NOETHERIAN MODULES AND NOETHERIAN INJECTIVE RINGS

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(Received January 6, 1964)

1. Preliminary dialogue. The author's interest in noetherian modules and rings is motivated by his desire to help in the development of a general structure theory for these modules and rings. Of great importance in this theory are the *uniform* submodules, a term introduced by A. W. Goldie in [4]. We begin with that definition.

A right R module M is *uniform* if $N_1 \cap N_2 \neq 0$ for any two nonzero submodules N_1 and N_2 of M. A right ideal of a ring R is *uniform*, if it is uniform as a right R module. All modules will be *right* modules over a ring.

An R module M is said to be *noetherian* if it satisfies the maximum condition for submodules, while a ring R is *right* noetherian, if it is noetherian as a right R module.

In §2, §3, and §4 we shall discuss properties of uniform and injective modules. In so doing, we shall extend some of the results of R. E. Johnson and E. T. Wong [13] to the case where the singular submodule is not zero. This has been done in part by L. Lesieur and R. Croisot [16], a portion of which we shall repeat here with proofs when it seems appropriate.

In §5, using the paper [18] by E. Matlis, we show that an R module is noetherian (and injective) iff it is the subdirect (direct) sum of uniform noetherian (and injective) submodules. In §6, we characterize right noetherian and injective rings as a direct sum of a finite number of completely primary rings.

For the most part, the general background and definition can be found in the papers by A. W. Goldie, R. E. Johnson, J. Lambek, E. Matlis, E. T. Wong, and the author as listed in the bibliography.

Let M be a (S, R) module, i.e., a left S module and a right R module over the rings S and R. If X, Y and Z are sets in S, M and R respectively, then the annihilations are defined as $X^r = \{m \in M \mid xm = 0, \text{ all } x \in X\}$, $Y^r = \{a \in R \mid ya = 0, \text{ all } y \in Y\}$, $Y^l = \{s \in S \mid sy = 0, \text{ all } y \in Y\}$ and $Z^l = \{m \in M \mid mz = 0, \text{ all } z \in Z\}$.

2. The lower singular submodule. If M is a right R module, then M is a (K, R) module, where $K = \mathrm{Hom}_R(M, M)$ are endomorphisms written on

(1) Research supported in part by the National Science Foundation.

the left side. We define $P=\{\alpha \in K \mid \alpha^r$ is large in $M\}$ and call PM the *lower singular submodule*.(2) This term was chosen since in 2.2 we shall show $PM \subseteq M^{\blacktriangle}$, the (right) singular submodule. By [16, p. 374], we have that P is a twosided ideal of K.

In order to keep continuity, we repeat a part of theorem 6.1 of [11, p. 538].

LEMMA 2.1. *Let N' be a large submodule of N over a ring A, then $(N' : x) = \{a \in A \mid xa \in N'\}$ is a large right ideal of A for any $x \in N$.*

PROOF. Let H be a right ideal of A. Suppose $H \cap (N' : x) = 0$, then $xH \cap N' = 0$. Since N' is large, then xH must be zero. Hence $H \subseteq (N' : x)$ $\cap H = 0$.

THEOREM 2.2. *Let M be a module over R, then the lower singular submodule PM is contained in the singular submodule $M^{\blacktriangle}$.*

PROOF. If $\alpha \in P$, then α^r is large. By 2.1 we have $(\alpha^r : x) = H$ is large in R for all $x \in M$. Hence $\alpha x H = 0$ which implies $\alpha x \in M^{\blacktriangle}$.

THEOREM 2.3. *Let M be uniform over R with singular submodule $M^{\blacktriangle} \neq M$, then $P = \{\alpha \in K \mid \alpha M \subseteq M^{\blacktriangle}\}$.*

PPOOF. Suppose $\alpha \notin P$ and $\alpha M \subseteq M^{\blacktriangle}$. Then $(\alpha x)^r = x^r$ since $\alpha^r = 0$. Hence x^r is large for all $x \in M$ and $M = M^{\blacktriangle}$, a contradiction. From 2.2, $PM \subseteq M^{\blacktriangle}$ and the proof is complete.

3. Relationships between $\mathrm{Hom}_R(M, M)$ and M.

In this section let $K = \mathrm{Hom}_R(M, M)$ and P denote the ideal of K as defined in §2. Since we are interested in noetherian modules, the next theorem follows from theorem 2.2 of [1] in the case when M is noetherian. However, we choose the theorem in its present form in order to separate properties.

