ON DERIVATIONS OF NILPOTENT LIE ALGEBRAS

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By E. Schenkman and N. Jacobson [1], it is shown that a nilpotent Lie algebra over a field of characteristic 0 always has a non-inner derivation. That is, denote by $\mathfrak{D}(N)$ and $\mathfrak{J}(N)$ the derivation algebra and the collection of all inner derivations of a nilpotent Lie algebra N respectively, then $\mathfrak{D}(N) \supsetneqq \mathfrak{J}(N)$. Clearly $\mathfrak{J}(N)$ is contained in the maximal nilpotent ideal $\mathfrak{N}$ of $\mathfrak{D}(N)$, hence $\mathfrak{J}(N)$ is contained in the radical $\mathfrak{R}$ of $\mathfrak{D}(N)$. In the present note we shall show that $\mathfrak{R}$ does not coincide with $\mathfrak{J}(N)$. That is, the Schenkman-Jacobson's result may be strengthened as follows:

THEOREM. *Let $\mathfrak{D}(N)$ be the derivation algebra of a nilpotent Lie algebra N over a field of characteristic* 0, *then the radical $\mathfrak{R}$ of $\mathfrak{D}(N)$ always contains a non-inner derivation.*

To prove the theorem we prepare the following lemma.

LEMMA. *Let N be a nilpotent Lie algebra whose center $Z(N)$ is contained in the derived subalgebra $[N, N]$, and let us suppose that for any ideal M with codimension* 1 *of N, the center $Z(M)$ of M is not contained in $[N, N]$. Then N belongs to either of following two types:*

Type (i). *N has an ideal M with codimension* 1 *and an element e such that*

$$e \notin M,\ [e,[N,N]] = 0 \quad \text{and} \quad [e, Z(M)] \subsetneqq Z(N)\,.$$

Type (ii). *N has a basis $\{e_{11}, e_{12}, e_{21}, e_{22}, \cdots, e_{r1}, e_{r2}, z_0\}$ which satisfies the conditions*

$$[e_{ij}, e_{kl}] = 0 \qquad for \qquad i \neq k\,,$$

$$[e_{i1}, e_{i2}] = z_0\,,$$

$$[e_{ij}, z_0] = 0\,.$$

PROOF. Let M be an arbitrary ideal with codimension 1. We note first that such an ideal M always contains the derived subalgebra $[N, N]$. In fact, let a be an element of N which does not belong to M, then it holds that for any scalars λ and μ,

$$[\lambda a+M, \mu a+M] \subset \lambda[a, M] + \mu[M, a] + [M, M] \subset M.$$

Now let e be an element of $Z(M)$ which does not belong to $[N, N]$, and we take an ideal M' which satisfies $N = \{e\} + M'$.[1] In virtue of $[e, [N, N]] = 0$, it holds that $Z(M') \cap [N, N] = Z(N)$. We extend the basis of $Z(N)$ to the one of $Z(M')$ by adjoining $z_1, \cdots, z_s$. Then $[e, z_i] \in Z(N)$, hence the restriction of ad e to the space $\{z_1, \cdots, z_s\}$ gives a homomorphism into $Z(N)$. Actually, this is an isomorphism. In fact, if $[e, \sum \alpha_i z_i] = 0$ then $\sum \alpha_i z_i \in Z(N)$ and $\sum \alpha_i z_i = 0$. Hence if this is not an onto-isomorphism, N has the type (i). Otherwise, $\dim Z(N) = s$ and $[e, z_1], \cdots, [e, z_s]$ constitute a basis of $Z(N)$. Let us set

$$M' = \{z_1, z_2, \cdots, z_s, m_1, \cdots, m_t, [N, N]\}$$

and

$$M'' = \{e, z_2, \cdots, z_s, m_1, \cdots, m_t, [N, N]\} .$$

Then we may suppose that $[z_1, Z(M'')] = Z(N)$, because N belongs to the type (i) if it not so. If we represent an element z'' of $Z(M'')$ which is not contained in $[N, N]$ such as

$$z'' = \alpha e + \beta_2 z_2 + \cdots + \beta_s z_s + \gamma_1 m_1 + \cdots + \gamma_t m_t + n \quad (n \in [N, N]),$$

then $[z_1, z''] = \alpha[z_1, e]$. The element of the right-hand side is independent of the choice of z'', except for a scalar multiple. Hence $\dim Z(N) = 1$, $s = 1$, and $\alpha \neq 0$. So we may replace e by z'' when we take the basis of M''. Thus we obtain

$$[z_1, m_i] = [z'', m_i] = [z_1, [N, N]] = [z'', [N, N]] = 0 .$$

$$Z(N) = \{[z_1, z'']\} .$$

Now we suppose that $t > 0$, i.e. the set $\{m_1, \cdots, m_t\}$ is not empty, and let f be the element of the center of the ideal $\{z_1, z'', m_2, \cdots, m_t, [N, N]\}$ which does not belong to $[N, N]$. If we express f so that

$$f = \alpha_1 z_1 + \alpha_2 z'' + \beta_2 m_2 + \cdots + \beta_t m_t + n \quad (n \in [N, N]),$$

