MULTIPLIERS FOR WALSH FOURIER SERIES

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Introduction. In the theory of trigonometric Fourier series (abbrev. TFS), it is well known that the behavior of a TFS is "ameliorated" by integrating (even by a fractional order) the generating function. But, the process of taking the α-th integral (in the sense of H. Weyl) of a function f is to consider the convolution of f with an integrable function whose Fourier coefficients are $(i|n|)^{-\alpha}$; this fact suggests us the possibility to define a corresponding operation in the dyadic group of N. J. Fine [1]. The purpose of the present paper is to investigate a class of multiplier transformations of Walsh Fourier series. (abbrev. WFS), which shares most of properties with fractional integration.

Let G be the dyadic group, with elements $x=(x_n)$, $x_n=0$ or 1 $(n=1,2,\cdots)$, $y=(y_n)$ etc., with the "addition" $\dot{+}$; the topology of G is defined by the neighborhoods $V_n=\{x\,;x_1=\cdots=x_n=0\}$ $(n=1,2,\cdots)$ of the identity element, or equivalently, by the distance $d(x,y)=\sum_{n=1}^{\infty}|x_n-y_n|2^{-n}$. The Rademacher functions $\phi_n(x)$ $(n=0,1,2,\cdots)$ are defined by $\phi_n(x)=(-1)^{x(n+1)}$ where $x(n+1)$ stands for x_{n+1}, and the Walsh functions, the characters of G, are given by

$$\psi_0(x)=1,$$

$$\psi_n(x)=\phi_{n(1)}(x)\phi_{n(2)}(x)\cdots\phi_{n(r)}(x)$$

$$\text{for}\quad n=2^{n(1)}+2^{n(2)}+\cdots+2^{n(r)}=1,\ n(1)>n(2)>\cdots>n(r)\geqq 0.$$

We refer the reader to Fine [1] for basic properties of Walsh functions.

1. Polynomials and formal series. A (Walsh) polynomial of degree n is a linear combination $\sum_{k=0}^{n-1}c_k\psi_k(x)$ with $c_{n-1}\neq 0$; the totality of polynomials of degree not exceeding n is denoted by $\mathfrak{P}_n$ and the union of the $\mathfrak{P}_n$'s by $\mathfrak{P}$. It is clear that $\mathfrak{P}$ (as well as each of $\mathfrak{P}_n$) forms a linear space.

We denote by $\mathfrak{F}$ the set of all formal (Walsh) series with complex coefficients. It is not difficult to introduce such topologies in $\mathfrak{P}$ and $\mathfrak{F}$ that they are the duals with these topologies, but we do not insist on this point.

Let $f_1(x)=\sum c_k\psi_k(x)$ and $f_2(x)=\sum d_k\psi_k(x)$ be two elements of $\mathfrak{F}$. We call the formal series

$$f_3(x)=\sum c_k d_k \psi_k(x)$$

the convolution of $f_1(x)$ and $f_2(x)$, and denote it by $(f_1 * f_2)(x)$. If both series happen to be WFS or Walsh-Fourier-Stieltjes series, this definition agrees with the ordinary one. It is clear that $\mathfrak{F}$ is a commutative algebra with convolution as multiplication, and $\mathfrak{P}$, $\mathfrak{P}_n$ are ideals of $\mathfrak{F}$. $\mathfrak{F}$ has a unit, $\delta(x)=\sum_{k=0}^{\infty}\psi_k(x)$, which is the Fourier-Stieltjes series of the Dirac measure situated at 0. Thus multiplier transformations are (restriction of) convolution transformation in $\mathfrak{F}$.

2. **Kernel functions**. We study here a special class of formal series, the kernels of our multiplier transformations. Let us write

$$I_\alpha(x)=1+\sum_{k=1}^{\infty}2^{-k(1)\alpha}\psi_k(x) \qquad (\alpha \text{ real}),$$

where $k(1)$ is the first dyadic exponent of k.