THEOREM 3.1. *If M is a uniform R module, then K/P is an integral domain and the elements not in P are right regular. In addition, if M is noetherian then $P = \{\alpha \in K \mid \alpha^n = 0\}$.*

PROOF. If $\alpha \notin P$, then $\alpha^r = 0$. If $\alpha, \beta \notin P$, then $(\alpha\beta)^r = \beta^r = 0$. Thus $\alpha\beta \notin P$ and K/P is an integral domain. Suppose $\alpha\gamma = 0$, $\alpha \notin P$. Since

(2) Of course, the idea for this ideal P comes from the definition of the singular submodule by R. E. Johnson.

$\alpha^r = 0$, then $\gamma M = 0$ and $\gamma = 0$.

Suppose M is noetherian, then we need only show that P is a nil ideal. For $\alpha \in P$, there is an integer s such that

$$\alpha^r \subseteq (\alpha^2)^r \subseteq \cdots \subseteq (\alpha^s)^r = (\alpha^{s+1})^r = \cdots .$$

Suppose $\alpha^m M \neq 0$ for all integers m, then

$$(\alpha^s)^r \cap \alpha^{s+1} M = 0 .$$

For let u be contained in this intersection, then $u = \alpha^{s+1} v$, $v \in M$. Then $0 = \alpha^s u = \alpha^{2s+1} v = \alpha^{s+1} v = u$. This contradicts the uniform property of M.

COROLLARY 3.2. *Every right or left unit of K is a unit. If $\alpha = \alpha\beta$ for $\alpha \notin P$ and $\beta \neq 0$, then β is a unit. In addition, the identity of K is the only idempotent.*

PROOF. Suppose $\alpha\beta = 1$, then $\alpha, \beta \notin P$. From $\alpha(1-\beta\alpha) = 0$, we have $\beta\alpha = 1$ and thus α and β are units. The remaining part is direct.

Now 3.1 provides the following theorem.

THEOREM 3.3. *If M is a uniform R module, then K is an integral domain if and only if the lower singular submodule $PM = 0$.*

In [13, p. 263] Johnson and Wong showed that K is an integral domain provided the singular submodule $M^\blacktriangle = 0$. The converse to the theorem of Johnson and Wong is not possible by the following example.

Let I denote the integers. In $I/(4)$ let M be ring composed of 0 and 2. Consider M as a right module over itself. Then M is uniform and $\mathrm{Hom}_R(M, M) \cong I/(2)$ while $M^\blacktriangle = M$.

In [13, p. 260] a module M is called *quasi-injective* if for each submodule N of M every R homomorphism of N into M can be extended to a R homomorphism of M into M. A ring R is said to be *H-local*, provided the elements not in H are the units of R, where H is an ideal of R. Therefore if R is H-local, then R/H is a division ring. In the literature this type of ring is sometimes called *completely primary*. However, we shall reserve this term for a slightly different ring in §6.

Using the methods of 1.7 of [13] we prove the first part of the next theorem. The last part follows from 3.1.

THEOREM, 3.4. *If M is a quasi-injective uniform R module, then K is* *a P-local ring. If M is noetherian then* $P = \{\alpha \in K \mid \alpha^n = 0\}$.(3)

PROOF. Let $\alpha \in K$, $\alpha \notin P$ then $\alpha^r = 0$. We define $\gamma \in \mathrm{Hom}_R(\alpha M, M)$ by $\gamma(\alpha x) = x$. Then γ extends to some $\beta \in K$ and $\beta\alpha = 1$. By 3.2 we have our result.

Let $\hat{M}$ denote the injective envelope of M, i.e., $\hat{M}$ is the unique minimal injective extension and the unique maximal essential extension of M. Let $\hat{K} = \mathrm{Hom}_R(\hat{M}, \hat{M})$ and $\hat{P} = \{\alpha \in \hat{K} \mid \alpha^r \text{ is large in } \hat{M}\}$.

THEOREM. 3.5. *The R module M is uniform if and only if* $\hat{K}$ *is a* $\hat{P}$*-local ring.*

PROOF. Since M is uniform, it follows that $\hat{M}$ is uniform. Thus $\hat{K}$ is a $\hat{P}$-local ring by 3.4.

