla_{\varepsilon}u_{\lambda_1\cdots\lambda_p} = \sum_{i=3}^{p} R_{\lambda_i}{}^{\varepsilon} v_{\lambda_3\cdots\varepsilon\cdots\lambda_p} + \sum_{2<j<i} R_{\lambda_j\lambda_i}{}^{\rho\sigma} v_{\lambda_3\cdots\rho\cdots\sigma\cdots\lambda_p} + 2(n-2p+2)\, v_{\lambda_3\cdots\lambda_p}$$

and from (6.1) we get

$$\Delta v = 2(n+1-2p)\, v\,.$$

Hence if our space is compact, we have an integral formula

$$2(n+1-2p)(v, v) + (dv, dv) + (\delta v, \delta v) = 0\,.$$

Thus we have $v=0$ if $2p<n+1$ and v is harmonic if $2p=n+1$. Now if we define w by

$$w = i(\eta)\, u: \qquad \eta^{\alpha} u_{\alpha\lambda_2\cdots\lambda_p}$$

then we have $\delta w = \delta i(\eta) u = -v$. Hence when $2p=n+1$, v being harmonic, we have $v=0$, too. Consequently we obtain

LEMMA 6.1. *Let u be a harmonic p-form, ($p \geqq 2$), in an n-dimensional compact Sasakian space. If $p \leqq (1/2)(n+1)$, then*

$$i(\eta)\, u \qquad \textit{is coclosed.}$$

7. Harmonic tensors in a compact Sasakian space. In this section we shall prove the following

THEOREM 7.1. *In an n dimensional compact Sasakian space, any harmonic p-form u is orthogonal to η, i.e., $i(\eta)u = 0$, provided that $p < (1/2)(n+1)$.*

If $p=1$, this is nothing but Theorem 4.1, so we shall assume $p \geqq 2$. To prove this theorem we introduce w, α and β by

$$w = i(\eta)\, u\ :\quad w_{\lambda_2\cdots\lambda_p} = \eta^{\alpha} u_{\alpha\lambda_2\cdots\lambda_p}\,,$$

$$\alpha = \eta\wedge w\ :\quad \alpha_{\lambda_1\cdots\lambda_p} = \sum_{i=1}^{p} (-1)^{i-1} \eta_{\lambda_i} w_{\lambda_1\cdots\hat{\lambda}_i\cdots\lambda_p}\,,$$

$$\beta = u - \alpha\ :\quad u_{\lambda_1\cdots\lambda_p} = \alpha_{\lambda_1\cdots\lambda_p} + \beta_{\lambda_1\cdots\lambda_p}\,,$$

where $\widehat{\lambda}_i$ means that λ_i is omitted.

We can easily see that β is orthogonal to η, $i(\eta)\beta=0$, and α being coclosed by Lemma 2.4 and Lemma 6.1 β is coclosed, too.

We shall need the following

LEMMA 7.2. *In an n dimensional compact Sasakian space, p-form $\alpha=\eta\wedge i(\eta)u$ and $\beta=u-\alpha$ are harmonic for any harmonic p-form u, provided that $p\leqq(1/2)(n+1)$.*

PROOF. As we have $\Delta\beta=-\Delta\alpha$, it holds that

$$\beta^{\lambda_1\cdots\lambda_p}(\Delta\beta)_{\lambda_1\cdots\lambda_p} = -\beta^{\lambda_1\cdots\lambda_p}(\Delta\alpha)_{\lambda_1\cdots\lambda_p}.$$