LEMMA 1. *Let $1\leqq p\leqq\infty$ and let q be its conjugate exponent,* i.e., $(1/p)+(1/q)=1$. *Then we have, for $\alpha>1/q$, $I_\alpha(x)\in L^p=L^p(G)$.*

PROOF. If $p=\infty$, then $q=1$ and $\alpha>1$ implies the absolute (and uniform) convergence of $I_\alpha(x)$, thus $I_\alpha(x)$ is the WFS of a continuous function, which is more than what is to be proved. On the other hand, it is well known that $D_{2^j}(x)$, the Dirichlet kernel of order 2^j, equals to 2^j or 0 according as $x\in V_j$ or not. Thus for $1\leqq p<\infty$, we have $\|D_{2^j}\|_p=2^{j(1-1/p)}=2^{j/q}$. Now

$$I_\alpha^{(n)}(x)\equiv(I_\alpha * D_{2^n})(x)=1+\sum_{j=0}^{n-1}2^{-j\alpha}\phi_j(x)D_{2^j}(x)$$

gives for $m>n$,

$$\|I_\alpha^{(m)}(x)-I_\alpha^{(n)}(x)\|_p=\Big\|\sum_{j=n}^{m-1}2^{-j\alpha}\phi_j(x)D_{2^j}(x)\Big\|_p$$

$$\leqq\sum_{j=n}^{m-1}2^{-j\alpha}\|D_{2^j}(x)\|_p=\sum_{j=n}^{m-1}2^{-j(\alpha-1/q)}\to 0 \qquad (m,n\to\infty).$$

Thus $I_\alpha^{(n)}(x)$ converges in L^p-norm to a function whose WFS is $I_\alpha(x)$, q.e.d.

Lemma 1 may be restated as follows:

$$I_\alpha(x) \in L^p \quad \textit{for} \quad p < 1/(1-\alpha) \qquad (0 < \alpha \leqq 1).$$

LEMMA 2. *If* $h \in V_n$, *we have*

$$\| \Delta_h I_\alpha \|_p \equiv \| I_\alpha(x \dotplus h) - I_\alpha(x) \|_p = O(2^{-n(\alpha - 1/q)}) \quad (\alpha > 1/q).$$

PROOF. $$\Delta_h I_\alpha = \sum_{j=0}^{\infty} 2^{-j\alpha} \sum_{k=2^j}^{2^{j+1}-1} \psi_k(x \dotplus h) - \sum_{j=0}^{\infty} 2^{-j\alpha} \sum_{k=2^j}^{2^{j+1}-1} \psi_k(x)$$

$$= \sum_{j=0}^{\infty} 2^{-j\alpha} \sum_{k=2^j}^{2^{j+1}-1} (\psi_k(h) - 1)\psi_k(x)$$

$$= \sum_{j=n}^{\infty} 2^{-j\alpha} \sum_{k=2^j}^{2^{j+1}-1} (\psi_k(h) - 1)\psi_k(x) \qquad (\because \psi_k(h) = 1,\ 0 \leqq k < 2^n)$$

$$= \sum_{j=n}^{\infty} 2^{-j\alpha} (\phi_j(x \dotplus h) D_{2^j}(x \dotplus h) - \phi_j(x) D_{2^j}(x)).$$

Thus, by Minkowski's inequality,

$$\| \Delta_h I_\alpha \|_p \leqq \sum_{j=n}^{\infty} 2^{-j\alpha} \| D_{2^j}(x \dotplus h) \|_p + \sum_{j=n}^{\infty} 2^{-j\alpha} \| D_{2^j}(x) \|_p$$

$$= 2 \sum_{j=n}^{\infty} 2^{-j\alpha} \cdot 2^{j/q} = 2 \sum_{j=n}^{\infty} 2^{-j(\alpha - 1/q)}, \qquad \text{q. e. d.}$$

LEMMA 3. *There is a positive constant* B_α *depending only on* α *such that*

$$\| I_{-\alpha}^{(m)} \|_p \leqq B_\alpha 2^{m(\alpha + 1/q)} \qquad (\alpha > 0).$$

PROOF. $$I_{-\alpha}^{(m)}(x) = 1 + \sum_{k=1}^{2^m - 1} 2^{k(1)\alpha} \psi_k(x) = 1 + \sum_{j=0}^{m-1} 2^{j\alpha} \phi_j(x) D_{2^j}(x).$$

Thus

$$\| I_{-\alpha}^{(m)}(x) \|_p \leqq 1 + \sum_{j=0}^{m-1} 2^{j\alpha} \| D_{2^j} \|_p = 1 + \sum_{j=0}^{m-1} 2^{j(\alpha + 1/q)} = O(2^{m(\alpha + 1/q)}), \qquad \text{q. e. d.}$$

The case $p=1$ is of particular importance for later applications.

