SUMMABILITY OF DOUBLE FOURIER SERIES

HIROSHI WATANABE

(Received November 8, 1964 and in revised form, February 24, 1965)

1. Let $f(x,y)$ be integrable in the square $|x| \leqq \pi$, $|y| \leqq \pi$, let its Fourier series be

$$(1) \qquad \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} c_{m,n} e^{i(mx+ny)},$$

where

$$c_{m,n} = \frac{1}{(2\pi)^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} f(x,y) e^{-i(mx+ny)}\, dx\, dy,$$

and let its (m,n)-th partial sums be

$$s_{m,n}(x,y) = \sum_{k=-m}^{m} \sum_{l=-n}^{n} c_{k,l}\, e^{i(kx+ly)}.$$

J. Marcinkiewicz [1], G. Grünwald [2] and J. G. Herriot [3] considered the summability of the sequence

$$(2) \qquad \{s_{n,n}(x,y)\}$$

and obtained the following Theorems A and B.

THEOREM A. *If $f(x,y) \in L^p$ $(p > 1)$, then*

$$\lim_{n\to\infty} H_n(x,y) = 0 \quad \textit{almost everywhere,}$$

and

$$\int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \{H^*(x,y)\}^p\, dx\, dy \leqq A_p \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} |f(x,y)|^p\, dx\, dy,$$

where

$$H_n(x,y) = \left(\frac{1}{n+1} \sum_{k=0}^{n} |s_{k,k}(x,y) - f(x,y)|^2\right)^{1/2},$$

and

$$H^*(x,y) = \sup_n H_n(x,y) .$$

THEOREM B. *If $f(x,y) \in L$, then*

$$\lim_{n\to\infty} \sigma_n(x,y) = f(x,y) \quad \textit{almost everywhere},$$

and for any $t > 0$

$$|\{\sigma^*(x,y) > t\}| \leqq \frac{A}{t} \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} |f(x,y)|\, dx\, dy ,$$

where

$$\sigma_n(x,y) = \frac{1}{n+1} \sum_{k=0}^{n} s_{k,}(_k x, y) ,$$

and

$$\sigma^*(x,y) = \sup_n |\sigma_n(x,y)| .$$

When $\varphi(x,y) \in L$ has a Fourier series of power series type, i.e.

$$\varphi(x,y) \sim \sum_{k=0}^{\infty}\sum_{l=0}^{\infty} c_{k,l}\, e^{i(kx+ly)} ,$$

we denote its partial sums by

$$t_{m,n}(x,y) = \sum_{k=0}^{m}\sum_{l=0}^{n} c_{k,l}\, e^{i(kx+ly)} ,$$

its $(C,1,1)$ means by

$$\tau_{m,n}(x,y) = \frac{1}{(m+1)(n+1)} \sum_{k=0}^{m}\sum_{l=0}^{n} t_{k,l}(x,y) ,$$

and we put

$$T^*(x,y) = \left(\sum_{n=0}^{\infty} \frac{|t_{n,n}(x,y) - \tau_{n,n}(x,y)|^r}{n+1}\right)^{1/r} ,$$

Then we have the following theorems.

THEOREM 1. *If* $\varphi(x,y)\log^+|\varphi(x,y)| \in L$ *and its Fourier series is of power series type, then we have for* $r \geqq 2$

$$\text{(i)} \qquad \left(\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \{T^*(x,y)\}^\mu \, dx\, dy\right)^{1/\mu}$$

$$\leqq A_{\mu,r}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} |\varphi(x,y)|\log^+|\varphi(x,y)|\, dx\, dy + A'_{\mu,r}, \quad 0<\mu<1 .$$

$$\text{(ii)} \qquad \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \{T^*(x,y)\}\, dx\, dy$$

$$\leqq B_r \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} |\varphi(x,y)|(\log^+|\varphi(x,y)|)^2\, dx\, dy + B'_r .$$

THEOREM 2. *If* $f(x,y) \in L$, *then we have*

$$\lim_{n\to\infty} \gamma_n(x,y) = f(x,y) \quad \textit{almost everywhere},$$

and

$$|\{\gamma^*(x,y) > t\}| < \frac{A}{t}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} |f(x,y)|\, dx\, dy ,$$

where

$$p_n = [n^k], \quad 0 < k \leqq 1 ,$$

$$\gamma_n(x,y) = \frac{1}{n+1}\sum_{m=0}^{n} s_{p_m, pm}(x,y),$$

and

$$\gamma^*(x,y) = \sup_n |\gamma_n(x,y)| .$$

This theorem is due to G. Maruyama [4] in one variable case.