Conversely, suppose $\hat{K}$ is a $\hat{P}$-local ring and $N_1 \cap N_2 = 0$ for nonzero $N_1, N_2 \subseteq M$. For $N = N_1 \oplus N_2$ (direct sum), there exists an endomorphism α of N such that $\alpha n_1 = n_1$ for $n_1 \in N_1$ and $\alpha n_2 = 0$ for $n_2 \in N_2$. Since $\hat{M}$ is injective, α extends to $\alpha^* \in \hat{K}$. Suppose $\alpha^* \in \hat{P}$, then $(\alpha^*)^r$ is large so that $(\alpha^*)^r \cap N_1 \neq 0$, which is not true. Suppose $\alpha^* \notin P$, then α^* is a monomorphism and thus α is a monomorphism. This also is impossible since $\alpha N_2 = 0$. Thus $N_1 \cap N_2 \neq 0$.

This theorem is essentially a consequence of 2.2 and 2.6 of the paper [18] by E. Matlis, although the modules in [18] are unitary. Here, since the proof and formulation is quite different, we feel that it is reasonable to include the proof.

With the ideal P in K two submodules immediately come to mind, namely P^r and PM. We showed that $PM \subseteq M^\blacktriangle$. In [16], L. Lesieur and R. Croisot defined the core $C(\hat{M}) = \bigcap_{\alpha \in \hat{P}} \alpha^r$ and the core $C(M) = C(\hat{M}) \cap M$. Note that $\hat{P}^r = C(\hat{M})$. Problem: Find necessary and sufficient conditions that $P^r = C(M)$ or $PM = M^\blacktriangle$.

4. Properties of essential and injective extensions.

For the most part we shall discuss the primary properties and characteristics of the ring of endomorphism of essential and injective extensions of our module.

Let $M \subseteq M^E$ be R modules, then M^E is an essential extension of M if $M \cap N \neq 0$ for every nonzero submodule N of M^E.

(3) The first part of this theorem appears in [16] and [18].

THEOREM 4.1. *Let M^E be an essential extension and $\widehat{M}$ denote the injective envelope of the uniform R module M. Let* $K^E = \mathrm{Hom}_R(M^E, M^E)$ *and* $P^E = \{\alpha \in K^E | \alpha^r \neq 0 \text{ in } M^E\}$, *then K^E/P^E is an integral domain and $\widehat{K}$ is a $\widehat{P}$-local ring. If M is noetherian, then* $P^E = \{\alpha \in K^E | \alpha^n M = 0$ *for some integer* $n\}$.

PROOF. Since M is uniform, so also is M^E and $\widehat{M}$. The first part now follows from 3.1 and 3.4.

If $\alpha^r \neq 0$ in M^E, then $\alpha^r \neq 0$ in M. If now M is noetherian, there is an integer s such that *in* M we have $\alpha^r \subseteq (\alpha^2)^r \subseteq \cdots \subseteq (\alpha^s)^r = (\alpha^{s+1})^r = \cdots$. If $\alpha^n M \neq 0$ for all integers n, then as in 3.1 we have

$$(\alpha^s)^r \cap \alpha^{s+1} M = 0 .$$

This contradicts the uniform poperty of M^E.

REMARK. In [14, p. 167] or [16, p. 303] we find defined a rational extension M^s of M. From this definition and the fact that a rational extension is essential, we may in 4.1 replace M^E by the rational extension M^s. The resulting theorem reads the same with $P^s = \{\alpha \in K^s | \alpha^n M^s = 0$ for some integer $n\}$.

Note that in 4.1 we characterize the non-units as these elements which to some integral exponent annihilate M, when M is noetherian. In addition, we can replace in theorem 2.5 of [18, p. 516] the word "indecomposable" by "uniform" by 3.5. Thus every injective, indecomposable module is uniform.

In our case the singular submodules have a "prime" or "pure" characteristic as described in

THEOREM 4.2. *Let $M^\blacktriangle$ be the singular submodule of the uniform module M. If $\alpha x \in M^\blacktriangle$ and $x \notin M^\blacktriangle$, then $\alpha M \subseteq M^\blacktriangle$ for* $\alpha \in \mathrm{Hom}_R(M, M)$.

PROOF. Let $\alpha x \in M^\blacktriangle$ and $x \notin M^\blacktriangle$. Suppose $\alpha \notin P$, then $\alpha^r = 0$. Now $(\alpha x)(\alpha x)^r = 0$ where $(\alpha x)^r$ is a large right ideal of R. Then $\alpha[x(\alpha x)^r] = 0$ and $x(\alpha x)^r = 0$. Therefore $x \in M^\blacktriangle$ which contradicts our supposition, and hence $\alpha \in P$. By 2.2, we have our result.