1) By $\{a, b, c, \cdots\}$ we mean the space generated by $a, b, c, \cdots$ linearly independently.

then from the relations $[f, z_1]=[f, z'']=0$ and $[z_1, z'']\neq 0$, we get $\alpha_1=\alpha_2=0$. Hence $t>0$ implies that $t\geqq 2$, and by changing the indices of m_i suitably, we may express N as follows:

$$N=\{m_1, z_1, z'', f, m_3, \cdots, m_t, [N, N]\}.$$

Again, let f' be the element of the center of the ideal $\{z_1, z'', m_1, m_3, \cdots, m_t, [N, N]\}$ which does not belong to $[N, N]$. Because of

$$f' \in \{m_1, m_3, \cdots, m_t, [N, N]\}$$

and

$$\bar{\in} \{m_3, \cdots\cdots, m_t, [N, N]\},$$

we may replace m_1 by f', and we may suppose that $[f, f']=[z_1, z'']$. By repeating this process and by changing the notations such as $z_1 \to e_{11}$, $z'' \to e_{12}$, $f \to e_{21}$, $f' \to e_{22}, \cdots$, we may conclude that our case is the type (ii).

PROOF OF THE THEOREM. We shall prove the theorem by constructing a non-inner derivation D_0 contained in a certain solvable ideal of $\mathfrak{D}(N)$. We devide the cases into following three (1), (2) and (3).

(1) *The center $Z(N)$ is not contained in $[N, N]$.*

We extend the basis of $Z(N)\cap[N, N]$ to the one of $Z(N)$ by adding $z_1, \cdots, z_k$. The subspace A generated by $z_1, \cdots, z_k$ is an abelian ideal of N and N is decomposed in a direct sum of ideals such as

$$N=A+M \quad (M\supset[N, N]),$$

and the center $Z[M]$ of M is contained in $[M, M]=[N, N]$. According to S. Togo [2], $\mathfrak{D}(N)$ is decomposed into the following form:

$$\mathfrak{D}(N)=\mathfrak{D}(A)+\mathfrak{D}(M)+\mathfrak{D}(A, M)+\mathfrak{D}(M, A),$$

where $\mathfrak{D}(A)$ is the set of all derivations which transform A into A and M into 0, and $\mathfrak{D}(A, M)$ is all of the derivations which transform A into $Z(M)$ and $[A, A]+M$ into 0. $\mathfrak{D}(M)$ and $\mathfrak{D}(M, A)$ are also defined similarly. Now let D_0 be the derivation which acts on A identically and transforms M into 0, then obviously it holds that

$$[D_0, \mathfrak{D}(A)]=[D_0, \mathfrak{D}(M)]=0,$$

$$[D_0, \mathfrak{D}(A, M)]=\mathfrak{D}(A, M), \qquad [D_0, \mathfrak{D}(M, A)]=\mathfrak{D}(M, A).$$

And if we denote by $\mathfrak{C}(M)$ the set of all derivations if $\mathfrak{D}(M)$ which transform M into $Z(M)$, then

$$\mathfrak{D}(M, A)\mathfrak{D}(A, M)(A) \subset \mathfrak{D}(M, A)(Z(M)) \subset \mathfrak{D}(M, A)([M, M]) = 0\,,$$

hence

$$[\mathfrak{D}(M, A), \mathfrak{D}(A, M)] \subset \mathfrak{C}(M)\,.$$

And by the same reason, we get

$$[\mathfrak{D}(M, A), \mathfrak{C}(M)] = \mathfrak{D}(M, A)\,\mathfrak{C}(M) = 0$$

and similarly

$$[\mathfrak{D}(A, M), \mathfrak{C}(M)] = 0\,.$$

Hence

$$\mathfrak{D}' = \{D_0, \mathfrak{D}(A, M), \mathfrak{D}(M, A), \mathfrak{C}(M)\}$$

is an ideal of $\mathfrak{D}(N)$, and it holds that

$$[\mathfrak{D}', \mathfrak{D}'] \subset \{\mathfrak{D}(A, M), \mathfrak{D}(M, A), \mathfrak{C}(M)\}$$
$$[[\mathfrak{D}', \mathfrak{D}'], [\mathfrak{D}', \mathfrak{D}']] \subset \mathfrak{C}(M)\,.$$

Since $\mathfrak{C}(M)$ is abelian, $\mathfrak{D}'$ is a solvable ideal. It is obvious that D_0 is a non-inner derivation, for any inner derivation sends the element of the center into 0.

(2) $Z(N) \subset [N, N]$ *and there exists an ideal* M *with codimension* 1 *whose center* $Z(M)$ *is contained in* $[N, N]$.