2. To prove Theorem 1, we need several lemmas.

LEMMA 1. *Let* $f_n(x) \in L$, *and let its conjugate function be* $\tilde{f}_n(x)$. *Then we have for* $r > 1$

(i) $$\int_{-\pi}^{\pi}\Big(\sum_n |\tilde{f}_n(x)|^r\Big)^{\mu/r} dx$$
$$\leqq A_{\mu,r}\Big(\int_{-\pi}^{\pi}\Big(\sum_n |f_n(x)|^r\Big)^{1/r} dx\Big)^{\mu}, \quad 0<\mu<1.$$

(ii) $$\int_{-\pi}^{\pi}\Big(\sum_n |\tilde{f}_n(x)|^r\Big)^{1/r} dx$$
$$\leqq A_r \int_{-\pi}^{\pi}\Big(\sum_n |f_n(x)|^r\Big)^{1/r} \log^+\Big(\sum_n |f_n(x)|^r\Big)^{1/r} dx + A'_r.$$

This is known, see G. Sunouchi [5].

LEMMA 2. *Let $f_n(x)\in L$, and let the k_n-th partial sum of its Fourier series be $s_{n,k_n}(x)$. Then we have for $r>1$,*

(i) $$\int_{-\pi}^{\pi}\Big(\sum_n |s_{n,k_n}(x)|^r\Big)^{\mu/r} dx$$
$$\leqq A_{r,\mu}\Big(\int_{-\pi}^{\pi}\Big(\sum_n |f_n(x)|^r\Big)^{1/r} dx\Big)^{\mu}, \quad 0<\mu<1,$$

(ii) $$\int_{-\pi}^{\pi}\Big(\sum_n |s_{n,k_n}(x)|^r\Big)^{1/r} dx$$
$$\leqq A_r \int_{-\pi}^{\pi}\Big(\sum_n |f_n(x)|^r\Big)^{1/r} \log^+\Big(\sum_n |f_n(x)|^r\Big)^{1/r} dx + A'_r.$$

This is derived from Lemma 1 in the usual way.

LEMMA 3. *Let $\phi(z)=\sum_{n=0}^{\infty} c_n z^n$, $z=\rho e^{ix}$, be regular in the interior of a unit circle If we put*

$$g_r(x)=\Big(\int_0^1 (1-\rho)^{r-1}|\phi'(z)|^r d\rho\Big)^{1/r},$$

then we have for $r\geqq 2$

$$\int_{-\pi}^{\pi}\{g_r(x)\}^p dx \leqq A_p \int_{-\pi}^{\pi}|\phi(e^{ix})|^p dx, \quad p>1.$$

This is due to G. Sunouchi [6].

LEMMA 4. *Let $\varphi(x) \in L$ have a Fourier series of power series type, i.e.*

$$\varphi(x) \sim \sum_{n=0}^{\infty} c_n e^{inx} . \tag{3}$$

We denote the partial sums and $(C, 1)$ *means of* (3) *by* $t_n(x)$ *and* $\tau_n(x)$, *respectively. If we put*

$$T^*(x) = \left(\sum_{n=0}^{\infty} \frac{|t_n(x) - \tau_n(x)|^r}{n+1} \right)^{1/r} ,$$

then for $r \geqq 2$, *we have*

$$\left\{ \int_{-\pi}^{\pi} (T^*(x))^\mu \, dx \right\}^{1/\mu} \leqq A_{r,\mu} \int_{-\pi}^{\pi} |\varphi(x)| \, dx \tag{i}$$

$$\int_{-\pi}^{\pi} T^*(x) \, dx \leqq B_r \int_{-\pi}^{\pi} |\varphi(x)| \log^+ |\varphi(x)| \, dx + B_r' . \tag{ii}$$

This lemma is proved by the same way as in A. Zygmund [7; p. 237-239] with exponent $r \geqq 2$, using Lemmas 2 and 3.