3. **Lemmas on the best approximation.** A function $f(x)$ on G is said to belong to the $\mathrm{Lip}^{(p)}\,\alpha(W)$ (resp. $\mathrm{lip}^{(p)}\alpha(W)$) if

$$\| f(x+h)-f(x)\|_p = O((d(h,0))^\alpha) \qquad (\text{resp. } o((d(h,0))^\alpha)).$$

This definition is essentially due to G. Morgenthaler [5]. A characterization of the class $\mathrm{Lip}^{(p)}\alpha(W)$ was given by us [9], which applies with little modification also to the class $\mathrm{lip}^{(p)}\alpha(W)$, i.e., we have

LEMMA 4. *The following four statements are equivalent*:

(1) $f(x) \in \mathrm{Lip}^{(p)}\,\alpha(W)$

(2) $\omega^{(p)}(2^{-n}; f) \equiv \sup\{\| f(x+h)-f(x)\|_p : h \in V_n\} = O(2^{-n\alpha})$

(3) $E_m{}^{(p)}(f) \equiv \inf\{\| f-p_m\|_p : p_m \in \mathfrak{P}_m\} = O(m^{-\alpha})$

(4) $\| f(x) - s_{2^n}(x; f)\|_p = O(2^{-n\alpha})$

similarly for the o-case.

As a corollary of Lemma 4, we have

LEMMA 5. *Let* $\alpha > 0$, $\beta > 0$, $r \geqq 1$, $s \geqq 1$ *and* $1/t \geqq (1/r) + (1/s) - 1$. *Then* $f \in \mathrm{Lip}^{(r)}\,\alpha(W)$ (*resp.* $\mathrm{lip}^{(r)}\,\alpha(W)$) *and* $g \in \mathrm{Lip}^{(s)}\,\beta(W)$ *together imply* $f * g \in \mathrm{Lip}^{(t)}\,(\alpha+\beta)(W)$ (*resp.* $\mathrm{lip}^{(t)}\,(\alpha+\beta)(W)$).

For the proof of these Lemmas, the reader is referred to [9] for the O-case; the o-case can be proved similarly.

4. Metric properties of multiplier transforms. Let us write

$$f_\alpha(x) = (I_\alpha * f)(x) \qquad \text{for } f \in L^1.$$

THEOREM 1. *The operation* $f \to f_\alpha$ *has the following properties*:

1°. $(f_\alpha)_\beta(x) = f_{\alpha+\beta}(x) \qquad f \in L^1,\ \alpha > 0,\ \beta > 0.$

2°. *If* $f \in \mathrm{Lip}^{(p)}\,\alpha(W)$ *then* $f_\beta \in \mathrm{Lip}^{(p)}(\alpha+\beta)(W)$

similarly for lip *class* $p \geqq 1,\ \alpha > 0,\ \beta > 0.$

3°. *If* f *is in* $\mathfrak{P}_n$ *and* $\alpha > 0$, *then there is a constant* A_α, *depending only on* α, *such that* $\| f_{-\alpha}\|_p \leqq A_\alpha n^\alpha \| f\|_p$.

4°. *If* $f \in L^p$ $(1 \leqq p < \infty)$ *or* C *and* $\alpha > 1/p$, *then*

$$f_\alpha \in \mathrm{lip}^{(\infty)}(\alpha - 1/p)(W),$$

PROOF. 1° is directly verified by an application of Fubini theorem. Ad 2°: Lemmas 2 and 4 imply $I_\beta \in \mathrm{Lip}^{(1)}\,\beta(W)$, and the result follows from

Lemma 5. 3° follows from Lemma 3 upon "truncating" the formal series $I_{-\alpha}$:

$$f_{-\alpha}(x) = (I_{-\alpha} * f)(x) = (I_{-\alpha}^{(m)} * f)(x) \qquad (m = n(1) + 1).$$

Consequently

$$\| f_{-\alpha} \|_p \leqq \| I_{-\alpha}^{(m)} \|_1 \, \| f \|_p$$
$$\leqq B_\alpha 2^{m\alpha} \| f \|_p \leqq A_\alpha n^\alpha \| f \|_p .$$

To prove 4°, observe that $\|f(x+h)-f(x)\|_p = o(1)$ $(h \to 0)$. Now a combination of Lemma 2 and Lemma 5 yields the requied result.