We conclude this section by extending 1.3 of [13] with $M^\blacktriangle \neq 0$.

THEOREM 4.3. *Let M be an arbitrary A module. Then M is quasi-injective if and only if for* $H = \mathrm{Hom}_A(M, M)$ *and* $\widehat{K} = \mathrm{Hom}_A(\widehat{M}, \widehat{M})$, *we have* $\widehat{K}/q_0 = H^*/q = H^{(4)}$, *where*

(4) Throughout this discussion we shall identify a ring with its isomorphic image.

$$H^* = \{k \in \hat{K} \mid kM \subseteq M\}, \quad q = \{h^* \in H^* \mid h^*M = 0\}$$

and

$$q_0 = \{k \in \hat{K} \mid kM = 0\} .$$

PROOF. If $H^*/q = \hat{K}/q_0$, then since M is a H^*/q module it is a $\hat{K}/q_0$ module. Hence, by [7, p. 3], M is a $\hat{K}$ module. Then, by theorem 1.1 of [13], M is a quasi-injective module.

Conversely, suppose M is quasi-injective. Then by 1.1 of [13] we have $\hat{K}=H^*$. Hence $\hat{H}^*/q = K/q_0 = H$.

5. Decomposition of noetherian modules. We start this section with a theorem which follows directly from the literature.

The term *irredundant subdirect sum* will be used and we refer the reader to L. Levy [19, p. 65].

THEOREM 5.1. *An R module M is noetherian if and anly if M is an irredundant subdirect sum of a finite number of uniform noetherian modules.*

PROOF. Since M is noetherian, then $0 = N_1 \cap \cdots \cap N_n$ where this representation is irredundant and where the N_i are $\cap$-irreducible. Hence, M is the irredundant subdirect sum of the uniform noetherian modules M/N_i.

Conversely, suppose M is a irredundant subdirect sum of uniform noetherian modules H_i, $i = 1, 2, \cdots, n$. Then M contains submodules N_i such that $M/N_i \cong H_i$, $i=1, 2, \cdots, n$, and $0 = N_1 \cap \cdots \cap N_n$. The result now follows from a theorem of Grundy's [6, p. 242].

One would now ask for what modules is this subdirect sum a direct sum. The injective unitary modules are such modules. This follows using 2.5 of Matlis [18] which we shall repeat here.

THEOREM 5.2. *If R is a right noetherian ring with identity, then every unitary injective R module M is a direct sum of uniform injective submodules.*

Since in a noetherian module we do not have infinite direct sums, we can use 5.1 and 5.2 to prove

THEOREM 5.3. *Let R be a right noetherian ring with identity. Then a unitary R module M is noetherian and injective iff M is a direct sum of a finite number of uniform and injective submodules.*

If J is an $\cap$-irreducible submodule of M, then M/J is uniform and we can rephase 2.3 of [18] for modules. This will hint at a noetherian type intersection theory for noetherian modules.

THEOREM 5.4. *Let M be a noetherian unitary module over a right noetherian ring R. For a submodule N of M, let $N = J_1 \cap \cdots \cap J_n$ be a representation of N as an irredundant intersection of $\cap$-irreducible submodules. Then $\widehat{M/N} = \widehat{M/J_1} \oplus \cdots \oplus \widehat{M/J_n}$, where the M/J_i are uniform modules for all i.*

At this point we summarize our results as follows.

THEOREM 5.5. *An R module M is noetherian iff M is a subdirect sum of uniform and noetherian R modules N_i, $i=1,2,\cdots,n$. Here $N_i \cong M/J_i$ where $0 = J_1 \cap \cdots \cap J_n$ for submodules J_i of M and $\mathrm{Hom}_R(N_i, N_i)/P_i$ is an integral domain, where P_i is the set of nil elements. In addition, if R is noetherian and M is unitary, then $\hat{M} = \widehat{M/J_1} \oplus \cdots \oplus \widehat{M/J_n}$ and $\mathrm{Hom}_R(\widehat{M/J_i}, \widehat{M/J_i})$ is $\hat{P}_i$-local.*

Certainly more can and will be said than is stated here. For example, theorem 2.2 of [16, p. 374] states that $\mathrm{Hom}_R(\hat{M}, \hat{M})/P$ is semisimple. However, since the great concern is with the structure of M, it would seem that we need more connecting links between a decomposition of M and that of $\hat{M}$.