Now after J. Marcinkiewicz [1], we can prove Theorem 1. We have

$$\begin{aligned} t_{n,n}(x,y) - \tau_{n,n}(x,y) &= t_n^{(1)}\{t_n^{(2)}(\varphi) - \tau_n^{(2)}(\varphi)\} \\ &\quad + \tau_n^{(2)} \{t_n^{(1)}(\varphi) - \tau_n^{(1)}(\varphi)\} \\ &\equiv I_n(x,y) + J_n(x,y) , \end{aligned}$$

where, in general, $t_k^{(1)}(g)$ or $\tau_k^{(1)}(g)$ means the partial sums or $(C, 1)$ means of Fourier series of $g(x,y)$ considered as a function of x only for any fixed y, respectively. $t_k^{(2)}(g)$ or $\tau_k^{(2)}(g)$ means those of y only for any fixed x, respectively. Then we have by Lemma 2 (i) and Lemma 4 (ii)

$$\left(\int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \left\{ \sum \frac{|I_n|^r}{n+1} \right\}^{\mu/r} dx \, dy \right)^{1/\mu}$$

$$\leqq (2\pi)^{1-\mu} \int_{-\pi}^{\pi} \left\{ A_{\mu,r}^{1/\mu} \int_{-\pi}^{\pi} \left(\sum_{k=0}^{\infty} |t_k^{(2)}(\varphi) - \tau_k^{(2)}(\varphi)|^r/(k+1) \right)^{1/r} dx \right\} dy$$

$$\leqq A_{\mu,r}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}|\varphi(x,y)|\log^{+}|\varphi(x,y)|\,dx\,dy + A'_{\mu,r}.$$

Concerning J_n, we have a similar calculation.

The proof of Theorem 1, (ii) is similar to that of above and we may omit it.

3. We now prove Theorem 2. We may suppose that $0 < k < 1$, because that the case $k = 1$ of Theorem 2 was proved by G. Grünwald [2]. We have

$$(5)\quad \gamma_n(x,y) = \frac{1}{n+1}\sum_{k=0}^{n} s_{p_k,p_k}(x,y)$$

$$= \frac{1}{\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f(x+u,y+v)\left[\frac{1}{4(n+1)\sin\frac{u}{2}\sin\frac{v}{2}}\right]D_n(u,v)\,du\,dv,$$

where

$$D_n(u,v) = \sum_{k=0}^{n}\sin\left(p_k+\frac{1}{2}\right)u\sin\left(p_k+\frac{1}{2}\right)v.$$

We consider two functions of a variable s such that

$$g_1(s) = p_i + \frac{1}{2} \quad \textit{for} \quad i \leqq s < i+1,$$

$$(6)\qquad g_2(s) = q_{i+1} + 1 + (s-i-1)(q_{i+1}-q_i) \quad \textit{for} \quad i \leqq s < i+1,$$

where

$$q_i = i^k, \quad 0 < k < 1.$$

Then we have

$$D_n(u,v) = \int_0^{n+1}\sin(g_1(s)u)\sin(g_1(s)v)\,ds.$$

If we put

$$\widetilde{D}_n(u,v) = \int_0^{n+1}\sin(g_2(s)u)\sin(g_2(s)v)\,ds,$$

then

$$\gamma_n(x,y) = \frac{1}{\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f(x+u,y+v)\left[\frac{\widetilde{D}_n(u,v)}{4(n+1)\sin\frac{u}{2}\sin\frac{v}{2}}\right] du\, dv$$

$$+ \frac{1}{\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f(x+u,y+v)\left[\frac{D_n(u,v)-\widetilde{D}_n(u,v)}{4(n+1)\sin\frac{u}{2}\sin\frac{v}{2}}\right] du\, dv$$

$$= \gamma_n^{(1)}(x,y) + \gamma_n^{(2)}(x,y)\,. \tag{7}$$

To estimate the last expressions we need the following lemma and some calculations.

LEMMA 5. *If $f(x,y) \in L$, then we have the inequalities*

$$\sigma^*(x,y) \leqq A(f^*(x,y) + f^{**}(x,y))\,,$$

and

$$|\{f^*(x,y) > t\}| \leqq \frac{A}{t}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} |f(x,y)|\, dx\, dy\,,$$

$$|\{f^{**}(x,y) > t\}| \leqq \frac{A}{t}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} |f(x,y)|\, dx\, dy\,,$$

where

$$f^*(x,y) = \sup_s \{f^*_{2^s}(x,y)\, 2^{-\frac{|s|}{2}}\}\,, \quad s = 0, \pm 1, \pm 2, \cdots$$

$$f^{**}(x,y) = \sup_s \{f^{**}_{2^s}(x,y)\, 2^{-\frac{|s|}{2}}\}\,, \quad s = 0, \pm 1, \pm 2, \cdots$$

and

$$f^*_\alpha(x,y) = \sup_{h>0} \frac{1}{4\alpha h^2}\int_{-\alpha h}^{\alpha h}\int_{-h}^{h} |f(x+u,y+v)|\, du\, dv\,,$$

$$f^{**}_\alpha(x,y) = \sup_{h>0} \frac{1}{4\alpha h^2}\int_{-\alpha h}^{\alpha h} dv \int_{-h+v}^{h+v} |f(x+u,y+v)|\, du\, dv\,.$$

The proof of this lemma is found in G. Grünwald [2].

