ON k-PARALLELIZABLE MANIFOLDS

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Introduction. A manifold is called k-parallelizable if the restriction of its tangent bundle to any k-skeleton is trivial ([1]). In such a manifold some of the characteristic classes vanish. Making use of this fact we shall study the properties of such manifolds together with the imbeddability and field of frames.

1. Throughout this paper we denote by X_n an orientable compact n-dimensional C^∞ manifold with a Riemann metric. We use the following notations:

$w_i \in H^i(X_n, Z_2)$...... Stiefel-Whitney class of dimension i,
$\overline{w}_i \in H^i(X_n, Z_2)$...... Dual Stiefel-Whitney class,

$$w = \sum_{i\geqq 0} w_i, \quad \overline{w} = \sum_{i\geqq 0} \overline{w}_i,$$

$$w \cdot \overline{w} = 1, \tag{1. 1}$$

$P_{4i} \in H^{4i}(X_n, Z)$...... Pontryagin class of dimension $4i$,
$\overline{P}_{4i} \in H^{4i}(X_n, Z)$...... Dual pontryagin class,

$$p = \sum_{i\geqq 0} (-1)^i P_{4i}, \quad \overline{p} = \sum_{i\geqq 0} \overline{P}_{4i},$$

$$p \cdot \overline{p} = 1, \tag{1. 2}$$

$$P_{4i} = \{(2\pi)^{2i}(2i)!\}^{-1} \sum_{s,t} \delta\begin{pmatrix} s_1 \cdots s_{2i} \\ t_1 \cdots t_{2i}\end{pmatrix} \Omega_{s_1t_1} \cdots \Omega_{s_{2i}t_{2i}}, \tag{1. 3}$$

$$P'_{4k} = \alpha_k \sum_i \Omega_{i_1i_2}\Omega_{i_2i_3} \cdots \Omega_{i_{2k}i_1}, \tag{1. 4}$$

where $\delta(\quad)$, Ω_{ij} or α_k denotes the generalized Kronecker symbol, the curvature form of X_n or some numerical constant respectively ([2]). It is known that P_{4i} is a polynomial of P'_{4i}'s ($k \leqq i$) and P'_{4i} is a polynomial of P_{4k}'s ($k \leqq i$). The following two theorems are fundamental for our purpose.

I. *For each q there exists a continuous field of tangent $n-q$ frames defined over the q dimensional skeleton K^q of X_n. In order that there may exist such a field on any K^{q+1}, it is necessary and sufficient that* $w_{q+1}=0$ ([3], p. 199).

II. *There always exists a continuous field of tangent $n-2m+2$ pseudo-frames over any K^t $(t<4m)$ of X_n. There exists such a field over any K^{4m} if and only if $P_{4m}=0$, $1\leqq m\leqq[n/4]$, where a k pseudo-frame means an ordered set of k vectors, at least $k-1$ of which are linearly independent* ([2]).

Since a s-frame is a s-pseudo-frame, we have from **I** and **II** the Theorem 1.

THEOREM 1. *If X_n is k-parallelizable $(1\leqq k\leqq n)$, then*

$$w_p=0 \quad (1\leqq p\leqq k) \tag{1.5}$$

and

$$P'_{4m}=0 \quad (4\leqq 4m\leqq k)\,, \tag{1.6}$$

i.e.

$$P_{4m}=0 \quad (4\leqq 4m\leqq k)\,. \tag{1.7}$$

Next we consider the case where X_n is differentiably imbedded in an Euclidean space whose dimension is small enough. Suppose that $X_n\subset E_{n+q}$, where E_{n+q} denotes the $n+q$ dimensional Euclidean space. Then we must have

$$\bar{P}_{4m}=0 \quad (2m\geqq q) \text{ ([5])} \tag{1.8}$$

and

$$\overline{w}_t=0 \quad (t\geqq q)\,, \text{ ([7])}. \tag{1.9}$$

THEOREM 2. *If X_n is $4m$-parallelizable and differentiably imbedded in the E_{n+2m+2} $(4\leqq 4m\leqq n)$, then $p=1$ and $w=1$,* i.e. *X_n is "bord".* ([4])

PROOF. We have from (1.9)

$$\overline{w}_t=0 \quad (t\geqq 2m+2)\,. \tag{1.10}$$

On the other hand we have from (1.5)

$$w_p=0 \quad (1\leqq p\leqq 4m)\,, \tag{1.11}$$

i.e.

$$\bar{w}_p = 0 \quad (1 \leqq p \leqq 4\mathrm{m}) \tag{1. 12}$$

by (1. 1). Since $m \geqq 1$ we have

$$\bar{w}_p = 0 \quad (1 \leqq p) \tag{1. 13}$$

i.e.

$$w_p = 0 \quad (1 \leqq p). \tag{1. 14}$$

Next we have from (1. 7)

$$P_{4t} = 0 \quad (1 \leqq t \leqq m), \tag{1. 15}$$

i.e.

$$\bar{P}_{4t} = 0 \quad (1 \leqq t \leqq m) \tag{1. 16}$$

by (1. 2). Meanwhile we have from (1. 8)

$$\bar{P}_{4s} = 0 \quad (s \geqq m+1). \tag{1. 17}$$

We see from (1. 16) and (1. 17) that

$$\bar{P}_{4t} = 0 \quad (1 \leqq t), \tag{1. 18}$$

i.e.

$$P_{4t} = 0 \quad (1 \leqq t). \tag{1. 19}$$

By virtue of (1. 14) and (1. 19) every Stiefel-Whitney number and Pontryagin number become zero. Hence X_n is "bord". ([4]) Q.E.D.