6. A structure for right noetherian and injective rings.

THEOREM 6.1. *Let R be a ring with identity, and U a uniform right ideal of R, then $P = \{a \in U \mid a^r \neq 0 \text{ in } U\}$ is a twosided ideal of U, U/P is an integral domain, and the elements not in P are right regular.*

PROOF. The proof follows closely the proof of 3.1.

COROLLARY 6.2. *If R is a uniform right noetherian non nil ring and $P = \{a \in R \mid a^r \neq 0\}$, then P is a nilpotent ideal and R/P is an integral domain.*

Observe that the elements not in P are right regular. One would be tempted to believe these elements to be left regular.

EXAMPLE. Let e_{11} be the 2×2 matrix with 1 in one-one position and zero elsewhere, and e_{12} the matrix with 1 in the one-two position and zero elsewhere. Then $\mathrm{I}e_{11} + \mathrm{I}e_{12}$, where I is the ring of integers, gives a ring which is uniform with $A.C.C.$ This ring has no identity and the element not in P are *not* left regular. The author has not found a ring such as this *with identity* where the elements not in P are not left regular.

Let P be an ideal of a ring A, then A is termed *P-completely primary* provided A/P is a division ring. In this definition A need not have an identity.

THEOREM 6.3. *Let R be a ring with identity and let U be a direct summand of R, where U is a uniform right ideal and injective over R.*[5] *Then U is a P-completely primary ring where $P=\{a\in U \mid a^r\neq 0$ in $U\}$.*

PROOF. Since $R=U\oplus V$, then $1=e_1+e_2$ and $U=e_1R$ where e_1 is a left identity of the ring U. From 6.1, for $a\notin P$ we have $a^r=0$ in U. Thus we define R-homomorphism $\theta: aU\rightarrow U$ by $\theta(au)=u$ for $u\in U$. Since U is injective over R, θ has an extension $\phi: U\rightarrow U$. Now $\phi^r=0$, for if $\phi^r\neq 0$, then $\phi^r\cap aU\neq 0$. This implies $\phi(au)=0$ for some $u\neq 0$, a contradiction. Since ϕ is an epimorphism, it is therefore an isomorphism. Thus ϕ^{-1} exists and $\phi^{-1}: U\rightarrow U$. However, $\phi^{-1}(u)=au$. Hence $aU=U$ and $ab=e_1$ for some $b\in U$. Thus each element not in P has a right inverse relative to e_1. We need only show that $\bar{e}_1$ in U/P is a right identity element. If $x\notin P$, then $(x-xe_1)x=0$ and $(\bar{x}-\bar{x}\bar{e}_1)\bar{x}=\bar{0}$ in U/P. Since U/P is an integral domain, it follows that $\bar{x}=\bar{x}\bar{e}_1$.

In this theorem note that $\mathrm{Hom}_R(U,U)$ is a P-local ring by 3.4 and hence e_1Re_1 is a local ring by [7, p. 51].

From 5.3, 6.1, and 6.3 we summarize.

THEOREM 6.4. *A ring R with identity is right noetherian and right injective over R iff R is a direct sum of a finite number of uniform right noetherian and injective over R right ideals H_i, $i=1,2,\cdots,n$, where each H_i is a P_i-completely primary ring. Here $P_i=\{a\in H_i \mid a^n=0\}$. In addition, $H_i=e_iR$ and e_iRe_i is a Q_i-local ring, where $Q_i=\{a\in e_iRe_i \mid a^n=0\}$.*

For 6.4 we have the following finite example. Let $R=I/(p^k)$ where I is the ring of integers, and p is a prime. Then R_2, the set of 2×2 matrices with elements in R, is noetherian (finite) and injective over R_2. Let E_{ij} be the matrix with 1 in the i,j position and zero elsewhere. Then $R_2=E_{11}R_2\oplus E_{22}R_2$. Here $E_{11}R_2$ is uniform and R_2 injective. The ideal P for $E_{11}R_2$ is $E_{11}(p)/(p^k)+E_{12}R$. Thus $(E_{11}R_2)/P\cong I/(p)$, and $E_{11}R_2E_{11}\cong I/(p^k)$.

(5) That is, injective as a right R module,

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