Now we estimate $\gamma_n^{(2)}(x,y)$ in (7) first. We have

$$D_n(u,v) - \widetilde{D}_n(u,v) = \int_0^{n+1} \{\sin(g_1(s)\,u)\sin(g_1(s)\,v) - \sin(g_2(s)\,u)\sin(g_2(s)\,v)\}\,ds$$

$$= -\int_0^{n+1} \{g_2(s) - g_1(s)\}\,\{u\cos(\xi(s)\,u)\sin(\xi(s)\,v) + v\sin(\xi(s)\,u)\cos(\xi(s)\,v)\}\,ds\,,$$

where

$$g_1(s) < \xi(s) < g_2(s)\,,$$

and from (7) we have

$$|\gamma_n^{(2)}(x,y)| \leqq \frac{A}{n+1}\int_0^{n+1}\{g_2(s) - g_1(s)\}\,ds \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} |f(x+u,y+v)|$$

$$\times \frac{|u\cos(\xi(s)u)\sin(\xi(s)v) + v\sin(\xi(s)u)\cos(\xi(s)v)|}{\left|4\sin\frac{u}{2}\sin\frac{v}{2}\right|}\,du\,dv$$

$$\leqq \frac{A'}{n+1}\int_0^{n+1} f^*_1(x,y)\,ds$$

(8) $$\leqq A'' f^*_1(x,y)\,,$$

since $g_2(s) - g_1(s)$ is bounded from the definition (6).

Finally, to estimate $\gamma_n^{(1)}(x,y)$, we proceed as follows. From (6) and (7) we have

$$\widetilde{D}_n(u,v) = \frac{1}{2}\int_0^{n+1}[\cos\{g_2(s)(u-v)\} - \cos\{g_2(s)(u+v)\}]\,ds$$

$$= \frac{1}{2}\sum_{i=0}^{n}\int_i^{i+1}[\cos\{q_{i+1}+1+(s-i-1)(q_{i+1}-q_i)\}(u-v)$$

$$- \cos\{q_{i+1}+1+(s-i-1)(q_{i+1}-q_i)\}(u+v)]\,ds$$

$$= \frac{1}{2}\sum_{i=0}^{n}\left\{\frac{\sin(q_{i+1}+1)(u-v) - \sin(q_i+1)(u-v)}{(q_{i+1}-q_i)(u-v)}\right.$$

$$\left. - \frac{\sin(q_{i+1}+1)(u+v) - \sin(q_i+1)(u+v)}{(q_{i+1}-q_i)(u+v)}\right\}$$

by partial summation

$$= \frac{1}{2}\sum_{i=0}^{n-1} \frac{\sin(q_{i+1}+1)(u-v) - \sin(q_0+1)(u-v)}{u-v} \Delta\left(\frac{1}{q_{i+1}-q_i}\right)$$

$$- \frac{1}{2}\sum_{i=0}^{n-1} \frac{\sin(q_{i+1}+1)(u+v) - \sin(q_0+1)(u+v)}{u+v} \Delta\left(\frac{1}{q_{i+1}-q_i}\right)$$

$$+ \frac{1}{2} \frac{\sin(q_{n+1}+1)(u-v)}{u-v} \frac{1}{q_{n+1}-q_n} - \frac{1}{2} \frac{\sin(q_{n+1}+1)(u+v)}{u+v} \frac{1}{q_{n+1}-q_n},$$

and we have

$$\widetilde{D}_n(u, v) = \frac{1}{2}\sum_{i=0}^{n-1}\left\{\frac{\sin(q_{i+1}+1)(u-v)}{u-v} - \frac{\sin(q_{i+1}+1)(u+v)}{u+v}\right\}\Delta\left(\frac{1}{q_{i+1}-q_i}\right)$$

$$(9) \qquad - \frac{1}{2}\left\{\frac{\sin(q_0+1)(u-v)}{u-v} - \frac{\sin(q_0+1)(u+v)}{u+v}\right\}\sum_{i=0}^{n-1}\Delta\left(\frac{1}{q_{i+1}-q_i}\right)$$

$$+ \frac{1}{2}\left\{\frac{\sin(q_{n+1}+1)(u-v)}{u-v} - \frac{\sin(q_{n+1}+1)(u+v)}{u+v}\right\}\frac{1}{q_{n+1}-q_n}.$$