Let us set $N = \{e\} + M$, $N^1 = N$, and $N^i = [N, N^{i-1}]$ for $i \geqq 2$. Now we suppose that

$$Z(M) \subset N^i \quad \text{and} \quad \not\subset N^{i+1}\,.$$

Then by our assumption it must be $i \geqq 2$. We choose an element z such as

$$z \in Z(M) \quad \text{and} \quad \notin N^{i+1}\,.$$

Then the linear transformation D_0 sending e to z and M to 0 is a derivation of N. In fact, for m, $m' \in M$,

$$D([\lambda e + m, \mu e + m']) = 0$$

and
$$[D(\lambda e + m), \mu e + m'] + [\lambda e + m, D(\mu e + m')]$$
$$= \lambda\mu[z, e] + \lambda\mu[e, z] = 0\,.$$

Now we suppose that D_0 is represented as $\operatorname{ad} q$. Then $[q, M] = D_0(M) = 0$. We express q as $\lambda e + m$ where $m \in M$. Since $[\lambda e + m, m] = 0$, if $\lambda \neq 0$, then we obtain $[e, m] = 0$. Hence $z = [\lambda e + m,\ e] = [m, e] = 0$. But this contradicts to our assumption $z \notin N^{i+1}$. Therefore $\lambda = 0$ and $q \in Z(M)$. Then

$$z = [q, e] \in [Z(M), e] \subset [N^i, N] = N^{i+1}\,,$$

which is impossible. Hence D_0 is not inner. Now,

$$\mathfrak{D}' = \{D \in \mathfrak{D}(N)\,;\ D(N) \subset [N, N]\}$$

is an ideal of $\mathfrak{D}(N)$ containing D_0. And this is nilpotent. In fact for $D, D' \in \mathfrak{D}'$,

$$DD'(N) \subset D([N, N]) \subset [D(N), N] \subset N^3\,, \quad \text{etc.}$$

(3) $Z(N) \subset [N, N]$, *and for any ideal* M *of codimension* 1, $Z(M)$ *is not contained in* $[N, N]$.

By the above lemma it is sufficient to treat two types (i) and (ii).

Type (i) Let us take the ideal M with codimension 1 such that

$$N = \{e\} + M\,, \qquad [e, [N, N]] = 0\,,$$

and
$$[e, Z(M)] \subsetneqq Z(N)\,.$$

Then we may choose the element z such that

$$z \in Z(N)\,, \ \text{ and } \ \notin [e, Z(M)]\,.$$

We denote by D_0 the endomorphism of N which sends e to z and M to 0. Similarly as in the case (2) we may show that D_0 is a derivation, but is not inner. In fact, if $D_0 = \operatorname{ad} q$, then $q \in Z(M)$ and $z = [q, e] \in [e, Z(M)]$ which is a contradiction. And we may prove that D_0 is contained in a nilpotent ideal in a quite similar way.

Type (ii) The endomorphism D_0 such that $e_{ij}\to e_{ij}$, $z_0\to 2z_0$ is a derivation and is not inner, for $D_0(N)$ is not contained in $[N, N]$. We denote by $\mathfrak{C}(N)$ the set of all derivations which send N into $Z(N)$, and let D be an arbitrary derivation. D_0 and D have the following matrix forms:

$$D_0 = \begin{pmatrix} 1 & & & & & \\ & 1 & & & 0 & \\ & & \cdot & & & \\ & & & \cdot & & \\ & & & & \cdot & \\ 0 & & & & 1 & \\ & & & & & 2 \end{pmatrix}, \qquad D = \begin{pmatrix} & & & & 0 \\ & & & & 0 \\ & & * & & \cdot \\ & & & & \cdot \\ & & & & \cdot \\ & & & & 0 \\ * & * & \cdots & * & * \end{pmatrix}.$$

Hence $[D, D_0]$ has only 0 elements except for the lowest row, namely $[D, D_0] \in \mathfrak{C}(N)$. Hence $\{D_0, \mathfrak{C}(N)\}$ is an ideal of $\mathfrak{D}(N)$. And its derived subalgebra is contained in abelian ideal $\mathfrak{C}(N)$. Therefore $\{D_0, \mathfrak{C}(N)\}$ is a solovable ideal.

Thus the theorem is proved.

REMARK. It may occur that the maximal nilpotent ideal $\mathfrak{N}$ of $\mathfrak{D}(N)$ consists of only inner derivations. For example, let N be a Lie algebra whose basis $\{e_1, e_2, e_3\}$ satisfies the following multiplication rules:

$$[e_1, e_2] = e_3, \qquad [e_1, e_3] = [e_2, e_3] = 0.$$

Then N belongs to the type (ii) mentioned above. And it is easily shown that $\mathfrak{N}$ is generated by $\mathrm{ad}\, e_1$ and $\mathrm{ad}\, e_2$.

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