On the other hand we have, taking account of (3.5), (3.6) and $i(\eta)\beta=0$,

$$\begin{aligned}\beta^{\lambda_1\cdots\lambda_p}R_{\lambda_i}{}^{\varepsilon}\alpha_{\lambda_1\cdots\varepsilon\cdots\lambda_p} &= \beta^{\lambda_1\cdots\lambda_p}R_{\lambda_i}{}^{\varepsilon}\Big[\sum_{k\neq i}(-1)^{k-1}\eta_{\lambda_k}w_{\lambda_1\cdots\hat{\lambda}_k\cdots\varepsilon\cdots\lambda_p}+(-1)^{i-1}\eta_\varepsilon w_{\lambda_1\cdots\hat{\lambda}_i\cdots\lambda_p}\Big]\\ &=(-1)^{i-1}\beta^{\lambda_1\cdots\lambda_p}R_{\lambda_i}{}^{\varepsilon}\eta_\varepsilon w_{\lambda_1\cdots\hat{\lambda}_i\cdots\lambda_p}=0,\end{aligned}$$

$$\begin{aligned}\beta^{\lambda_1\cdots\lambda_p}R_{\lambda_j\lambda_i}{}^{\rho\sigma}\alpha_{\lambda_1\cdots\rho\cdots\sigma\cdots\lambda_p} &= \beta^{\lambda_1\cdots\lambda_p}R_{\lambda_j\lambda_i}{}^{\rho\sigma}\Big[\sum_{k\neq i,j}(-1)^{k-1}\eta_{\lambda_k}\alpha_{\lambda_1\cdots\hat{\lambda}_k\cdots\rho\cdots\sigma\cdots\lambda_p}\\ &\quad+(-1)^{j-1}\eta_\rho\alpha_{\lambda_1\cdots\hat{\lambda}_j\cdots\sigma\cdots\lambda_p}+(-1)^{i-1}\eta_\sigma\alpha_{\lambda_1\cdots\rho\cdots\hat{\lambda}_i\cdots\lambda_p}\Big]=0.\end{aligned}$$

Thus we can get

$$\begin{aligned}\beta^{\lambda_1\cdots\lambda_p}(\Delta\beta)_{\lambda_1\cdots\lambda_p} &= -\beta^{\lambda_1\cdots\lambda_p}\nabla^\varepsilon\nabla_\varepsilon\alpha_{\lambda_1\cdots\lambda_p}\\ &=2\beta^{\lambda_1\cdots\lambda_p}\sum_{i=1}^{p}(-1)^{i+1}\varphi_{\lambda_i}{}^{\varepsilon}\nabla_\varepsilon w_{\lambda_1\cdots\hat{\lambda}_i\cdots\lambda_p}.\end{aligned}\tag{7.1}$$

As w is closed by Lemma 2.2, we have for a fixed i,

$$\nabla_\varepsilon w_{\lambda_1\cdots\hat{\lambda}_i\cdots\lambda_p}=\sum_{j\neq i}\nabla_{\lambda_j}w_{\lambda_1\cdots\varepsilon\cdots\hat{\lambda}_i\cdots\lambda_p},$$

where the subscript ε appears at the j-th position if $j<i$ and at the $(j-1)$-th position if $j>i$. Hence if, for fixed i and $j(i\neq j)$, we define $A_{(i)}^{\lambda_j}$ by

$$A_{(i)}^{\lambda_j}=\beta^{\lambda_1\cdots\lambda_p}\varphi_{\lambda_i}{}^{\varepsilon}w_{\lambda_1\cdots\varepsilon\cdots\hat{\lambda}_i\cdots\lambda_p},$$

then it follows that

$$\begin{aligned}\beta^{\lambda_1\cdots\lambda_p}\varphi_{\lambda_i}{}^{\varepsilon}\nabla_{\varepsilon}w_{\lambda_1\cdots\hat{\lambda}_i\cdots\lambda_p} &= \beta^{\lambda_1\cdots\lambda_p}\varphi_{\lambda_i}{}^{\varepsilon}\sum_{j\neq i}\nabla_{\lambda_j}w_{\lambda_1\cdots\varepsilon\cdots\hat{\lambda}_i\cdots\lambda_p} \\ &= \sum_{j\neq i}[\nabla_{\lambda_j}A^{\lambda_j}_{(i)} - \beta^{\lambda_1\cdots\lambda_p}w_{\lambda_1\cdots\varepsilon\cdots\hat{\lambda}_i\cdots\lambda_p}\nabla_{\lambda_j}\varphi_{\lambda_i}{}^{\varepsilon}] \\ &= \sum_{j\neq i}\nabla_{\lambda_j}A^{\lambda_j}_{(i)}\,.\end{aligned}$$