If we put

$$\widetilde{\sigma}_n(x, y) = \frac{1}{\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f(x+u, y+v)\frac{1}{8(n+1)\sin\frac{u}{2}\sin\frac{v}{2}}$$

$$\times \left\{\frac{\sin(n+1)(u-v)}{u-v} - \frac{\sin(n+1)(u+v)}{u+v}\right\} du\, dv$$

and

$$\widetilde{\sigma}^*(x, y) = \sup_n |\widetilde{\sigma}_n(x, y)|,$$

then we have from (9)

$$\gamma_n^{(1)}(x, y) = \frac{1}{n+1}\left[\sum_{i=0}^{n-1}\widetilde{\sigma}_{q_{i+1}}(x, y)(q_{i+1}+1)\Delta\left(\frac{1}{q_{i+1}-q_i}\right)\right.$$

$$\left. - \widetilde{\sigma}_{q_0+1}(x, y)(q_0+1)\sum_{i=0}^{n-1}\Delta\left(\frac{1}{q_{i+1}-q_i}\right) + \widetilde{\sigma}_{q_{n+1}}(x, y)(q_{n+1}+1)\frac{1}{q_{n+1}-q_n}\right]$$

and

$$|\gamma_n^{(1)}(x,y)| \leqq \tilde{\sigma}^*(x,y)\frac{1}{n+1}\left[\sum_{i=0}^{n-1}(q_{i+1}+1)\left|\Delta\left(\frac{1}{q_{i+1}-q_i}\right)\right|\right.$$

$$\left.+ (q_0+1)\sum_{i=0}^{n-1}\left|\Delta\left(\frac{1}{q_{i+1}-q_i}\right)\right| + (q_{n+1}+1)\frac{1}{q_{n+1}-q_n}\right].$$

It is easily seen that the expression in the bracket is finite since $q_i = i^k$, $0<k<1$, so we have

$$(10) \qquad |\gamma_n^{(1)}(x,y)| \leqq A\tilde{\sigma}^*(x,y) \leqq A \quad \sigma^*(x,y),$$

since $\tilde{\sigma}_n(x,y)$ is essentially equal to

$$\sigma_n(x,y) = \frac{1}{\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f(x+u,y+v)\frac{1}{8(n+1)\sin\frac{u}{2}\sin\frac{v}{2}}$$

$$\times\left\{\frac{\sin(n+1)(u-v)}{2\sin\frac{u-v}{2}} - \frac{\sin(n+1)(u+v)}{2\sin\frac{u+v}{2}}\right\} du\,dv.$$

Combining (7), (8) and (10) with Lemma 5 we get the inequality

$$\gamma^*(x,y) \leqq A\{f^*(x,y) + f^{**}(x,y)\}.$$

And finally we have, again by Lemma 5,

$$|\{\gamma^*(x,y) > t\}| \leqq \frac{A}{t}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}|f(x,y)|\,dx\,dy.$$

The convergence almost everywhere of $\gamma_n(x,y)$ to $f(x,y)$ is derived from the above inequality by the usual way. Thus we complete the proof.

References

[1] J. Marcinkiewicz, Sur une méthode remarquable de sommation des séries doubles de Fourier, Annali della Scuola Norm. Sup. di Pisa Ser. II, 8(1939), 149–160.

[2] G. Grünwald, Über die Summabilität der Fourierschen Reihe, Acta Sci. Math. de Szeged, 10 (1941), 55–63.

[3] J. G. HERRIOT, Nörlund summability of double Fourier series, Trans. Amer. Math. Soc., 52(1942), 72-94.
[4] G. MARUYAMA, Summability of Fourier series, Tôhoku Math. Journ., 47(1940), 255-260.
[5] G. SUNOUCHI, Strong summability of Walsh Fourier series, Tôhoku Math. Journ., 16 (1964), 228-237.
[6] G. SUNOUCHI, On the strong summability of Fourier series, Proc. Amer. Math. Soc., 1(1950), 526-533.
[7] A. ZYGMUND, Trigonometric series II, Cambridge, (1959).

TOKAI UNIVERSITY, TOKYO.