The next theorem and its proof shows that our multiplier transformation is very close to fractional integration (cf. Zygmund [13]).

THEOREM 2. *If* $f \in L^p$ $(p > 1)$, $\alpha = \frac{1}{p} - \frac{1}{q} > 0$ *then* $f_\alpha \in L^q$ *and* $\| f \|_q \leqq A_{p,\alpha} \| f \|_p$.

PROOF. We begin with the special case $1 < p < 2$, $q = 2$, $\alpha = \frac{2-p}{2p}$. We may and do suppose that the mean value of $f(x)$ is 0. Our assertion is now equivalent to

$$\left(\sum_{\nu=1}^{\infty} 2^{-2\nu(1)\alpha} \, | c_\nu |^2 \right)^{1/2} \leqq A_\alpha \| f \|_p ,$$

where c_ν are the Fourier coefficients of f.

The left-hand member does not exceed, by Hölder's inequality,

$$A_\alpha \left(\sum \nu^{-2\alpha} \, | c_\nu |^2 \right)^{1/2} \leqq \left\{ \left(\sum | c_\nu |^{p'} \right)^{1/p'} \left(\sum \nu^{p-2} \, | c_\nu |^p \right)^{1/p} \right\}^{1/2}$$
$$\leqq \| f \|_p^{1/2} A_\alpha \| f \|_p^{1/2} = A_\alpha \| f \|_p$$

by well-known inequalities of Hausdorff-Young and Paley. (cf. [14], Chapter XII, Theorems (2. 8) and (5. 1)).

Now the Theorem is true for $\frac{1}{p} = \frac{1}{2} + \frac{\alpha}{2}, \frac{1}{q} = \frac{1}{2} - \frac{\alpha}{2}$.

For let g be a polynomial $g(x) = \sum d_\nu \psi_\nu(x)$, with $\| g \|_p = 1$. We have

$$\left| \int f_\alpha(x) \overline{g(x)} \, dx \right| = \left| \sum 2^{-\nu(1)\alpha} c_\nu \bar{d}_\nu \right|$$
$$\leqq A_\alpha \left(\sum \nu^{-\alpha} \, | c_\nu |^2 \right)^{1/2} \left(\sum \nu^{-\alpha} \, | d_\nu |^2 \right)^{1/2}$$

which does not exceed $A_\alpha \|f\|_p$ by the preceding case.

The proof will be complete if we prove the following Theorem:

THEOREM 3. *Let* $f \in L^1$, $0 < \alpha < 1$. *Then the operation* $f \to f_\alpha$ *is of weak type* $\left(1, \frac{1}{1-\alpha}\right)$. *That is, there exists a constant* A_α, *depending on* α *only, such that for any* $y > 0$,

$$m(\{x ; |f_\alpha(x)| > y\}) \leqq \left(\frac{A_\alpha}{y} \|f\|_1\right)^{\frac{1}{1-\alpha}}.$$

PROOF. We need the following lemmas:

LEMMA 6. *Let* z *be a positive number greater than* $\|f\|_1$. *Then the following decomposition is possible*:

(i) $f(x) = v(x) + w(x), \quad w(x) = \sum w_{ij}(x)$,

(ii) $|v(x)| \leqq 2z$ *for almost every* x,

(iii) $\|v\|_1 \leqq \|f\|_1$,

(iv) $\sum \|w_{ij}\|_1 \leqq 4\|f\|_1$,

(v) *there exist* $x_{ij} \in G$ *and neighborhood* V_i *of* 0, *such that* w_{ij} *vanishes outside* $V_i(x_{ij})$,

$V_i(x_{ij})$ *ane mutually disjoint*,

$$\sum_{i,j} m(V_i(x_{ij})) \leqq \frac{1}{z} \|f\|_1;$$

(vi) $\int w_{ij}(x)\,dx = 0$ *for every pair* (i,j).