THEOREM 3. *If X_n is $4m$-parallelizable and differentiably imbedded in the E_{n+2m+4} $(4 \leqq 4m \leqq n)$, then*

$$p = \sum_{t \geqq 0} (-1)^t \bar{P}^t_{4m+4}, \tag{1. 20}$$

and

$$w = \begin{cases} 1 & (m \geqq 2) \\ (1+\bar{w}_5)^{-1} & (m=1). \end{cases} \tag{1. 21}$$

If moreover $n \not\equiv 0 \bmod 4(m+1)$ and $m \geqq 2$, then X_n is "bord".

PROOF. We have from (1. 7) and (1. 8)

(1. 22) $$P_{4t} = 0 \quad (1 \leqq t \leqq m),$$

i.e.

(1. 23) $$\bar{P}_{4t} = 0 \quad (1 \leqq t \leqq m)$$

and

(1. 24) $$\bar{P}_{4s} = 0 \quad (s \geqq m+2).$$

Therefore, the only non zero dual Pontryagin class is $\bar{P}_{4m+4}$. Hence we have from (1. 2)

(1. 25) $$p = \sum_{t \geqq 0} (-1)^t P_{4t} = \frac{1}{1+\bar{P}_{4m+4}}.$$

Next we have from (1. 5) and (1. 9)

(1. 26) $$w_t = 0 \quad (1 \leqq t \leqq 4m), \quad \text{i.e.} \quad \bar{w}_t = 0 \ (1 \leqq t \leqq 4m)$$

and

(1. 27) $$\bar{w}_k = 0 \quad (k \geqq 2m+4).$$

Hence we have

(1. 28) $$\begin{cases} \bar{w}_t = 0 & t \geqq 1 \quad (m \geqq 2), \\ \bar{w}_t = 0 & t \neq 0, 5, 6 \quad (m = 1). \end{cases}$$ Q.E.D.

For example we consider the case where $X_{16} \subset E_{22}$ and X_{16} is 4-parallelizable. In this case we have from (1. 20)

(1. 29) $$\begin{cases} P_4 = P_{12} = 0, \\ P_{16} = P_8^2 \end{cases}$$

which leads to

(1. 30) $$\tau = \frac{2 \cdot 181}{3^4 \cdot 5^2 \cdot 7} P_8^2[X_{16}]$$

and

(1. 31) $$A = \frac{2^5}{3^4 \cdot 5^2 \cdot 7} P_8^2[X_{16}], \quad ([5])$$

where τ or A denotes the index or the A-genus respectively. We see that the index of this manifold is divisible by 724, for if $X_{16} \subset E_{22}$ the A-genus is divisible by 2^6 ([6]), i.e. $P_8^2[X_{16}]$ is even by (1. 31). Thus we have the

COROLLARY. *If X_{16} is 4-parallelizable and differentiably imbedded in the E_{22}, then its index is divisible by 724.*

2. In order that X_n may admit a continuous field of tangent k-frames it is necessary from **I** and **II** that

$$w_i = 0 \quad (n-k+1 \leqq i \leqq n) \tag{2. 1}$$

and

$$P'_{4m} = 0 \quad (n \geqq 4m \geqq 2(n-k+1)), \tag{2. 2}$$

because we can construct a $k+1$ pseudo-frame from a k-frame.

THEOREM 4. *If X_n is $4m$-parallelizable $(4 \leqq 4m \leqq n)$ and moreover admits a continuous field of tangent $n-2m-1$ field, then*

$$w_i = 0 \quad (1 \leqq i \leqq n) \tag{2. 3}$$

and

$$P_{4i} = 0 \quad (1 \leqq i \leqq [n/4]), \tag{2. 4}$$

i.e. X_n is "bord".

PROOF. Since X_n is $4m$-parallelizable, we have

$$w_i = 0 \quad (1 \leqq i \leqq 4m) \tag{2. 5}$$

and

$$P'_{4i} = 0 \quad (1 \leqq i \leqq m). \tag{2. 6}$$

The second assumption leads to

$$w_j = 0 \quad (2m + 2 \leqq j \leqq n) \tag{2. 7}$$

and

$$P'_{4t} = 0 \quad (m + 1 \leqq t \leqq [n/4]). \tag{2. 8}$$

Therefore we have (2. 3) and (2. 4). Q.E.D.

THEOREM 5. *If X_n is $4m$-parallelizable $(4 \leqq 4m \leqq n - 4)$ and admits a continuous field of tangent $(n-2m-3)$-frame, then p is generated by $P_{4(m+1)}$ and*

$$(2.9)\qquad \begin{cases} w = 1 \quad (m \geqq 2), \\ w = 1 + w_5 \quad (m = 1). \end{cases}$$

If moreover $n \not\equiv 0 \mod 4(m+1)$ *and* $m \geqq 2$, *then* X_n *is "bord".*

PROOF. First we have

$$(2.10)\qquad w_i = 0 \quad (1 \leqq i \leqq 4m)$$

and

$$(2.11)\qquad P'_{4j} = 0 \quad (1 \leqq j \leqq m).$$

Next we have from the second assumption

$$(2.12)\qquad w_k = 0 \quad (2m+4 \leqq k \leqq n)$$

and

$$(2.13)\qquad P'_{4t} = 0 \quad (t \geqq m+2).$$

Therefore the only non zero P' is $P'_{4(m+1)}$. Hence we have

$$(2.14)\qquad P_{4i} = 0 \quad (i \not\equiv 0 \mod (m+1)).$$

Q.E.D.

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