Substituting the last equation into (7.1) we get $(\Delta\beta, \beta) = 0$, from which it follows that β and hence α are harmonic. Q.E.D.

PROOF OF THEOREM 7.1. As $\alpha = \eta\wedge w$ is harmonic, we have $d\eta\wedge w = 0$. Hence it follows that

$$\varphi_{\lambda_1\lambda_2}w_{\lambda_3\cdots\lambda_{p+1}} - \sum_{i=3}^{p+1}\varphi_{\lambda_1\lambda_i}w_{\lambda_3\cdots\lambda_2\cdots\lambda_{p+1}} - \sum_{j=3}^{p+1}\varphi_{\lambda_j\lambda_2}w_{\lambda_3\cdots\lambda_1\cdots\lambda_{p+1}} + \sum_{j<i}\varphi_{\lambda_j\lambda_i}w_{\lambda_3\cdots\lambda_1\cdots\lambda_2\cdots\lambda_{p+1}} = 0.$$

Transvecting the last equation with $\varphi^{\lambda_1\lambda_2}$ we can get $(n+1-2p)w = 0$, so if $n+1 > 2p$, we have $w=0$. Q.E.D.

In the case when $2p = n+1$, we have from Lemma 2.2, 6.1 and 7.2 the following

THEOREM 7.3. *In an n dimensional compact Sasakian space, if u is a harmonic* $(1/2)(n+1)$*-form, then* $i(\eta)u$ *and* $\eta\wedge i(\eta)u$ *are harmonic.*

8. **An operator** Φ. In an n dimensional compact Sasakian space we shall introduce an operator

$$\Phi: u \longrightarrow \Phi u = \overset{*}{u}$$

which is defined by

$$\Phi u: \quad \overset{*}{u}_{\lambda_1\cdots\lambda_p} = \sum_{i=1}^{p}\varphi_{\lambda_i}{}^{\alpha}u_{\lambda_1\cdots\alpha\cdots\lambda_p}\,,$$

where u is a p-form and the subscript α appears at the i-th position.

The purpose of this section is to prove the following

THEOREM 8.1. *In an n dimensional compact Sasakian space, if u is a harmonic p-form and* $p < (1/2)(n+1)$, *then* Φu *is harmonic too.*

PROOF. If $p=1$, this is nothing but Theorem 4.2, so we shall assume $p \geqq 2$. Let u be harmonic and assume that $p<(1/2)(n+1)$, then we have $\Delta u=0$ and $i(\eta)u=0$.

We shall show in the following that $(\Delta \overset{*}{u})_{\lambda_1\cdots\lambda_p}\overset{*}{u}^{\lambda_1\cdots\lambda_p}=0$. At first on taking account of $\varphi_{\lambda_i}{}^{\alpha}=\nabla_{\lambda_i}\eta^{\alpha}$, (3.2), (3.4) and $i(\eta)u=0$ we can get easily

$$\nabla^{\varepsilon}\nabla_{\varepsilon}\overset{*}{u}_{\lambda_1\cdots\lambda_p}=\sum_{i=1}^{p}\varphi_{\lambda_i}{}^{\alpha}\nabla^{\varepsilon}\nabla_{\varepsilon}u_{\lambda_1\cdots\alpha\cdots\lambda_p}$$