This Lemma is due to S. Igari [3], and is a modification of the "decomposition lemma" of L. Hörmander [2].

LEMMA 7. *With the notations of the previous lemma, we have*

$$w_\alpha(x) = 0 \qquad \text{for} \qquad x \notin \bigcup_{i,j} V_i(x_{ij}) \equiv E.$$

PROOF. Fix a pair (i,j), and consider $u = w_{ij}$, $a = x_{ij}$. It is sufficient to prove that $u(x+a) = 0$ for $x \notin V = V_i$.

Now

$$u_\alpha(x \dotplus a) = \int u(t)\, I_\alpha(x \dotplus a \dotplus t)\, dt$$

$$= \int_{V(a)} u(t)\,(I_\alpha(x \dotplus a \dotplus t) - I_\alpha(x))\, dt$$

$$= \int_V u(t \dotplus a)\,(I_\alpha(x \dotplus t) - I_\alpha(x))\, dt\,.$$

Let us evaluate $I_\alpha(x \dotplus t) - I_\alpha(x)$ for $x \notin V$, $t \in V$. We have seen in the proof of Lemma 2, that, for $t \in V = V_i$,

$$I_\alpha(x \dotplus t) - I_\alpha(x)$$
$$= \sum_{j=i}^{\infty} 2^{-j\alpha}(\phi_j(x \dotplus t)\, D_{2^j}(x \dotplus t) - \phi_j(x)\, D_{2^j}(x))\,.$$

But, $x \notin V_i$, $t \in V_i$ implies $x \dotplus t \notin V_i$ (V_i being a subgroup of G). Since D_{2^j} vanishes outside V_j, all of the summands vanish, and so does $I_\alpha(x \dotplus t) - I_\alpha(x)$.

PROOF OF THEOREM 3. We may suppose $\|f\|_1 = 1$.
It is sufficient to prove the following two facts:

1°. $$m(\{x;\ |v_\alpha(x)| > y\}) \leqq A_\alpha y^{1/(\alpha-1)}$$

2°. $$m(\{x;\ |w_\alpha(x)| > y\}) \leqq A_\alpha y^{1/(\alpha-1)}\,.$$

Or, 2° is evident from Lemma 6, (v) and Lemma 7, put $z = y^{1/(1-\alpha)}$. To prove 1°, we use the special case of Theorem 2 already established. In fact

$$m(\{x;\ |v_\alpha(x)| > y\}) \leqq y^{-q}\int |v_\alpha(x)|^q\, dx \qquad \left(\frac{1}{q} = \frac{1}{2} - \frac{\alpha}{2}\right)$$

$$\leqq y^{-p} A_\alpha \left(\int |v(x)|^p\, dx\right)^{q/p}$$

$$\leqq A_\alpha y^{-q} z^{(p-1)q/p} \|v\|_1^{q/p} \leqq A_\alpha y^\beta \|f\|_1 = A_\alpha y^\beta$$

where $\beta = -q + (p-1)\,q/p(1-\alpha) = -1/(1-\alpha)$, q. e. d.

The proof of Theorem 2 is completed by an application of Marcinkiewicz interpolation Theorem ([14] Chapter XII, Theorem (4.6)), since $I_\alpha \in L^1$ implies $\|f_\alpha\|_\infty \leqq A_\alpha \|f\|_\infty$.

In the theory of TFS, it is well known that a formally integrated Fourier series converges uniformly. This is not the case for $f_1(x) = (I_1 * f)(x)$, $f \in L^1$, though there is a partial substitute, as indicates the following theorem.

THEOREM 4. *Let* $f \in L^1$, $f(x) \sim \sum c_k \psi_k(x)$. *Then we have*

$$\| L(x; f)\|_p \leqq A_p \| f \|_1 \qquad (0 < p < \infty),$$

where

$$L(x; f) = \sup_n | s_n(x; f) | = \sup_n |(f_1 * D_n)(x)|$$

and A_p *depends only on* p.

PROOF. Putting $m = n(1)$ we have

$$\begin{aligned} s_n(x; f_1) &= c_0 + \sum_{\nu=1}^{n-1} 2^{-\nu(1)} c_\nu \psi_\nu(x) \\ &= c_0 + \sum_{\nu=1}^{2^m-1} 2^{-\nu(1)} c_\nu \psi_\nu(x) + \sum_{\nu=2^m}^{n-1} 2^{-\nu(1)} c_\nu \psi_\nu(x) \\ &= c_0 + \sum_{j=0}^{m-1} 2^{-j} \delta_j(x; f) + 2^{-m} \sum_{\nu=2^m}^{n-1} c_\nu \psi_\nu(x) = c_0 + S_1 + S_2, \quad \text{say.} \end{aligned}$$

where $\delta_j(x; f) = s_{2^{j+1}}(x; f) - s_{2^j}(x; f) = \int f(x \dotplus t) \phi_j(t) D_{2^j}(t)\, dt$. Since $|c_\nu| \leqq \|f\|_1$ for every ν, it is clear that $\| S_2 \|_\infty \leqq \| f \|_1$. On the other hand,

$$|\delta_j(x; f)| \leqq \int |f(x \dotplus t)|\, D_{2^j}(t)\, dt$$

implies, for $p \geqq 1$,

$$\| \delta_j(x; f)\|_p \leqq \| f \|_1 \| D_{2^j} \|_p \leqq 2^{j/q} | f \|_1,$$

where $q = p' = p/(p-1)$. This inequality, combined with

$$\begin{aligned} | s_{2^m}(x; f_1)| &\leqq | c_0 | + \sum_{j=0}^{m-1} 2^{-j} | \delta_j(x; f)| \\ &\leqq | c_0 | + \sum_{j=0}^{\infty} 2^{-j} | \delta_j(x; f)| \end{aligned}$$

gives

$$\begin{aligned} \| \sup_m | s_{2^m}(x; f_1)| \|_p &\leqq | c_0 | + \sum_{j=0}^{\infty} 2^{-j} \|\delta_j(x; f)\|_p \\ &\leqq \| f \|_1 + \sum_{j=0}^{\infty} 2^{-j(1-1/q)} \| f \|_1 = A_p \| f \|_1 . \end{aligned}$$

This yields the required estimate for S_1, and the proof of complete.

The theorem ceases to be true for $p=\infty$; in fact, consider the series $\sum_{\nu=1}^{\infty}\frac{\psi_\nu(x)}{\log(\nu+1)}$, which is the Fourier series of an integrable function $f(x)$, for which $f_1(x)$ is not bounded in any neighborhood of 0 (S. Yano [12]).

5. Series with random signs. Another substitute, yielding the uniform convergence of multiplier transforms, is obtained by considering series with random signs. The following theorem is the Walsh analogue of a result of Paley and Zygmund (see [14], Chapter V, Theorem (8, 34)).

THEOREM 5. (i) *Suppose* $\sum_{\nu=0}^{\infty} a_\nu^2<\infty$. *Then the "random Walsh series"* $\sum_{\nu=0}^{\infty} a_\nu\phi_\nu(t)\psi_\nu(x)$ *has, for almost all* t, *partial sums of magnitude* $o((\log n)^{1/2})$, *uniformly in* x.

(ii) *If* $\sum a_\nu^2(\log\nu)^{1+\varepsilon}<\infty$ *for some* $\varepsilon>0$, *then, for almost all* t, *the series* $\sum_{\nu=0}^{\infty} a_\nu\phi_\nu(t)\psi_\nu(x)$ *converges uniformly in* x.

The proof of this theorem is a repetition of that of the trigonometric case due to Salem and Zygmund, the only difference being the use of a fact that a Walsh polynomial is of constant value when it is restricted to a suitable neighborhood of a point. Thus we omit the proof, referring the reader to Zygmund [14], Chapter V, pp. 219-220. The following corollary, however, seems to be new.