and by virtue of $\Delta u=0$ we have

$$\nabla^{\varepsilon}\nabla_{\varepsilon}\overset{*}{u}_{\lambda_1\cdots\lambda_p}=\sum_{i}\varphi_{\lambda_i}{}^{\alpha}\Big[R_{\alpha}^{\sigma}u_{\lambda_1\cdots\sigma\cdots\lambda_p}+\sum_{j\neq i}R_{\lambda_j}{}^{\sigma}u_{\lambda_1\cdots\sigma\cdots\alpha\cdots\lambda_p}$$

$$+\sum_{k<i}R_{\lambda_k\alpha}{}^{\rho\sigma}u_{\lambda_1\cdots\rho\cdots\sigma\cdots\lambda_p}+\sum_{k>i}R_{\alpha\lambda_k}{}^{\sigma\rho}u_{\lambda_1\cdots\sigma\cdots\rho\cdots\lambda_p}$$

$$+\sum_{\substack{k<j\\ k,j\neq i}}R_{\lambda_k\lambda_j}{}^{\rho\sigma}u_{\lambda_1\cdots\rho\cdots\alpha\cdots\sigma\cdots\lambda_p}\Big].$$

If we take account of (3.10), then the last equation is written as the following form,

$$\nabla^{\varepsilon}\nabla_{\varepsilon}\overset{*}{u}_{\lambda_1\cdots\lambda_p}-\sum_{i=1}^{p}R_{\lambda_i}{}^{\sigma}\overset{*}{u}_{\lambda_1\cdots\sigma\cdots\lambda_p}=\sum_{i=1}^{p}T^{(i)}_{\lambda_1\cdots\lambda_p}, \tag{8.1}$$

where we have put

$$T^{(i)}_{\lambda_1\cdots\lambda_p}=-\sum_{k\neq i}\varphi_{\lambda_i}{}^{\varepsilon}R_{\varepsilon\lambda_k\rho\alpha}u_{\lambda_1\cdots}{}^{\rho}{}_{\cdots}{}^{\alpha}{}_{\cdots\lambda_p}+\sum_{\substack{k<j\\ k,j\neq i}}\varphi_{\lambda_i}{}^{\alpha}R_{\lambda_k\lambda_j}{}^{\rho\sigma}u_{\lambda_1\cdots\rho\cdots\alpha\cdots\sigma\cdots\lambda_p},$$

where scripts ρ, α and σ of u are at the k-th, i-th and j-th positions respectively.

Next we shall compute $T^{(i)}_{\lambda_1\cdots\lambda_p}\overset{*}{u}^{\lambda_1\cdots\lambda_p}$. By virtue of the first Bianchi's identity, (3.8) and the skew-symmetric property of u and $\overset{*}{u}$, we have

$$\begin{aligned}
&\varphi_{\lambda_i}{}^{\varepsilon}R_{\varepsilon\lambda_k\rho\alpha}u_{\lambda_1\cdots}{}^{\rho}{}_{\cdots}{}^{\alpha}{}_{\cdots\lambda_p}\overset{*}{u}^{\lambda_1\cdots\lambda_p}\\
&\quad=\varphi_{\lambda_i}{}^{\varepsilon}(-R_{\varepsilon\rho\alpha\lambda_k}-R_{\varepsilon\alpha\lambda_k\rho})u_{\lambda_1\cdots}{}^{\rho}{}_{\cdots}{}^{\alpha}{}_{\cdots\lambda_p}\overset{*}{u}^{\lambda_1\cdots\lambda_p}\\
&\quad=-2\varphi_{\lambda_i}{}^{\varepsilon}R_{\varepsilon\rho\alpha\lambda_k}u_{\lambda_1\cdots}{}^{\rho}{}_{\cdots}{}^{\alpha}{}_{\cdots\lambda_p}\overset{*}{u}^{\lambda_1\cdots\lambda_p}\\
&\quad=-2[-R_{\alpha\lambda_k\lambda_i\varepsilon}\varphi_{\rho}{}^{\varepsilon}+\varphi_{\alpha\lambda_i}g_{\lambda_k\rho}-\varphi_{\alpha\rho}g_{\lambda_k\lambda_i}-\varphi_{\lambda_k\lambda_i}g_{\alpha\rho}+\varphi_{\lambda_k\rho}g_{\alpha\lambda_i}]\\
&\qquad\qquad\times u_{\lambda_1\cdots}{}^{\rho}{}_{\cdots}{}^{\alpha}{}_{\cdots\lambda_p}\overset{*}{u}^{\lambda_1\cdots\lambda_p}
\end{aligned}$$

$$= 2R_{\alpha\lambda_k\lambda_i\varepsilon}\varphi_\rho{}^\varepsilon u_{\lambda_1\cdots}{}^{\rho}{}_{\cdots}{}^{\alpha}{}_{\cdots\lambda_p}\overset{*}{u}{}^{\lambda_1\cdots\lambda_p}$$

$$= R_{\alpha\varepsilon\lambda_i\lambda_k}\varphi_\rho{}^\varepsilon u_{\lambda_1\cdots}{}^{\rho}{}_{\cdots}{}^{\alpha}{}_{\cdots\lambda_p}\overset{*}{u}{}^{\lambda_1\cdots\lambda_p} = R_{\lambda_k\lambda_i}{}^{\alpha\varepsilon}\varphi_\varepsilon{}^\rho u_{\lambda_1\cdots\rho\cdots\alpha\cdots\lambda_p}\overset{*}{u}{}^{\lambda_1\cdots\lambda_p}$$

Thus we can get

$$\text{(8. 2)} \qquad \sum_{i=1}^{p} T^{(i)}_{\lambda_1\cdots\lambda_p}\overset{*}{u}{}^{\lambda_1\cdots\lambda_p} = -\sum_i\sum_{k\neq i} R_{\lambda_k\lambda_i}{}^{\alpha\varepsilon}\varphi_\varepsilon{}^\rho u_{\lambda_1\cdots\rho\cdots\alpha\cdots\lambda_p}\overset{*}{u}{}^{\lambda_1\cdots\lambda_p}$$

$$+\sum_i\sum_{\substack{k<j\\ k,j\neq i}} R_{\lambda_k\lambda_j}{}^{\rho\sigma}\varphi_{\lambda_i}{}^\alpha u_{\lambda_1\cdots\rho\cdots\alpha\cdots\sigma\cdots\lambda_p}\overset{*}{u}{}^{\lambda_1\cdots\lambda_p}$$

$$= \sum_{k<j} R_{\lambda_k\lambda_j}{}^{\rho\sigma}\overset{*}{u}_{\lambda_1\cdots\rho\cdots\sigma\cdots\lambda_p}\overset{*}{u}{}^{\lambda_1\cdots\lambda_p}.$$

Hence, from (8.1) and (8.2), we have $(\Delta\overset{*}{u})_{\lambda_1\cdots\lambda_p}\overset{*}{u}{}^{\lambda_1\cdots\lambda_p}=0$ and this completes the proof. Q.E.D.

Let u be a harmonic p-form, $p<(1/2)(n+1)$, in an n dimensional compact Sasakian space, then we know that $\Phi u, \Phi^2 u, \cdots$ are harmonic p-forms and hence the following p p-forms

$$\sum_{i=1}^{p}\varphi_{\lambda_i}{}^\alpha u_{\lambda_1\cdots\alpha\cdots\lambda_p}, \quad \sum_i\sum_{j\neq i}\varphi_{\lambda_j}{}^\beta\varphi_{\lambda_i}{}^\alpha u_{\lambda_1\cdots\beta\cdots\alpha\cdots\lambda_p}, \quad \cdots$$

$$\cdots, \quad \varphi_{\lambda_p}{}^{\alpha_p}\cdots\varphi_{\lambda_1}{}^{\alpha_1}u_{\alpha_1\cdots\alpha_p}$$

are harmonic.

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