COROLLARY. *There exists a set* E *of Haar measure* 1 *such that for any* $f\in L^1$, $f(x)\sim\sum a_\nu\psi_\nu(x)$ *and for any* $\alpha>1/2$, $t\in E$ *implies the uniform convergence of the formal series*

$$f_{\alpha,t}(x)=a_0\phi_0(t)+\sum_{\nu=1}^{\infty}2^{-\nu(1)\alpha}a_\nu\phi_\nu(t)\psi_\nu(x)\,.$$

PROOF. From Theorem 3 (ii), the series

$$I_{\alpha,t}(x)=\phi_0(t)+\sum_{\nu=1}^{\infty}2^{-1(1)\alpha}\phi_\nu(t)\psi_\nu(x)$$

converges, for a fixed $\alpha>1/2$ and for almost all t (say for $t\in E_\alpha$), uniformly in x, representing consequently a continuous function $I_{\alpha,t}(x)$. Or, it is easily seen that the sets E_α are increasing with respect to α. Put $E=\bigcap\{E_\alpha;\ \alpha$ rational, $\alpha>1/2\}$. Then $E\subset E_\alpha$ for $\alpha>1/2$ with α rational or irrational and E is of measure 1. It is now sufficient to observe that $f_{\alpha,t}=f*I_{\alpha,t}$ and $s_n(x;f_{\alpha,t})=(s_n(\cdot\,;I_{\alpha,t})*f)(x)$; the uniform convergence of $s_n(x;I_{\alpha,t})$ proves our assertion.

6. Multiplier $\{\nu^{-\alpha}\}$. The above theorems remain true if we consider $\nu^{-\alpha}$ instead of $2^{-\nu(1)\alpha}$. Let

$$J_\alpha(x)=1+\sum_{\nu=1}^{\infty}\nu^{-\alpha}\psi_\nu(x)\qquad(\alpha>0).$$

Repeated use of Abel transformations shows that $J_\alpha\in L^1$ and Theorem 1 is re-proved easily. The special case of Theorem 2 requires no change, and Theorem 3 will be based on the fact $J_\alpha(x)\leqq A_\alpha H_\alpha(x)$, where $H_\alpha(x)\equiv 2^{n(1-\alpha)}$ $(x\in V_n-V_{n+1})$, $n=0,1,2,\cdots$, $(0<\alpha<1)$. Lemma 7, with $H_\alpha(x)$ in place of $J_\alpha(x)$, remains true and the rest is similarly carried on.

If one could prove that the formal series

$$1+\sum_{\nu=1}^{\infty}\frac{2^{-\nu(1)}}{\nu}\psi_\nu(x)$$

should be a Walsh-Fourier- Stieltjes series, one would have a unified treatment of the two classes of multipliers $2^{-\nu(1)\alpha}$ and $\nu^{-\alpha}$; but the present author has been unable to prove this statement. However, for functions belonging to L^p $(1<p<\infty)$, we have

THEOREM 6. *Let* $\lambda_0=1$, $\lambda_\nu=2^{\nu(1)\alpha}/\nu^\alpha$ $\nu=1,2,\cdots$ *where* α *is a fixed real number, and let* $f\in L^p$, $1<p<\infty$, $f(x)\sim\sum c_\nu\psi_\nu(x)$. *Then* $\sum\lambda_\nu c_\nu\psi_\nu(x)$ *is the Fourier series of a function* Λf *in* L^p *and*

$$\|\Lambda f\|_p\leqq A_{\alpha,p}\|f\|_p.$$

This theorem is a special case of the Walsh analogue of a theorem of J. Marcinkiewicz [4] (see also [14], Chapter XV, P. 232) and proved similarly. The main step (corresponding to [14], Chapter XV, Lemma (2. 15)) has already been proved by G. Sunouchi ([6], Theorem 1).

7. Application to the theory of approximation. If a 2π-periodic function $f(x)$ has its TFS $\sum A_\nu(x)$, the formal trigonometric series $\sum\nu^\lambda A_\nu(x)$ plays an important role in the process of (trigonometric) approximation to $f(x)$ (see e.g. [8]). A similar fact holds for WFS. Let $f(x) \in L^1$ and let its WFS be $\sum c_\nu \psi_\nu(x)$. If $g_\nu(n)$ ($\nu = 0, 1, 2, \cdots$) is the sequence of Walsh-Fourier-Stieltjes coefficients of a bounded measure $\mu^{(n)}$ on G, with $g_0(n) = \int d\mu^{(n)} = 1$, we have multiplier transforms

$$P_n(x) = P_n(x\,; f) = (f * \mu^{(n)})(x) \sim \sum_{\nu=0}^{\infty} c_\nu g_\nu(n) \psi_\nu(x)\,,$$

where the parameter n need not be discrete.

If there exist a positive non-increasing function $\varphi(n)$ and a class K of functions in such a way that

(I) $\quad \|f - f * \mu^{(n)}\|_p = o(\varphi(n))$ implies $f(x) = \text{constant}$;

(II) $\quad \|f - f * \mu^{(n)}\|_p = O(\varphi(n))$ implies $f(x) \in K$;

(III) $\quad f(x) \in K$ implies $\|f - f * \mu^{(n)}\|_p = O(\varphi(n))$

then we say that the method of approximation with multiplier transforms defined by $\mu^{(n)}$ is saturated with the order $\varphi(n)$ and with the class K.

Suppose that there exist a positive constant c and sequence $\{\rho^{(\nu)}\}$, $\nu = 1, 2, \cdots$ for which

$$\lim_{n\to\infty} \frac{1 - g_\nu(n)}{\varphi(n)} = c\rho(\nu) \qquad (\nu = 1, 2, \cdots)\,,$$

then we can prove, by a standard weak compactness argument (we may take here the 2^N-th patial sum of the WFS of $(f - f * \mu^{(n)})$ instead of $(C,1)$-means, used in the case of TFS) that our method is saturated with the order $\varphi(n)$ and the class of those functions $f(x)$ for which

$$(*) \qquad \left\| \sum_{\nu=1}^{2^N-1} c_\nu \rho^{(\nu)} \psi_\nu(x) \right\|_p = O(1)$$

provided that the assertion (III) is verified by the properties of $\mu^{(n)}$. The relation $(*)$ is equivalent to, respectively,

$\sum c_\nu \rho(\nu) \psi_\nu(x)$ is the WFS of a bounded function $(p = \infty)$

$\sum c_\nu \rho(\nu) \psi_\nu(x)$ is the WFS of a function in L^p $\quad (1 < p < \infty)$

$\sum c_\nu \rho(\nu) \psi_\nu(x)$ is the Walsh-Fourier-Stieltjes series of a bounded measure on G $(p = 1)$.

For most of the well-known summability methods, the sequence $\rho(\nu)$ is of the form ν^{λ}, where λ is a positive number, and (III) is proved by a direct estimation. If we denote by $W^{\lambda}=W^{(p)\lambda}$ the class of all WFS for which $(*)$ holds with $\rho(\nu)=\nu^{\lambda}$, we have the following

THEOREM 7. *Let* $\lambda>0$ *and let* $T=(T_n)$ *be a linear approximation process with*

$$(1) \qquad \| T_n(f)(x) \|_p \leqq M_1 \| f \|_p$$

$$(2) \qquad \| f(x) - T_n(f)(x) \|_p = M_2 n^{-\lambda} \| f^{[\lambda]} \|_p \quad \textit{for} \quad f \in W^{\lambda}.$$

Then $f \in \mathrm{Lip}^{(p)} \alpha(W) \quad 0<\alpha<\lambda$ *implies*

$$\| f(x) - T_n(f)(x) \|_p = O(n^{-\alpha}),$$

where $f^{[\lambda]}$ *is (the function or the measure represented by) the formal series*

$$\sum_{\nu=1}^{\infty} c_\nu \nu^{\lambda} \psi_\nu(x).$$

This theorem was first proved by G. Sunouchi [7] in the theory of the trigonometric approximation; a different proof (with a slight generalization), which applies also for Walsh system, is found in Watari [10].

COROLLARY. *If* $f(x) \in \mathrm{Lip}^{(p)} \alpha(W)$ $1<p<\infty$, $0<\alpha<1$, *then for any* $\beta>0$ $\|\sigma_n^{\beta}(x;f)-f(x)\|_p = O(n^{-\alpha})$, *where* $\sigma_n^{\beta}(x;f)$ *denotes the n-th* (C,β) *means of the WFS of* $f(x)$.

For the proof it suffices to see that the approximation by σ_n^{β} is saturated with the order $1/n$ and the class $\{f : f^{[1]} \in L^p\}$; this fact being a consequence of Paley's decomposition theorem and multiplier theorem of Marcinkiewicz (see Theorem 6 above).

This result was proved, under an additional condition $\beta>\alpha$, by S. Yano [11]. For the trigonometric system, this is due to G. Sunouchi [7].

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